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“Wisdom begins in wonder.”

Socrates

"You can't direct the wind – but you can adjust the sails."

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CH 0 – PROLOGUE

THE REAL NUMBERS

You may have seen many different kinds of numbers in your previous math courses. Consider these examples:

$$7 \qquad 0 \qquad -9 \qquad 2.835 \qquad -\frac{15}{4} \qquad \frac{2}{3} \qquad \frac{7}{33} \qquad \pi \qquad -\sqrt{2}$$

As varied as all these numbers may seem, they actually have one critical <u>common</u> characteristic: They can all be written as ***decimal numbers***, some repeating and some non-repeating:

$7 = 7.0$	Repeating decimal (zeros forever)
$0 = 0.0$	Repeating decimal (zeros forever)
$-9 = -9.0$	Repeating decimal (zeros forever)
2.835	Repeating decimal (zeros forever)
$-\frac{15}{4} = -3.75$	Repeating decimal (zeros forever)
$\frac{2}{3} = 0.6666666\ldots$	Repeating decimal (6's forever)
$\frac{7}{33} = 0.21212121\ldots$	Repeating decimal (21's forever)
$\pi = 3.14159265\ldots$	Non-repeating decimal
$-\sqrt{2} = -1.41421356\ldots$	Non-repeating decimal

The repeating decimals are called ***rational numbers***, and the non-repeating decimals are called ***irrational numbers.***

Note that some of the decimals repeat a block of digits forever (the ***rational*** numbers), while some don't repeat (the ***irrational*** numbers). Nevertheless, they are all decimals.

But what about a number like $\sqrt{-9}$? This number must be a number whose square is –9. Now, what number do we know which, when squared, would come out –9? Does 3 work? No, since $3^2 = 9$. Does –3 work? No, since $(-3)^2 = 9$, also. We conclude that there is <u>no</u> decimal in the world that can represent the number $\sqrt{-9}$, or for that matter, the square root of any negative number.

To distinguish between the numbers that are decimals and numbers like $\sqrt{-9}$, which can never be written as a decimal, the term ***real number*** was given to the decimals, and the term ***imaginary number*** was given to numbers like $\sqrt{-9}$.

Hundreds of years ago, mathematicians thought it was obvious which numbers were real and which were imaginary. But this demonstrates a rather arrogant attitude. After all, to a beginning algebra student, a real number like $\sqrt{2}$ (which is an infinite, non-repeating decimal) may not seem "real" at all. Moreover, imaginary numbers, like $\sqrt{-1}$, seem very real to people (such as electronics engineers) who use them every day. The bottom line is, the terms *real* and *imaginary* are completely arbitrary -- one person's reality is another's imagination. But we're stuck with the terms, so we might as well learn them.

The ***Real Numbers*** is the combination of the *Rational Numbers* and the *Irrational Numbers*.

In summary, we call any number that can be written as a decimal a ***real number***, regardless of whether it repeats or not. The set of real numbers is often denoted by writing $\mathbb{R}$.

Homework

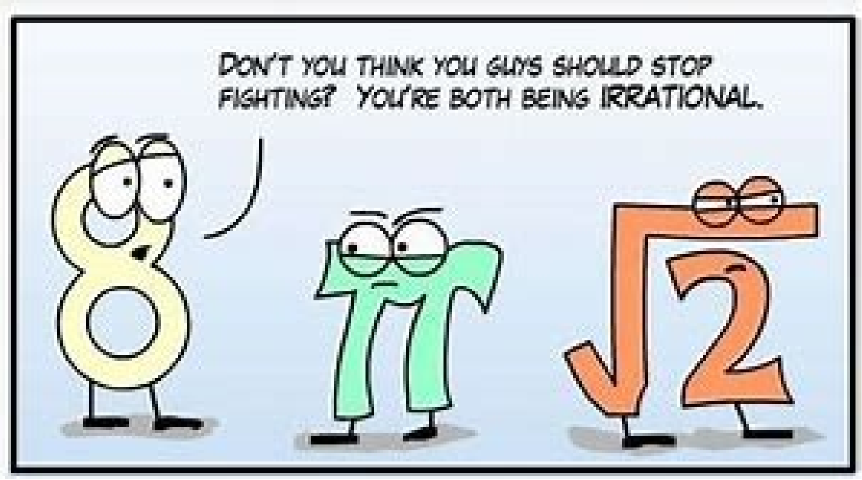
1. Classify each number as **real** or **imaginary**:

a. 123
b. -42
c. 0
d. 2.3
e. $\sqrt{-8}$
f. $\sqrt{144}$
g. $-\sqrt{81}$
h. $\sqrt{10}$
i. -23.78
j. $-\pi$
k. $\sqrt{3}$
l. $\sqrt{-121}$
m. 0.239057
n. $2.787878...$
o. 3.092748526
p. $3.1428669...$
q. $\sqrt{-(-8765)}$
r. $\sqrt{-0.25}$
s. $-\sqrt{-25}$
t. $\sqrt{-(-71)}$
u. 10^6

2. Put the following 13 real numbers in ascending order (smallest to biggest):

$$2\pi, \quad \sqrt{5}, \quad \sqrt{0}, \quad \frac{11}{3}, \quad 3.0808..., \quad -\pi,$$

$$\frac{1}{101}, \quad -\sqrt{3}, \quad 2^3, \quad 3^2, \quad -1, \quad \sqrt{25}, \quad \sqrt{1}$$



❑ *THREE OF THE THINGS WE DO TO REAL NUMBERS*

Opposite

The ***opposite*** of a real number is found by changing the sign of the number. For example, the opposite of 7 is –7, the opposite of $-\pi$ is π, and the opposite of 0 is 0 (since 0 doesn't really have a sign). The opposite of n is $-n$, and the opposite of $-n$ is n. Also notice that the sum of a number and its opposite is always 0; for example, $17 + (-17) = 0$.

When considering numbers on the real number line, two numbers are opposites of each other if they're the same distance from 0, but on opposite sides of 0. [Note that although 0 is the opposite of 0, it's kind of hard to justify the claim that they're on "opposite" sides of 0.]

Homework

3. What is the **opposite** of each number?

   a. 17 b. 0 c. –3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.
   b. The opposite of 0 is ____.
   c. The opposite of a negative number is always _____.
   d. The opposite of a positive number is always _____.
   e. The sum of a real number and its opposite is always _____.

5. Using the formula $y = -x$, find the y-value for the given x-value:

   a. $x = 9$ b. $x = -3$ c. $x = 0$ d. $x = \pi$ e. $x = -\sqrt{2}$

Reciprocal

The ***reciprocal*** of a real number is found by dividing the number into 1. Equivalently, the reciprocal of x is $\frac{1}{x}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. Every real number has a reciprocal except 0; the reciprocal of 0 would be $\frac{1}{0}$, which is undefined, as explained later in the Prologue.

Notice that the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative. In addition, **the product of any real number with its reciprocal is always 1**; for example, $\frac{2}{7} \cdot \frac{7}{2} = 1$.

Homework

6. Find the reciprocal of each real number:

   a. 5 b. $\frac{2}{9}$ c. $-\frac{7}{3}$ d. 1 e. 0 f. $\frac{1}{\pi}$ g. $-\sqrt{3}$

7. a. T/F: Every number has a reciprocal.

   b. The reciprocal of 0 is ____.

   c. The reciprocal of a negative number is always _____.

   d. The reciprocal of a positive number is always _____.

   e. The product of a real number and its reciprocal is always ____.

8. Using the formula $y = \frac{1}{x}$, answer each question:

   a. If $x = 14$, then $y =$ ____.

   b. If $x = \frac{2}{3}$, then $y =$ ____.

   c. If $x = -99$, then $y =$ ____.

   d. If $x = -\frac{5}{4}$, then $y =$ ____.

   e. If $x = 0$, then $y =$ ____.

Absolute Value

The ***absolute value*** of a real number is its distance to 0 on the number line. The absolute value of 9 is 9, because the number 9 is 9 units from 0 on the number line. The absolute value of –5 is 5, because the number –5 is 5 units from 0 on the number line. As for the real number 0, its absolute value is 0, since the number 0 is 0 units away from 0.

The notation for absolute value is two vertical bars around the number. So, for example, the absolute value of –12 is written $|-12|$, and equals 12. Here are three more examples:

$|35| = 35$ $\quad |-2.7| = 2.7$ $\quad |0| = 0$

In computer programming, $|-12|$ would be written
abs(12)

Homework

9. Evaluate each expression:

a. $|5\pi|$ b. $|-23.9|$ c. $|0|$ d. $|7-9|$ e. $|-\sqrt{7}|$

10. a. T/F: Every number has an absolute value.
b. The absolute value of 0 is ____.
c. The absolute value of a negative number is always _____.
d. The absolute value of a positive number is always _____.

11. Which two of the following operations can be applied to all real number?

Opposite Reciprocal Absolute Value

12. Find the **absolute value** of each number:

a. 72 b. –99 c. 0 d. π e. $-\pi$ f. $-\sqrt{2}$

13. Evaluate each expression:

a. $|17-7|$ b. $|3-25|$ c. $|2(3)-6(1)|$ d. $|2\pi+3\pi|$

14. Using the formula $y = |x|$, answer each question:

a. If $x = 33$, then $y =$ ____.

b. If $x = 0$, then $y =$ ____.

c. If $x = -25$, then $y =$ ____.

d. If $y = 17$, then $x =$ ____ or ____.

e. If $y = 0$ then $x =$ ____.

f. If $y = -5$, then $x =$ ____.

☐ *FRACTIONS*

EXAMPLE 1:

A. **Express $\frac{7x}{2}$ as the product of two quantities:**

Solution: The easiest way to see this process is to just do it and then check that it's right. Here's what I claim:

$$\frac{7x}{2} = \boldsymbol{\frac{7}{2}x}$$

And here's the reason:

$$\frac{7}{2}x = \frac{7}{2}\cdot\frac{x}{1} = \frac{7\cdot x}{2\cdot 1} = \frac{7x}{2} \quad ✓$$

B. **Express $\mathbf{\frac{7x+9}{-5}}$ as the sum or difference of two fractions.**

Solution: What two fractions have a sum of $\frac{7x+9}{-5}$? The –5 tells us that we could use fractions with a denominator of –5. Since the numerator is $7x + 9$, we can make one of the numerators $7x$ and the other one 9. That is

$$\frac{7x+9}{-5} = \frac{7x}{-5} + \frac{9}{-5} = -\frac{7}{5}x - \frac{9}{5}$$

C. **Combine the ideas of parts A and B to split up $\mathbf{\frac{8x-5}{7}}$.**

Solution:

$$\frac{8x-5}{7} = \frac{8x}{7} - \frac{5}{7} = \mathbf{\frac{8}{7}x - \frac{5}{7}}$$

D. **Express $\mathbf{\frac{8x+16}{16}}$ as the sum of two quantities.**

Solution:

$$\frac{8x+16}{16} = \frac{8x}{16} + \frac{16}{16} = \frac{x}{2} + 1 = \mathbf{\frac{1}{2}x + 1}$$

Homework

15. Express each fraction as the product of two quantities:

a. $\frac{9x}{4}$ b. $\frac{17x}{2}$ c. $\frac{3y}{16}$ d. $\frac{-33a}{17}$ e. $\frac{5n}{-23}$

f. $\frac{3x}{6}$ g. $\frac{-6x}{-2}$ h. $\frac{22m}{33}$ i. $\frac{-9x}{-15}$ j. $\frac{-39z}{52}$

16. Express each fraction as the sum or difference of two quantities, using parts C and D of Example 1 as a guide:

a. $\frac{3x+8}{4}$ b. $\frac{-9x+18}{6}$ c. $\frac{y-1}{8}$ d. $\frac{3n+15}{-5}$

e. $\frac{-3w-24}{2}$ f. $\frac{45x-75}{15}$ g. $\frac{33x+44}{55}$ h. $\frac{-31x-17}{-20}$

Operations with Fractions

$$-\frac{2}{3}-\frac{1}{2} = -\frac{4}{6}-\frac{3}{6} = -\mathbf{\frac{7}{6}}$$

$$\frac{4}{5}-\left(-\frac{2}{3}\right) = \frac{4}{5}+\frac{2}{3} = \frac{12}{15}+\frac{10}{15} = \mathbf{\frac{22}{15}}$$

$$\frac{2}{9}-7 = \frac{2}{9}-\frac{63}{9} = -\mathbf{\frac{61}{9}}$$

$$\left(\frac{2}{3}\right)\left(-\frac{5}{7}\right) = -\mathbf{\frac{10}{21}}$$

$$-\frac{4}{7}\div-2 = -\frac{4}{7}\times-\frac{1}{2} = \frac{4}{14} = \mathbf{\frac{2}{7}}$$

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{5}\right) = \mathbf{\frac{1}{120}}$$

$$\frac{-8}{\frac{1}{3}} = -\frac{8}{1}\div\frac{1}{3} = -\frac{8}{1}\times\frac{3}{1} = -\mathbf{24}$$

$$\frac{-\frac{9}{4}}{-2} = -\frac{9}{4}\div-2 = -\frac{9}{4}\div-\frac{2}{1} = -\frac{9}{4}\times-\frac{1}{2} = \mathbf{\frac{9}{8}}$$

Note: A negative sign can “float.” For instance,

$$\frac{-30}{6} = \frac{30}{-6} = -\frac{30}{6}$$

since all of these fractions have the value –5.

Powers and Square Roots of Fractions

An **exponent** still means what it always has, so these next examples should be clear.

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3}\cdot\frac{2}{3} = \mathbf{\frac{4}{9}} \qquad \left(-\frac{1}{4}\right)^3 = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) = -\mathbf{\frac{1}{64}}$$

$$\left(-\frac{9}{4}\right)^1 = -\mathbf{\frac{9}{4}} \qquad \left(\frac{1}{2}\right)^8 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \mathbf{\frac{1}{256}}$$

As for the **square root sign**, we still ask the question: What number (that’s not negative) times itself gives the number in the radical sign?

$\sqrt{\frac{9}{25}} = \mathbf{\frac{3}{5}}$ This is true because $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

$\sqrt{\frac{1}{144}} = \mathbf{\frac{1}{12}}$ This is because $\frac{1}{12}\times\frac{1}{12} = \frac{1}{144}$.

$\sqrt{-\frac{4}{49}}$ does <u>not</u> exist as a real number, because $-\frac{4}{49}$ is a negative number, and square roots of negative numbers are outside the real numbers. It’s an imaginary number.

$\sqrt{\frac{-4}{-49}}$ <u>does</u> exist as a real number, because the fraction is actually a positive number: $\sqrt{\frac{-4}{-49}} = \sqrt{\frac{4}{49}} = \mathbf{\frac{2}{7}}$.

Homework

Perform the indicated operation:

17. a. $-\frac{1}{2}-\frac{4}{5}$ b. $-\frac{1}{3}-\left(-\frac{1}{3}\right)$ c. $\frac{2}{3}-\left(-\frac{5}{6}\right)$

d. $-\frac{4}{5}+\frac{2}{3}$ e. $9-\frac{4}{5}$ f. $-1-\frac{2}{3}$

g. $\frac{8}{3}-5$ h. $-\frac{2}{3}-(-1)$ i. $-\frac{1}{4}-\frac{2}{7}$

18. a. $\left(-\frac{1}{2}\right)\left(-\frac{5}{6}\right)$ b. $\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)$ c. $-\frac{5}{6}\cdot-\frac{6}{5}$

d. $-\frac{2}{3}\div-\frac{3}{2}$ e. $\frac{1}{2}\div-9$ f. $7\div-\frac{3}{4}$

g. $\dfrac{-\frac{2}{3}}{-\frac{1}{9}}$ h. $\dfrac{\frac{4}{5}}{-8}$ i. $\dfrac{-\frac{4}{5}}{\frac{5}{8}}$

19. True/False: $\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}$ [assuming $b \neq 0$]

20. a. $\left(-\frac{2}{3}\right)^2$ b. $\left(-\frac{1}{2}\right)^3$ c. $\left(-\frac{1}{3}\right)^4$

d. $\left(-\frac{14}{19}\right)^1$ e. $\left(-\frac{1}{2}\right)^5$ f. $\left(-\frac{2}{3}\right)^6$

g. $\left(\frac{99}{-99}\right)^{99}$

21. a. $\sqrt{\frac{81}{100}}$ b. $\sqrt{\frac{36}{64}}$ c. $\sqrt{\frac{1}{4}}$

d. $\sqrt{\frac{1}{9}}$ e. $\sqrt{\frac{121}{144}}$ f. $\sqrt{-\frac{25}{81}}$

g. $\sqrt{\frac{-256}{-289}}$ h. $\sqrt{-\frac{14}{17}-\left(-\frac{14}{17}\right)}$

❑ *ORDER OF OPERATIONS*

Order of Operations
Parentheses and Brackets [( )]
Exponents
Multiply & Divide (left to right)
Add & Subtract (left to right)

Note: Certainly $(-5)^2 = 25$, since both the 5 and the minus sign are being squared [i.e., $(-5)^2 = (-5)(-5) = 25$]. However, consider the expression

$$-5^2$$

Do we square the –5? The answer is NO; the exponent attaches to the 5 <u>only</u>. The justification is the Order of Operations, which states that exponents (near the top of the chart) are to be done before we deal with negative signs (which are at the bottom of the chart). So, although $(-5)^2 = 25$, we must agree that

$$-5^2 = -25$$

Homework

22. Evaluate (simplify) each expression:

a. $3 \cdot 10^2 - (8 - 4)^3 - 3 \times 2$

b. $(5 - 3)^2 + (10 - 7)^3$

c. $[3 + 2(5)] - 1 + 3 \cdot 10$

d. $2(10 - 5)^2 - 12 \div 3$

e. $[2(10 - 5)]^3 \div (10 \cdot 10^2)$

f. $(1 + 4)^2 - (4 + 1)^2$

g. $[(3^2 - 2^2)^3 - 80] \div (36 / 4)$

h. $3 \cdot 4^2 - (13 - 12)^3$

i. $10 + 8(8 - 1)^2 - 3 - 2 - 1$

j. $[8^2 - 2^3 + 3 \cdot 4 - 2(7)]^2$

k. $[20 - (5 - 2)^2]^2 - 2 \cdot 3 \cdot 4$

l. $[13 - (8 - 3) + (10 - 2)]^3$

23. Evaluate each expression for the given values:

a. $(x + y)^2$ for $x = 2$ and $y = 1$

b. $x^2 + y^2$ for $x = 10$ and $y = 5$

c. $x^2 + xy + y^2$ for $x = 3$ and $y = 6$

d. $(x + y)(x - y)$ for $x = 10$ and $y = 2$

e. $x^2 - y^2$ for $x = 12$ and $y = 10$

☐ TERMS

If the final operation in an expression is multiplication, then the expression consists of one term. The expression $A(B + C - D)$ consists of one term.

If the final operation consists of additions and/or subtractions, then the expression consists of at least two terms – the number of terms is found by counting the things being added or subtracted. The expression $ax - b + L(R + Q)$ consists of three terms: the ax, the b, and the $L(R + Q)$.

Homework

24. Determine the number of **terms** in each expression:

a. abc
b. $a + b - c$
c. $xyz - w$
d. $(x - 3)^2$
e. $x^2 + 25$
f. $ab + ac - xy$
g. $(a + b)(x - y)$
h. $rst - qrw$
i. $(rst)(qrw)$
j. $(rst)(qrw) - 1$
k. $[(rst)(qrw) - 1]^5$
l. $25 - (x + y - z)^2$
m. $a - b$
n. $x + y^2 + z$
o. $20 - mnpq$
p. $a + x - c + QT$
q. $(y + xm)^2A + B$
r. $(a - b)^3 - (c - d)^3 - w$
s. $abcd + x - y + wxyz$
t. $[a(b - c)ed - 7]^4$
u. $w(u - x)^3 - abc(def - mnpq)$

☐ DIVISION WITH ZEROS

This is the result of dividing by zero.

It's a mathematical fact of life that the only number that's never allowed to be in the denominator (bottom) of a fraction is zero. Sometimes this is phrased

"Never divide by zero."

Why the big deal?

Recall from elementary school that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work.

Zero on the Bottom

Now let's put a zero on the bottom and see what happens:

$$\frac{9}{0} = ???$$

Let's try an answer of 0; unfortunately $0 \cdot 0 = 0$, not 9.

How about we try an answer of 9? Then $9 \cdot 0$ is also 0, not 9.

Could the answer be π? No; $\pi \cdot 0 = 0$, not 9.

In fact, any number we surmise as the answer will have to multiply with 0 to make a product of 9. But this is impossible, since any number times 0 is always 0, never 9. In short, no number in the whole world will work in this problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0; in fact, since $0 \cdot 0 = 0$, a possible answer is 0.

Let's try an answer of 5; because $5 \cdot 0 = 0$, another possible answer is 5.

Could π possibly work? Since $\pi \cdot 0 = 0$, another possible answer is π.

Is there any end to this madness? Apparently not, since any number we conjure up will multiply with 0 to make a product of 0. In short, every number in the whole world will work in this problem.

Summary:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is no answer to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are infinitely many answers to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

Thus,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

> "Black holes are where God divided by zero."
>
> *Steven Wright*

Homework

25. Evaluate each expression, and explain your conclusion:

a. $\frac{0}{15}$ b. $\frac{32}{0}$ c. $\frac{0}{0}$

26. Evaluate each expression:

a. $\frac{0}{17}$ b. $\frac{0}{-9}$ c. $\frac{6-6}{17+3}$ d. $\frac{3^2-8-1}{100}$

e. $\frac{98}{0}$ f. $\frac{-44}{0}$ g. $\frac{7+8}{2^3-8}$ h. $\frac{7^2-40}{-23+23}$

i. $\frac{0}{0}$ j. $\frac{-9+9}{10-10}$ k. $\frac{5^2-25}{0^2+0^3}$ l. $\frac{4\cdot5-2\cdot10}{3^3-9}$

27. $\frac{0}{\pi} = 0$ because

    a. 0 is the only number multiplied by π that will produce 0.
    b. no number times π equals 0.
    c. every number times π equals 0.

28. $\frac{0}{0}$ is undefined because

    a. no number times 0 equals 0.
    b. every number times 0 equals 0.
    c. any number divided by itself is 1.

29. $\frac{7}{0}$ is undefined because

    a. 0 is the only number multiplied by 0 that will produce 7.
    b. no number times 0 equals 7.
    c. every number times 0 equals 7.

30. a. The numerator of a fraction is 0. What can you conclude?
    b. The denominator of a fraction is 0. What can you conclude?

❑ LINEAR EQUATIONS AND FORMULAS

Solve for x: $\mathbf{2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)}$

Solution: The steps are

1) Distribute
2) Combine like terms
3) Solve the simplified equation

$$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$$

$$\Rightarrow \quad 6x - 14 - 5 + 15x = 4x - 1 + x + 7 \qquad \text{(distribute)}$$

$$\Rightarrow \quad 21x - 19 = 5x + 6 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 21x - \mathbf{5x} - 19 = 5x - \mathbf{5x} + 6 \qquad \text{(subtract } 5x \text{ from each side)}$$

$$\Rightarrow \quad 16x - 19 = 6 \qquad \text{(simplify)}$$

$$\Rightarrow \quad 16x - 19 \mathbf{+ 19} = 6 \mathbf{+ 19} \qquad \text{(add 19 to each side)}$$

$$\Rightarrow \quad 16x = 25 \qquad \text{(simplify)}$$

$$\Rightarrow \quad \frac{16x}{\mathbf{16}} = \frac{25}{\mathbf{16}} \qquad \text{(divide each side by 16)}$$

$$\Rightarrow \quad \boxed{x = \frac{25}{16}} \qquad \text{(simplify)}$$

<u>Solve for x:</u> $\boldsymbol{\dfrac{nx - w}{y + z} = e - f}$

<u>Solution:</u> Notice the use of parentheses in the solution.

$$\frac{nx - w}{y + z} = e - f \qquad \text{(original formula)}$$

$$\Rightarrow \quad \frac{nx - w}{y + z}(y + z) = (e - f)(y + z) \qquad \text{(multiply each side by } y + z\text{)}$$

$$\Rightarrow \quad nx - w = (e - f)(y + z) \qquad \text{(simplify)}$$

$$\Rightarrow \quad nx = (e - f)(y + z) + w \qquad \text{(add } w \text{ to each side)}$$

$$\Rightarrow \quad \boxed{x = \frac{(e - f)(y + z) + w}{n}} \qquad \text{(divide each side by } n\text{)}$$

Homework

31. Solve each equation:

a. $-4(a - 6) + (-5a - 3) = 6(2a + 1) - (5a + 4)$

b. $2(-8e - 6) - 8(-3e - 2) = 3(-8e - 7) - 4(-2e + 9)$

c. $5(-9r - 5) + 3(8r + 3) = -2(8r - 3) - 3(7r + 7)$

d. $9(-7j - 6) - 7(-5j + 3) = 6(8j + 1) + 5(5j - 8)$

e. $-6(-9d + 6) + 3(-3d - 9) = -8(-d - 9) - 8(-3d - 4)$

32. Solve each formula for *x*:

a. $x - c = d$ b. $2x + b = R$ c. $abx = c$

d. $\frac{x}{u} = N$ e. $x(y + z) = a$ f. $\frac{x}{n} = c - d$

g. $\frac{x}{a+b} = m - n$ h. $\frac{x}{c-Q} = c + Q$ i. $\frac{x}{R} = a - b + c$

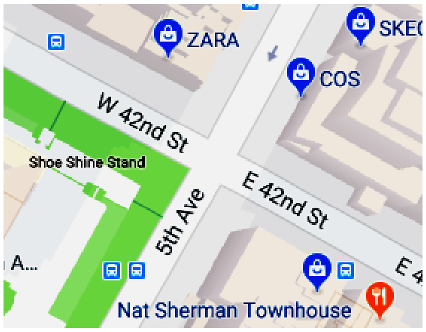
j. $x(b_1 + b_2) = A$ k. $\frac{x}{a} - e = m$ l. $\frac{x+a}{b} = y$

m. $\frac{ax - by}{c} = z$ n. $\frac{cx - a}{y + z} = h - g$ o. $\frac{ax + b}{c} - d = Q$

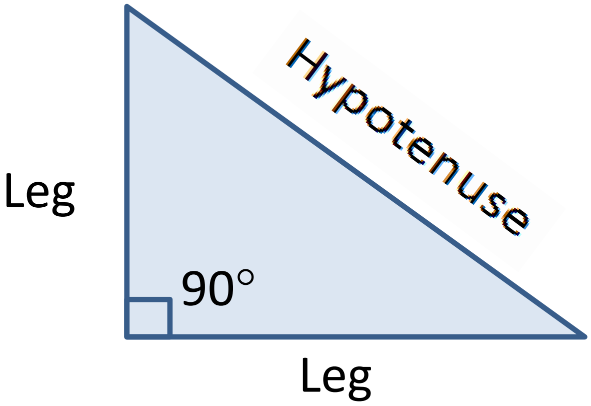
p. $\frac{9x + u - w}{Q + R} = m + n$ q. $9x - 7y + 13 = 0$

❑ THE PYTHAGOREAN THEOREM

The Right Triangle

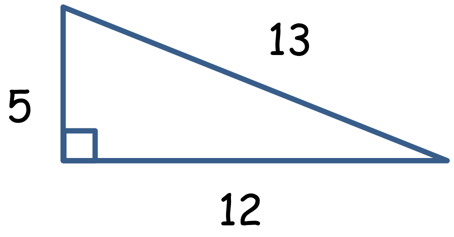


An angle of 90° is called a ***right angle***, and when two things meet at a right angle, we say they are ***perpendicular***. For example, the angle between the floor and the wall is 90°, and so the floor is perpendicular to the wall. And in Manhattan, 5th Avenue is perpendicular to 42nd Street.



If we have a triangle with a 90° angle in it, we call the triangle a ***right triangle***. The two sides which form the right angle (90°) are called the ***legs*** of the right triangle, and the side opposite the right angle is called the ***hypotenuse*** (accent on the 2nd syllable). It also turns out that the hypotenuse is always the longest side of a right triangle.

The Pythagorean Theorem



A Classic Right Triangle

Ancient civilizations discovered that a triangle with sides 5, 12, and 13 would actually be a right triangle -- that is, a triangle with a 90° angle in it.
[By the way, is it obvious that the hypotenuse must be the side of length 13?]

But what if just the two legs are known? Is there a way to calculate the length of the hypotenuse? The answer is yes, and the formula dates back to 600 BC, the time of Pythagoras and his faithful followers.

To discover this formula, let's rewrite the three sides of the above triangle:

leg = 5　　leg = 12　　hypotenuse = 13

Here's the secret: Use the idea of <u>squaring</u>. If we square the 5, the 12, and the 13, we get 25, 144, and 169; that is,

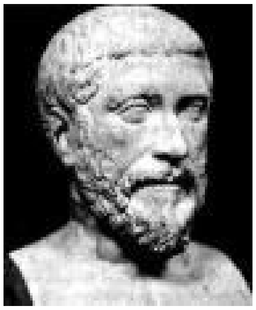
$5^2 = 25 \qquad 12^2 = 144 \qquad 13^2 = 169$

and we notice that the <u>sum</u> of 25 and 144 is 169:

$$25 + 144 = 169$$

In other words, a triangle with sides 5, 12, and 13 forms a right triangle precisely because

$$5^2 + 12^2 = 13^2$$



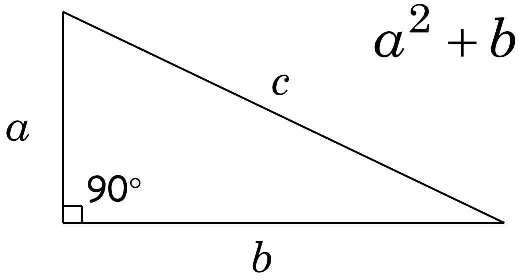
Now let's try to express this relationship in words -- it appears that

> *When you square the legs of a right triangle and add them together, you get the square of the hypotenuse.*

As a formula, we can state it this way:

> If a and b are the legs of a right triangle and c is the hypotenuse, then
>
> $$a^2 + b^2 = c^2$$

The Pythagorean Theorem



$$a^2 + b^2 = c^2$$

The sum of the squares of the legs is equal to the square of the hypotenuse

Solving Right Triangles

EXAMPLE 2: **The legs of a right triangle are 6 and 8. Find the hypotenuse.**

Solution: We begin by writing the Pythagorean Theorem. Then we plug in the known values, and finally determine the hypotenuse of the triangle.

$a^2 + b^2 = c^2$ (the Pythagorean Theorem)

$\mathbf{6}^2 + \mathbf{8}^2 = c^2$ (substitute the known values)

$36 + 64 = c^2$ (square each leg)

$100 = c^2$ (simplify)

What number, when squared, results in 100? A little experimentation yields the solution 10 (since $10^2 = 100$). Our conclusion is that

$$\boxed{\text{The hypotenuse is 10}}$$

Note: The equation $100 = c^2$ also has the solution $c = -10$ [since $(-10)^2 = 100$]. But a negative length makes no sense, so we stick with the positive solution, $c = 10$.

EXAMPLE 3: **Find the hypotenuse of a right triangle whose legs are 5 and 7.**

Solution: This is very similar to Example 2.

$a^2 + b^2 = c^2$ (the Pythagorean Theorem)

$\mathbf{5}^2 + \mathbf{7}^2 = c^2$ (substitute the known values)

$25 + 49 = c^2$ (square each leg)

$74 = c^2$ (simplify)

Is there a whole number whose square is 74? No, there's not, because $8^2 = 64$, which is too small, while $9^2 = 81$, which is too big. We see that the solution for c is somewhere between 8 and 9. But where between 8 and 9?

Finding a number whose square is 74 is the kind of problem that has plagued and enticed mathematicians, scientists, and philosophers for literally thousands of years. They'd really be irked if they knew that we can find an excellent approximation of this number using a cheap calculator. Enter the number 74 followed by the ***square root*** key, which is labeled something like $\sqrt{x}$. Thus, the hypotenuse is $\sqrt{74}$ (read: the positive square root of 74), and your calculator should have the result 8.602325267, or something close. [With fancier calculators, enter the square root key first, then the 74.]

This means that $\sqrt{74}$ is an **irrational** number.

But your calculator doesn't tell the whole story. The fact is, the square root of 74 has an <u>infinite</u> number of digits following the decimal point, and they <u>never</u> have a repeating pattern. Thus, we'll have to round off the answer to whatever's appropriate for the problem. For this problem, we'll round to the third digit past the point.

$$\boxed{\text{The hypotenuse is } 8.602}$$

Homework

33. In each problem, the two legs of a right triangle are given. Find the **hypotenuse**.

a. 3, 4	b. 5, 12	c. 10, 24	d. 30, 16
e. 7, 24	f. 12, 16	g. 30, 40	h. 9, 40
i. 12, 35	j. 20, 21	k. 48, 55	l. 13, 84
m. 17, 144	n. 11, 60	o. 51, 140	p. 24, 143

34. Find the **hypotenuse** of the triangle with the given legs. Use your calculator and round your answers to the hundredths place.

a. 2, 5	b. 4, 6	c. 1, 7	d. 5, 8
e. 2, 6	f. 3, 5	g. 4, 7	h. 7, 8

Solutions

1. a. Real b. Real c. Real d. Real
e. Imaginary f. Real g. Real h. Real
i. Real j. Real k. Real l. Imaginary
m. Real n. Real o. Real p. Real
q. Real r. Imaginary s. Imaginary t. Real
u. Real

2. $-\pi, -\sqrt{3}, -1, \sqrt{0}, \frac{1}{101}, \sqrt{1}, \sqrt{5}, 3.0808\ldots, \frac{11}{3}, \sqrt{25}, 2\pi, 2^3, 3^2$

3. a. -17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative e. 0

5. a. -9 b. 3 c. 0 d. $-\pi$ e. $\sqrt{2}$

6. a. $\frac{1}{5}$ b. $\frac{9}{2}$ c. $-\frac{3}{7}$ d. 1 e. Undefined f. π g. $\sqrt{3}$

7. a. False; 0 has no reciprocal. b. Undefined c. negative
d. positive e. 1

8. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

9. a. 5π b. 23.9 c. 0 d. 2 e. $\sqrt{7}$

10. a. T b. 0 c. positive d. positive

11. Opposite & Absolute Value (the reciprocal of 0 does not exist)

12. a. 72 b. 99 c. 0 d. π e. π f. $\sqrt{2}$

13. a. 10 b. 22 c. 0 d. 5π

14. a. 33 b. 0 c. 25 d. 17 or -17 e. 0 f. No solution

15. a. $\frac{9}{4}x$ b. $\frac{17}{2}x$ c. $\frac{3}{16}y$ d. $-\frac{33}{17}a$
e. $-\frac{5}{23}n$ f. $\frac{1}{2}x$ g. $3x$ h. $\frac{2}{3}m$
i. $\frac{3}{5}x$ j. $-\frac{3}{4}z$

16. a. $\frac{3}{4}x+2$ b. $-\frac{3}{2}x+3$ c. $\frac{1}{8}y-\frac{1}{8}$ d. $-\frac{3}{5}n-3$
e. $-\frac{3}{2}w-12$ f. $3x-5$ g. $\frac{3}{5}x+\frac{4}{5}$ h. $\frac{31}{20}x+\frac{17}{20}$

17. a. $-\frac{13}{10}$ b. 0 c. $\frac{3}{2}$ d. $-\frac{2}{15}$ e. $\frac{41}{5}$
f. $-\frac{5}{3}$ g. $-\frac{7}{3}$ h. $\frac{1}{3}$ i. $-\frac{15}{28}$

18. a. $\frac{5}{12}$ b. -1 c. 1 d. $\frac{4}{9}$ e. $-\frac{1}{18}$
f. $-\frac{28}{3}$ g. 6 h. $-\frac{1}{10}$ i. $-\frac{32}{25}$

19. True

20. a. $\frac{4}{9}$ b. $-\frac{1}{8}$ c. $\frac{1}{81}$ d. $-\frac{14}{19}$
e. $-\frac{1}{32}$ f. $\frac{64}{729}$ g. -1

21. a. $\frac{9}{10}$ b. $\frac{3}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{11}{12}$
f. Not a real number g. $\frac{16}{17}$ h. 0

22. a. 230 b. 31 c. 42 d. 46 e. 1 f. 0
g. 5 h. 47 i. 396 j. 2916 k. 97 l. 4096

23. a. $(x+y)^2 = (2+1)^2 = 3^2 = 9$
b. $x^2+y^2 = 10^2+5^2 = 100+25 = 125$
c. 63 d. 96 e. 44

24. a. 1 b. 3 c. 2 d. 1 e. 2 f. 3 g. 1 h. 2
i. 1 j. 2 k. 1 l. 2 m. 2 n. 3 o. 2 p. 4
q. 2 r. 3 s. 4 t. 1 u. 2

25. a. $\frac{0}{15} = 0$ since $0 \times 15 = 0$, and 0 is the only number that accomplishes this.

b. $\frac{32}{0}$ is undefined because any number times 0 is 0, never 32; thus NO number works.

c. $\frac{0}{0}$ is undefined because any number times 0 is 0; thus EVERY number works.

26. a. 0 b. 0 c. 0 d. 0 e. Undefined f. Undefined
g. Undefined h. Undefined i. Undefined j. Undefined
k. Undefined l. 0

27. a. **28**. b. **29**. b.

30. a. You can't conclude anything -- it depends on what's on the bottom. If the bottom is a non-zero number (like 7), then $\frac{0}{7} = 0$. If the bottom is zero, then $\frac{0}{0}$ is undefined.

b. This time we can conclude that the fraction is undefined, since division by 0 is undefined, no matter what's on the top of the fraction.

31. a. $a = \frac{19}{16}$ b. $e = -\frac{61}{24}$ c. $r = \frac{1}{16}$
d. $j = -\frac{41}{101}$ e. $d = \frac{167}{13}$

32. a. $x = d + c$ b. $x = \frac{R-b}{2}$ c. $x = \frac{c}{ab}$
d. $x = Nu$ e. $x = \frac{a}{y+z}$ f. $x = n(c-d)$
g. $x = (m-n)(a+b)$ h. $x = (c+Q)(c-Q)$ i. $x = R(a-b+c)$

j. $x = \dfrac{A}{b_1 + b_2}$ k. $x = a(m + e)$ l. $x = by - a$

m. $x = \dfrac{cz + by}{a}$ n. $x = \dfrac{(h - g)(y + z) + a}{c}$

o. $x = \dfrac{c(Q + d) - b}{a}$ p. $x = \dfrac{(m + n)(Q + R) + w - u}{9}$

q. $x = \dfrac{7y - 13}{9}$

33. a. 5 b. 13 c. 26 d. 34 e. 25 f. 20
g. 50 h. 41 i. 37 j. 29 k. 73 l. 85
m. 145 n. 61 o. 149 p. 145

34. a. 5.39 b. 7.21 c. 7.07 d. 9.43
e. 6.32 f. 5.83 g. 8.06 h. 10.63

> “The greatest mistake you can make in life is to be continually fearing you will make one.”
>
> – Elbert Hubbard (1856–1915)

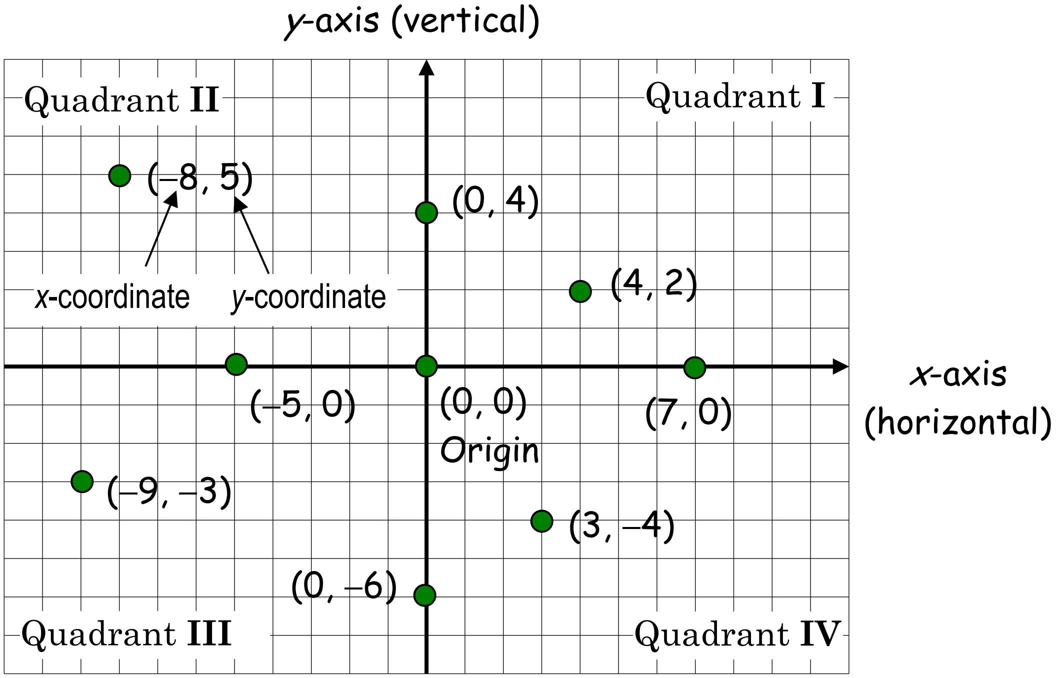
CH 1 – INTRO TO GRAPHING

❑ INTRODUCTION

A picture is worth a thousand words. This is especially true in math, where many ideas are very abstract. The French mathematician-philosopher René Descartes (who said, "I think, therefore I am") devised a way for us to visually represent the solutions of certain kinds of equations. It's called the "Cartesian coordinate system." The term *Cartesian* is the Latin form of the name Descartes.

To create the coordinate system we take two number lines (one called the *x*-axis and one called the *y*-axis, line them up perpendicular to each other (90° angle between them), and we have a two-dimensional coordinate system.

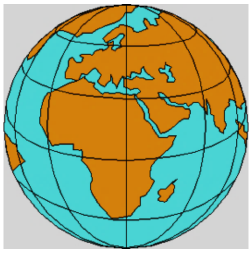
❑ THE CARTESIAN COORDINATE SYSTEM



Observations on the Cartesian Coordinate System

1. A two-dimensional coordinate system (like the figure above) represents a ***plane***. The horizontal axis is called the **x-axis** in math, but will be called other things in other subjects. Similarly, the vertical **y-axis** will be called something else in other subjects.

2. The ***ordered pair* (x, y)** represents a single ***point*** in the plane. The numbers x and y in the ordered pair are the ***coordinates*** of the point. Notice that a single point (ordered pair) consists of two coordinates. The coordinates of a point on the Earth are called its longitude and latitude.



3. The point (0, 0), where the axes intersect (cross), is called the ***origin***.

4. The first coordinate of the point (x, y) represents the distance to the right or left from the origin. The second coordinate represents the distance up or down. For example, the point (3, –4) is plotted by starting at the origin, moving 3 units to the right, and then moving 4 units down.

5. The ***quadrants*** are numbered I (one) through IV (four), starting in the upper-right region and going counterclockwise.

   In Quadrant I, both coordinates (the x and y) are positive.
   In Quadrant II, x is negative and y is positive.
   In Quadrant III, both coordinates are negative.
   In Quadrant IV, x is positive and y is negative.

6. Every point on the x-axis has a y-coordinate of 0.
   Every point on the y-axis has an x-coordinate of 0.

Homework

1. Is the Cartesian coordinate system described in this chapter 1-, 2-, or 3-dimensional?

2.
   a. The point (7, –3) lies in Quadrant ____.
   b. The point (–8, –9) lies in Quadrant ____.
   c. The point (–1, 6) lies in Quadrant ____.
   d. The point (π, 17) lies in Quadrant ____.

3.
   a. The point (17, 0) lies on the ___ axis.
   b. The point (0, –20) lies on the ___ axis.
   c. The point (0, 0) is called the ____ and lies on the ___ axis.

4.
   a. In which quadrants are the signs of x and y the same?
   b. In which quadrants are the signs of x and y opposites?

5.
   a. A point lies on the x-axis. What can you say for sure about the coordinates of the point? Hint: The following are points on the x-axis: (7, 0), (–23, 0) and (π, 0).

   b. A point lies on the y-axis. What can you say for sure about the coordinates of the point?

   c. A point lies on both axes. What can you say for sure about the coordinates of the point?

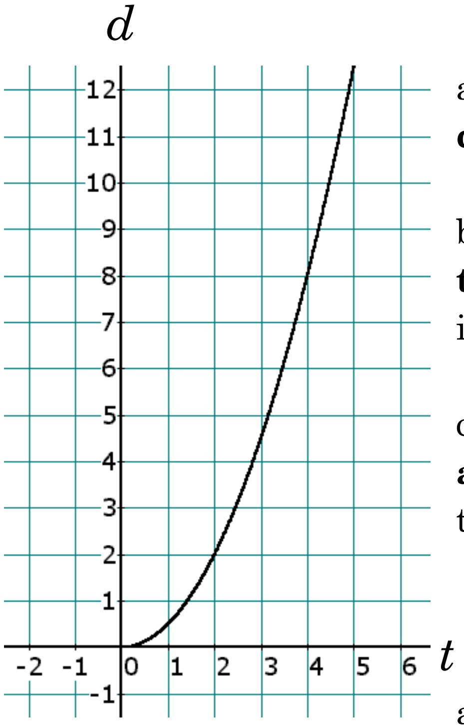
6. Does the notation (2, –7) represent two points or one point containing two coordinates?

GETTING AWAY FROM X AND Y

We call the horizontal and vertical axes the x-axis and the y-axis. Why use these letters? x and y are good algebra letters -- they don't stand for anything in particular, so they can represent anything. But outside this classroom, people need better variables for the things they're analyzing.

For example, when graphing degrees Fahrenheit and degrees Celsius, we might use F and C as the names of the axes. For a supply and demand curve from economics, why use x and y when we can use S and D?

EXAMPLE 1: **The graph below shows the distance in meters (d) a car has traveled based on how long it's been traveling in seconds (t).**



a. **Is the horizontal axis time or distance?** time

b. **As time increased, did the distance the car traveled increase or decrease?** increase

c. **What distance had the car traveled after 4 sec?** When $t = 4$, $d = 8$, so the car traveled 8 m.

d. **About how many seconds did it take the car to travel 6 m?** Looking at the d axis and locating the value $d = 6$, we see that the t-value is about 3.4. So it took about 3.4 sec to travel the 6 m.

Homework

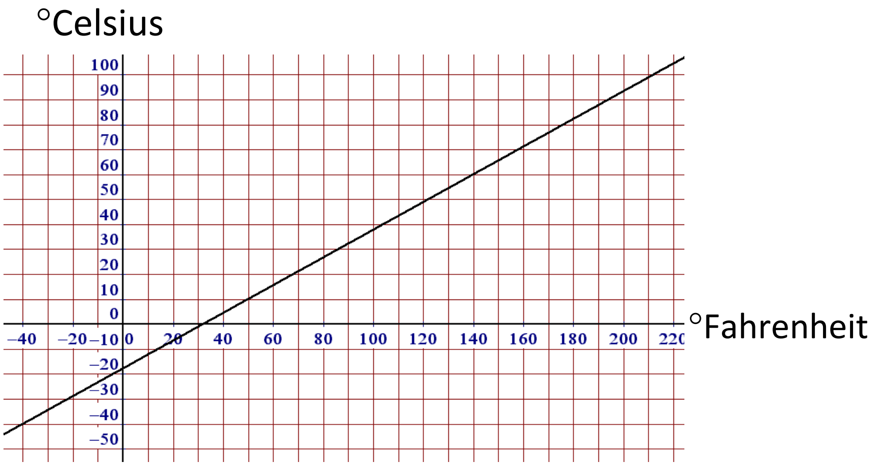
7. This problem is a continuation of Example 1.

   a. After 1 second, had the car moved more than 1 m or less than 1 m?

   b. Why do you think there is no graph in the second quadrant?

   c. How far did the car travel between $t = 3$ and $t = 5$?

8. The graph below depicts the relationship between two temperature scales, Fahrenheit and Celsius.



   a. The Fahrenheit temperatures are on the (horizontal, vertical) axis.

   b. The Celsius temperatures are on the (horizontal, vertical) axis.

   c. The graph passes through every quadrant except _____.

   d. A temperature of 140°F corresponds to what temperature on the Celsius scale? Hint: Locate 140°F on the horizontal axis; then go straight up until you hit the line. Now go to the left until you hit the vertical axis. What Celsius temperature do you see?

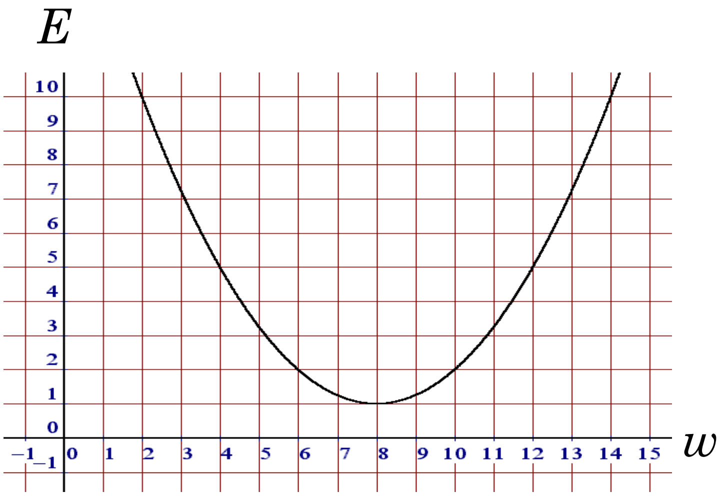
   e. What Celsius temperature corresponds to 50°F?

f. 212°F is the boiling point of water (assuming pure water and normal atmospheric pressure). Use the graph to estimate the boiling point of water on the Celsius scale.

g. 0°C is the freezing point of water. Use the graph to estimate the freezing point of water on the Fahrenheit scale.

h. This one's kind of hard: There's exactly one point on the line where the Fahrenheit and Celsius temperatures match. What temperature is that?

9. The following graph shows the relationship between E, the expenses, and w, the number of widgets sold.



a. Find the expense when 2 widgets are sold.

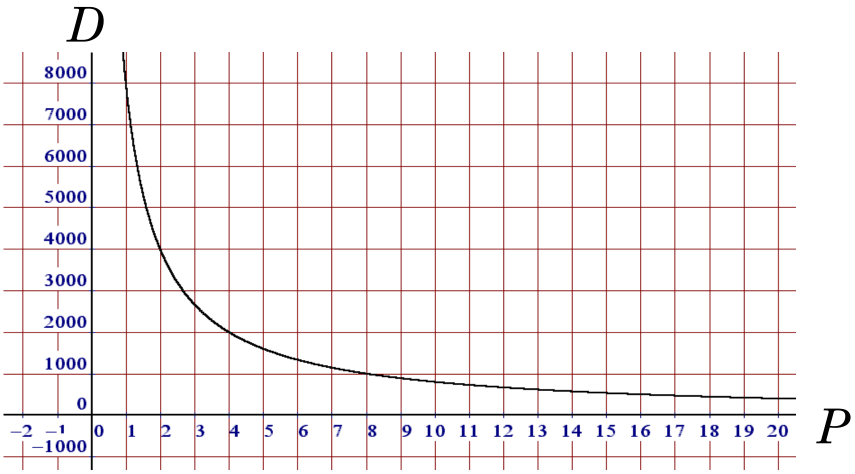
b. Find the expense when 8 widgets are sold.

c. Find the expense when 12 widgets are sold.

d. If the expense is \$2, what are the two possible values for the number of widgets sold?

e. What production level (what number of widgets) would produce the least expense?

10. The next graph shows the relationship between the demand for a deluxe widget and its price:



a. If the price of a deluxe widget is \$4, what is the demand?

b. When the price is just \$1, how many widgets are expected to be sold?

c. If the price jumps to \$16, what is the expected demand?

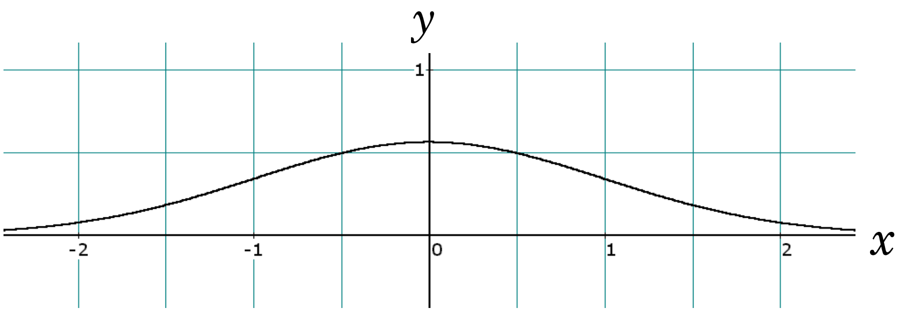
d. If the demand is 8000 widgets, what is the price of a widget?

e. As the price increases, the demand ________.

f. As the price decreases, the demand ________.

g. The entire graph is contained within which quadrant?

11. Now we come to the famous "bell-shaped curve," one of the most important graphs in statistics:



a. When $x = 0$, y is a little bigger than ___.

b. The graph reaches its maximum y-value when $x =$ ___.

c. When $x = 0.5$, y is approximately ___.

d. When $x = -0.5$, y is approximately ___.

e. Assume that the point (3, 0.006) lies on the bell-shaped curve. What is the y-value when $x = -3$?

f. Assume that the graph goes forever to the right and forever to the left. As x moves farther and farther to the right, the y-value is always (positive, negative) and gets infinitely close to ____.

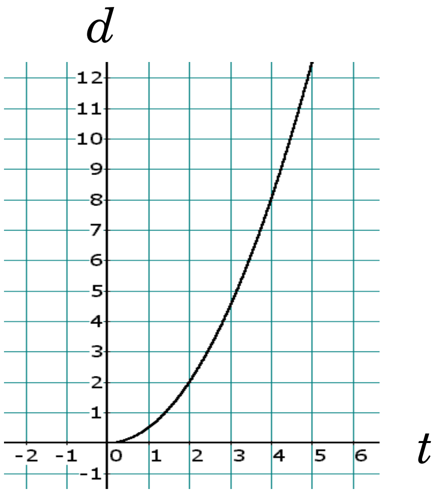
g. As x moves farther and farther to the left, the y-value is always (positive, negative) and gets infinitely close to ____.

h. The graph has symmetry with respect to which axis?

Practice Problems

12. The point $(\pi, -\pi)$ lies in which quadrant?

13. The graph below shows the distance in meters a car has traveled based on how long it's been traveling in seconds.



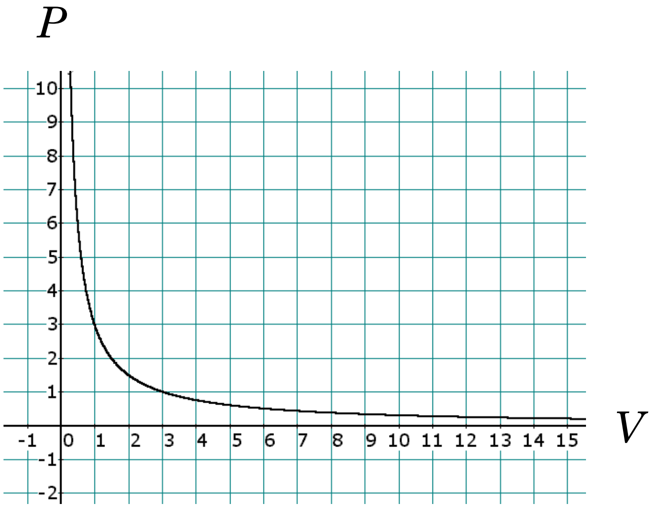
a. After 5 seconds, the car had traveled ___ meters.

b. Is the vertical axis time or distance?

c. After 3 seconds, the car had traveled ___ meters.

d. Between $t = 2$ and $t = 4$, how far did the car travel?

14. The following graph shows the relationship between the pressure and the volume of a gas:



a. When the volume is 3, the pressure is ____.
b. When the volume is 6, the pressure is ____.
c. As the volume increases, the pressure _____.
d. As the volume decreases, the pressure _____.

Solutions

1. The Cartesian coordinate system in this course is 2-dimensional. In later courses you will encounter a 3-dimensional coordinate system, and if you major in math you will find even higher-dimensional coordinate systems.

2. a. IV b. III c. II d. I

3. a. x b. y c. origin; it's on <u>both</u> axes.

4. a. I and III b. II and IV

5. a. that the y-coordinate must be 0.
b. that the x-coordinate must be 0.

c. both coordinates must be 0, since it's the origin.

6. one point containing two coordinates

7. a. Looking at the point where $t = 1$, we see that the d-value is between 0 and 1, so the car moved less than 1 m.

b. A point in the second quadrant would indicate a negative time value, and we usually measure time starting at zero.

c. The distance traveled at $t = 3$ is around 4.5 m, and the distance at $t = 5$ is about 12.5. Thus, the distance traveled between the two times was about 8 m.

8. a. horizontal b. vertical c. II d. 60°C e. 10°C f. 100°C
g. 32°F h. –40°

9. a. \$10 b. \$1 c. \$5 d. 6, 10 e. 8

10. a. 2000 widgets b. 8000 widgets c. 500 widgets d. \$1
e. decreases f. increases g. I

11. a. 0.5 b. 0 c. 0.5 d. 0.5 e. 0.006
f. positive; 0 g. positive; 0 h. y-axis

12. IV

13. a. about 12 b. distance c. 4.5 d. 6 m

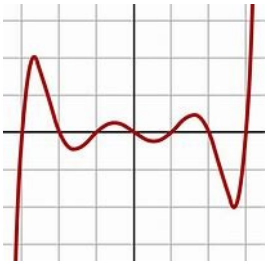
14. a. 1 b. 0.5 c. decreases d. increases

"A well-educated mind will always have more questions than answers."

Helen Keller (1880 – 1968)

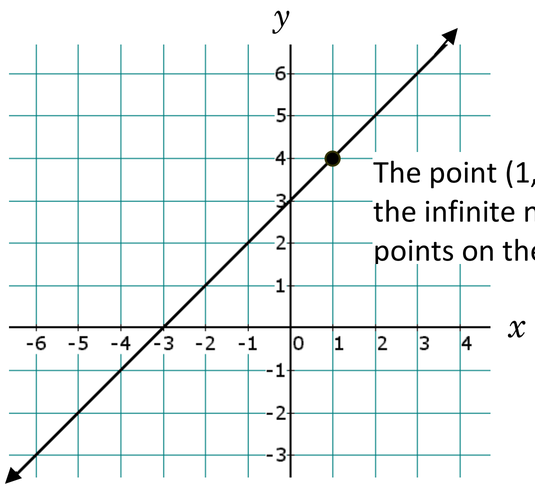
CH 2 – FROM GRAPH TO EQUATION

❑ *INTRODUCTION*



In this chapter I'm going to give you a graph, but only a small portion of the graph, generally around the origin. We will first analyze points that we can actually see (for instance, I will ask what y-value is associated with a certain x-value). Then I will ask you what y-value is associated with an x-value that you can't even see on the graph! How will we do this? We will use the given graph to create an algebraic formula to describe the graph, and then use that formula to predict y-values from <u>any</u> x-value I give you.

<u>EXAMPLE 1:</u> **Consider the following graph:**



The point (1, 4) is one of the infinite number of points on the line.

a. **Calculate the y-value when $x = 100$.**

b. **Calculate the y-value when $x = 2.7$.**

c. **Calculate the x-value when $y = 1000$.**

Solution: We certainly don't want to extend (draw) the graph all the way to $x = 100$. So our approach in this chapter will be to

1. determine the equation of the given graph by analyzing the relationship between x and y, and
2. use that formula to deduce the y-value for the given x-value.

The first step will be to create an x-y table by reading the coordinates off the graph. For example, in the given graph, we see that when $x = 1$, $y = 4$; thus the point (1, 4) is on the graph.

x	–4	–3	–2	–1	0	1	2	3
y	–1	0	1	2	3	4	5	6

Remember that the arrowheads on the graph are used to indicate that the graph (the line) goes forever in both directions. It then follows that x can be any number at all. For example, although it's not easy to see from the graph, if we choose x to be 1.5, it can be guessed that $y = 4.5$. Similarly, we could let $x = \pi$ and then see that y would be a little bigger than 6.

Guessing certainly isn't very accurate, and that's even when we have a graph to look at. It would be even more difficult to find points on the line if we had to extend the graph ourselves. This is why it's so very important to create an *equation* from the given graph. Assuming that we end up with the right equation, we are guaranteed exact y-values for any given x-value.

Now let's examine the equation of the line. Looking at the x- and y-values in the table, we might see that the y-value is always 3 more than the x-value. We are therefore led to the formula

$$y = x + 3$$

Check this formula against all the x-y pairs in the table and you will see that is really works. Now we can answer the three questions:

a. If $x = 100$, then $y = x + 3 = 100 + 3 = \mathbf{103}$.

b. If $x = 2.7$, then $y = x + 3 = 2.7 + 3 = \mathbf{5.7}$.

c. Be careful here; note that y is given and we want to find x.

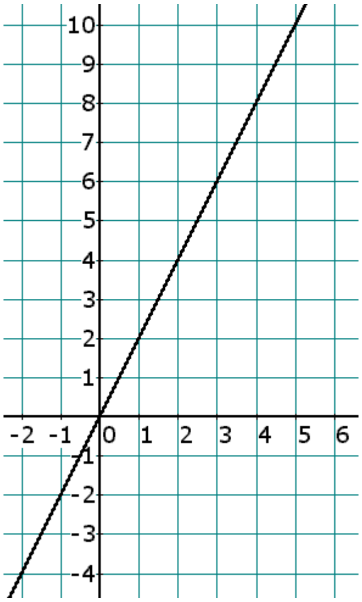
$$
\begin{array}{lll}
 & y = x + 3 & \text{(the formula for the line)} \\
\Rightarrow & 1000 = x + 3 & \text{(let } y = 1000) \\
\Rightarrow & 1000 \underline{-3} = x + 3 \underline{-3} & \text{(subtract 3 from each side)} \\
\Rightarrow & \mathbf{997} = x & \text{(solve for } x)
\end{array}
$$

In summary, the following three points lie on the line:

(100, 103) (2.7, 5.7) (997, 1000)

Homework

1. Given the graph, create a formula and then use that formula to answer the questions:



a. When $x = 3$, $y =$ ___.

b. When $x = 0$, $y =$ ___.

c. When $x = -4$, $y =$ ___.

d. When $x = 278$, $y =$ ___.

e. When $x = -900$, $y =$ ___.

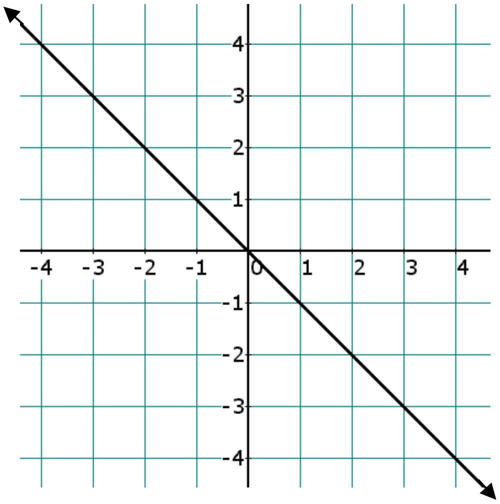
f. When $x = 18\pi$, $y =$ ___.

g. When $y = 10$, $x =$ ___.

h. When $y = 250$, $x =$ ___.

i. When $y = 22\pi$, $x =$ ___.

EXAMPLE 2: **Consider the following graph:**



a. **Calculate the *y*-value when *x* = 1200.**

b. **Calculate the *y*-value when *x* = –123.**

c. **Calculate the *x*-value when *y* = –25π.**

Solution: Again, the points in which we're interested are not visible on the graph, so let's create a table based on the graph, and then use that table to construct a formula.

x	–4	–3	–2	–1	0	1	2	3
y	4	3	2	1	0	–1	–2	–3

We see that the y-value is simply the ***opposite*** of the x-value:

$$y = -x$$

Notice that when $x = 0$, $y = -0 = 0$ (which yields the origin); therefore, 0 has an opposite, namely 0. We conclude that every number has an opposite. We can now answer the three questions:

a. If $x = 1200$, then $y = -x =$ **–1200**.

b. If $x = -123$, then $y = -x = -(-123) =$ **123**.

c. As in the previous example, the y-value is given and we're asked for the x-value. So let's begin with the formula, plug in the given value of y, and then solve for x:

$$y = -x \qquad \text{(the formula for the line)}$$

$$\Rightarrow \quad -25\pi = -x \qquad \text{(let } y = -25\pi\text{)}$$

$$\Rightarrow \quad 25\pi = x \qquad \text{(divide or multiply each side by } -1\text{)}$$

Therefore, the following three points lie on the line:

$$\boxed{(1200, -1200) \quad (-123, 123) \quad (25\pi, -25\pi)}$$

Homework

2. The graph of $y = -x$ lies within Quadrants ___ and ___ and at the origin.

3. What is the **opposite** of each number?

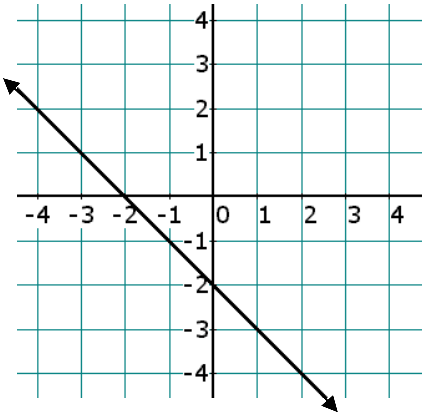
   a. 17 b. 0 c. -3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.
   b. The opposite of 0 is ____.
   c. The opposite of a negative number is always _____.
   d. The opposite of a positive number is always _____.

5. Consider the formula $y = -x + 5$. Calculate the y-value for the given x-value:

   a. 8 b. 99 c. -10 d. -5 e. 0 f. π g. $-\pi$

6. Given the graph, create a formula and then use that formula to answer the questions:



a. When $x = 2$, $y =$ ___.

b. When $x = 0$, $y =$ ___.

c. When $x = -4$, $y =$ ___.

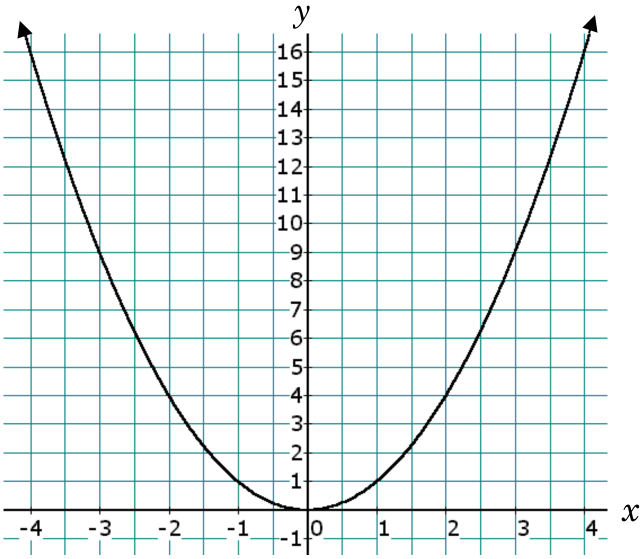
d. When $x = 25$, $y =$ ___.

e. When $x = -200$, $y =$ ___.

f. When $y = 32$, $x =$ ___.

g. When $y = -300$, $x =$ ___.

EXAMPLE 3: **Consider the following graph:**



a. Calculate the y-value when $x = 1{,}000$.

b. Calculate the y-value when $x = 1.2$.

c. Calculate the x-values when $y = 625$.

Solution: This graph is much different from the previous graph. It passes through the origin, and the rest of the graph resides only in Quadrants I and II. Like the bell-shaped curve from the homework in Chapter 1, it has symmetry with the y-axis. Let's try to read off some ordered pairs from the graph:

x	–4	–3	–2	–1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Now, how is the y-value related to the x-value? When $x = 0$, $y = 0$; this is rather useless, as is the pair (1, 1). But look at the ordered pairs (2, 4), (3, 9) and (4, 16). There's something going on here -- it appears that the y-value is simply the square of the x-value:

$$y = x^2$$

Lest we jump to conclusions, let's double-check the rest of the pairs in the table. For example, is the square of –4 equal to 16? Yes. Is the square of –1 equal to 1? Yes. Is the square of 0 equal to 0? Yes. Our formula $y = x^2$ is looking better and better. Notice that even though x can be any number at all, the y-value is never negative; its values start at 0 and go all the way toward infinity. Now to answer the original questions:

a. If $x = 1{,}000$, then $y = x^2 = 1{,}000^2 = \mathbf{1{,}000{,}000}$.

b. If $x = 1.2$, then $y = x^2 = 1.2^2 = \mathbf{1.44}$.

c. Notice that this part of the problem asks for the x-values, plural. I guess this means that a y-value of 625 might have two x-values associated with it. This seems reasonable from the table -- if given a y-value of 9, we notice that both 3 and –3 are x-values that yield a y-value of 9. In other words, if $y = 9$, then $x = 3$ or $–3$. So, what two numbers, when squared, would yield a result of 625? Well, $25^2 = 625$, and of course $(–25)^2 = 625$.

In short, the following four ordered pairs lie on the curve:

(1000, 1000000) (1.2, 1.44) (25, 625) (–25, 625)

Homework

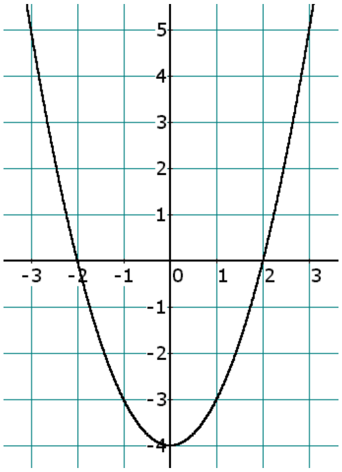
7. The graph of $y = x^2$ lies within Quadrants ___ and ___ and at the origin.

8. Using the formula $y = x^2$, answer each question:

a. If $x = 50$, then $y =$ ___.
b. If $x = -25$, then $y =$ ___.
c. If $x = 0$, then $y =$ ___.
d. If $y = 49$, then $x =$ ___ or ___.
e. If $y = 144$, then $x =$ ___ or ___.
f. If $y = -9$, then $x =$ ___.

9. If $y = 2x^2 - 3x - 1$, then what is y if $x = -5$?
Recall: The Order of Operations requires that exponents be done before multiplication, which is to be done before any adding or subtracting.

10. Consider the graph at the right:



a. Fill in the following x-y table.

x	–3	–2	–1	0	1	2	3
y							

b. Find a formula which relates x and y.
Hint: There's an x^2 in the formula.

c. When $x = 100$, $y =$ ___.

d. When $x = -20$, $y =$ ___.

e. What is the lowest point on the graph?

f. How many y-values are associated with the x-value 1,000?

g. How many x-values are associated with the y-value 1,000?

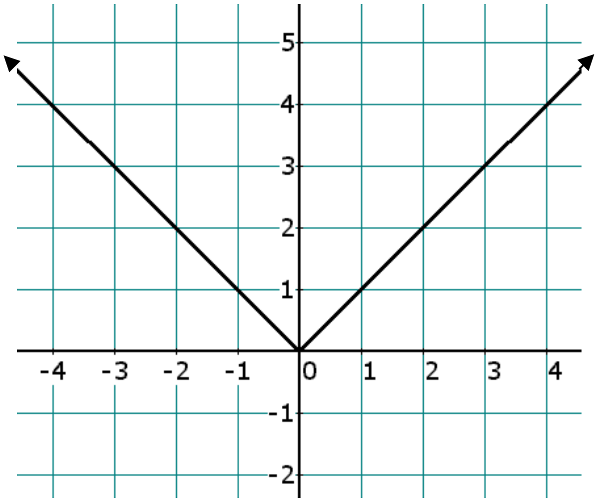
h. If $y = -3$, then $x =$ ___ or ___.

i. If $y = 96$, then $x =$ ___ or ___.

j. If $y = 221$, then $x =$ ___ or ___.

k. If $y = -6$, then $x =$ ___.

EXAMPLE 4: **Consider the following graph:**



a. Calculate the y-value when $x = 239$.

b. Calculate the y-value when $x = -777$.

c. Calculate the x-values when $y = 250$.

Solution: It's not a line and it's not curvy. So what is it? As usual, we begin with a table whose entries are read from the graph:

x	-4	-3	-2	-1	0	1	2	3	4
y	4	3	2	1	0	1	2	3	4

How is the y-value connected to the x-value? Well, when x is 0 or positive ($x \geq 0$), the y-value is the same as the x-value. But when x is negative ($x < 0$), the y-value is the opposite of the x-value.

Though the graph is straightforward (it's just the shape of the letter "V"), there's no simple, recognizable formula for the relationship between x and y. Or is there? You might recall from the Prologue the notion of absolute value: We say that

y is the ***absolute value*** of x,

and we write

$$y = |x|$$

Therefore, the vertical bar symbol around the x means "compute the absolute value of x," which means the following:

1. If $x \geq 0$, then $|x| = x$
   The absolute value of a positive number or 0 is the number itself.

2. If $x < 0$, then $|x| = -x$
   The absolute value of a negative number is the opposite of the number.

We should now be ready to answer the following questions:

a. For $x = 239$, $y = |x| = |239| = \mathbf{239}$.

b. For $x = -777$, $y = |x| = |-777| = \mathbf{777}$.

c. We are asked for the x-values (plural) when $y = 250$. In other words, we have to solve the equation

$$250 = |x|,$$

which asks: What numbers have an absolute value of 250? Well, since $|250| = 250$ and $|-250| = 250$, the two possible x-values are **250** and **–250**. We conclude that the following four points lie on the graph:

(239, 239)	(–777, 777)	(250, 250)	(–250, 250)

Be sure you can visualize the location of these four points on the graph.

Homework

11. The graph of $y = |x|$ lies within Quadrants ___ and ___ and at the origin.

12. Find the **absolute value** of each number:

a. 72　　b. –99　　c. 0　　d. π　　e. $-\pi$　　f. $-\sqrt{2}$

13. Evaluate each expression:

a. $|17 - 7|$　　b. $|3 - 25|$　　c. $|2(3) - 6(1)|$　　d. $|2\pi + 3\pi|$

14. Using the formula $y = |x|$, answer each question:

a. If $x = 33$, then $y =$ ____.

b. If $x = 0$, then $y =$ ____.

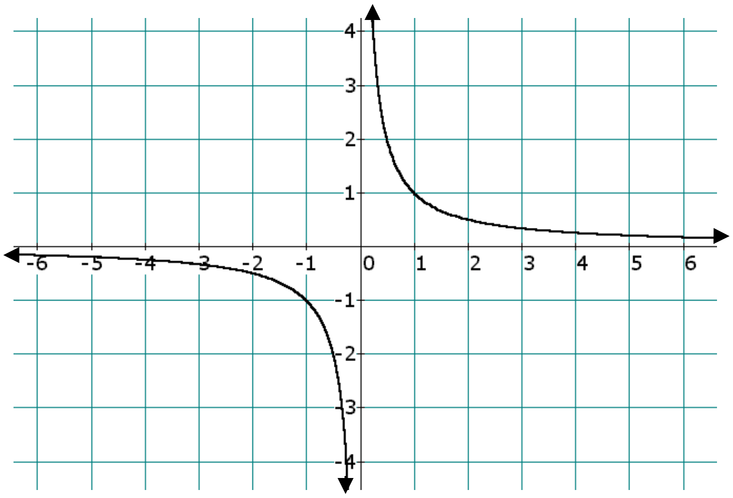
c. If $x = -25$, then $y =$ ____.

d. If $y = 17$, then $x =$ ____ or ____.

e. If $y = 0$ then $x =$ ____.

f. If $y = -5$, then $x =$ ____.

EXAMPLE 5: **Consider the following graph:**



a. Calculate the y-value when $x = 50$.

b. Calculate the y-value when $x = -1/10$.

c. Calculate the y-value when $x = 0$.

Solution: We have another curvy graph, but this one's in two separate pieces. The most important characteristic of this graph is that x is never 0; that is, the graph never touches the y-axis (even though it gets infinitely close!).

As in the two previous examples, the questions asked of us cannot be answered by looking at the graph -- we must find a formula using the points we can see on the graph, and then use that formula to predict the y-values for the given x-values. To find some points on this graph, you'll have to trust me to a certain extent, because it's not easy reading fractional numbers from such a rough picture. See if you can agree with the following claims:

When $x = 1$, it's pretty clear that $y = 1$. When $x = 2$, y looks like it's about 1/2 (or 0.5). And trust me, when $x = 3$, $y = 1/3$, and when $x = 4$, $y = 1/4$.

Now let $x = 1/2$; do you see that $y = 2$? How about when $x = 1/3$, then $y = 3$? This quite nicely analyzes the first quadrant. Note again that x cannot be 0, because if you start at the origin (where $x = 0$), you can go up or down the y-axis as far as you'd like and you'll never run into the graph. It's time for a summary:

x	1/3	1/2	1	2	3	4
y	3	2	1	1/2	1/3	1/4

What is going on here? Using the point (3, 1/3) as a guide, we conjecture that the y-value is found by dividing 1 by the x-value; in other words, flip over the x-value to get the y-value:

$$y = \frac{1}{x}$$

Let's check our formula for $x = 1$ and $x = 1/3$:

$$x = 1 \quad \Rightarrow \qquad y = \frac{1}{x} = \frac{1}{1} = 1 \ ✓$$

$$x = \frac{1}{3} \ \Rightarrow y = \frac{1}{x} = \frac{1}{\frac{1}{3}} = \frac{1}{1} \times \frac{3}{1} = 3 \ ✓$$

Now for a check of a negative x-value. Consider $x = -3$; the formula gives

$$y = \frac{1}{-3} = -\frac{1}{3},$$

which seems very reasonable from the graph. We are now convinced that we have the right formula: $y = \frac{1}{x}$. We call this the **reciprocal** formula, and we'll use it to answer the original questions:

a. If $x = 50$, then $y = \frac{1}{x} = \mathbf{\frac{1}{50}}$, or **0.02**.

b. If $x = -1/10$, then $y = \frac{1}{x} = \frac{1}{-\frac{1}{10}} = \frac{1}{1} \times -\frac{10}{1} = \mathbf{-10}$.

c. If $x = 0$, then $y = \frac{1}{x} = \frac{1}{0}$, which is **undefined** the Prologue). Notice that this is perfectly consistent with the earlier observation that x can never be 0 on the graph, and we see here that y can never be 0 in the formula, either.

Homework

15. The graph of $y = \frac{1}{x}$ lies entirely within Quadrants ___ and ___.

16. As x grows larger and larger, y is always (positive, negative), but getting (larger, smaller).

17. Using the formula $y = \frac{1}{x}$, answer each question:

a. If $x = 14$, then $y =$ ____.

b. If $x = \frac{2}{3}$, then $y =$ ____.

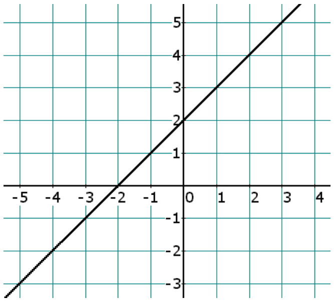
c. If $x = -99$, then $y =$ ____.

d. If $x = -\frac{5}{4}$, then $y =$ ____.

e. If $x = 0$, then $y =$ ____.

Practice Problems

18. Consider the following graph:



a. When $x = 3$, $y =$ _____.

b. When $x = 99$, $y =$ _____.

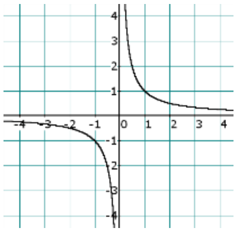
c. When $x = -45$, $y =$ _____.

d. When $y = 132$, $x =$ _____.

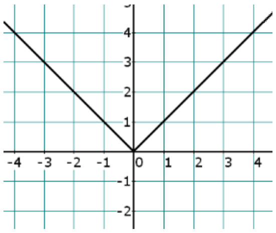
e. When $y = -33$, $x =$ _____.

19. What is the equation for each graph?

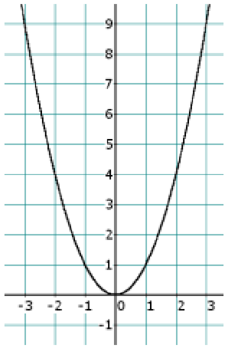
a.



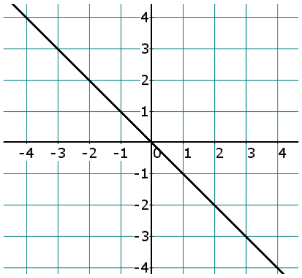
b.



c.



d.



20. Let $y = x^2$.

a. If $x = 17$, $y =$ _____

b. If $x = -13$, $y =$ _____

c. If $y = 16$, $x =$ _____

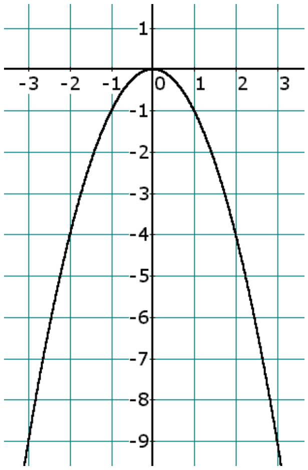
d. If $y = 0$, $x =$ _____

e. If $y = -9$, $x =$ _____

f. If $x = \pi$, $y =$ _____

g. Find the points on the graph where the two coordinates match.

21. Consider the graph:



a. If $x = 0$, then $y =$ _____

b. If $x = 10$, then $y =$ _____

c. If $x = -12$, then $y =$ _____

d. If $y = -225$, then $x =$ _____

e. If $y = 9$, then $x =$ _____

22. Evaluate: a. $|17|$ b. $|-\pi|$ c. $|0|$

23. Evaluate: $\dfrac{|-7|+|7|}{\frac{1}{2}+\frac{1}{3}}$

24. If $y = \dfrac{1}{x}$, and if $x = \dfrac{1}{10}$, what is y?

Solutions

---

1. a. 6 b. 0 c. -8 d. 556 e. -1800
f. 36π g. 5 h. 125 i. 11π

2. II and IV

3. a. -17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative

5. a. -3 b. -94 c. 15 d. 10
e. 5 f. $-\pi + 5$ g. $\pi + 5$

6. a. -4 b. -2 c. 2 d. -27
e. 198 f. -34 g. 298

7. I and II

8. a. 2,500 b. 625 c. 0 d. 7, -7 e. 12, -12
f. No answer (no real number squared could be negative)

9. 64

10. a. 5 0 -3 -4 -3 0 5 b. $y = x^2 - 4$ c. 9,996
d. 396 e. $(0, -4)$ f. 1 g. 2 h. -1 1 i. 10 -10
j. 15 -15 k. No answer

11. I and II

12. a. 72 b. 99 c. 0 d. π e. π f. $\sqrt{2}$

13. a. 10 b. 22 c. 0 d. 5π

14. a. 33 b. 0 c. 25 d. 17, -17 e. 0 f. No answer

15. I and III

16. positive; smaller

17. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

18. a. 5 b. 101 c. -43 d. 130 e. -35

19. a. $y = \frac{1}{x}$ b. $y = |x|$ c. $y = x^2$ e. $y = -x$

20. a. 289 b. 169 c. ±4 d. 0 e. Does not exist
f. π^2 g. Hint: There are two such points.

21. a. 0 b. −100 c. −144 d. ±15 e. Does not exist

22. a. 17 b. π c. 0

23. $\frac{84}{5}$ **24.** 10

> "The greatest use of life is to spend it for something that will outlast it."

William James (the father of modern psychology)

CH 3 – FROM EQUATION TO GRAPH

☐ INTRODUCTION

In this chapter we make pictures out of equations. Specifically, we take an equation containing the variables x and y, find solutions to that equation, plot those solutions in a Cartesian coordinate system, and then "connect the dots" to create a final graph of the equation.

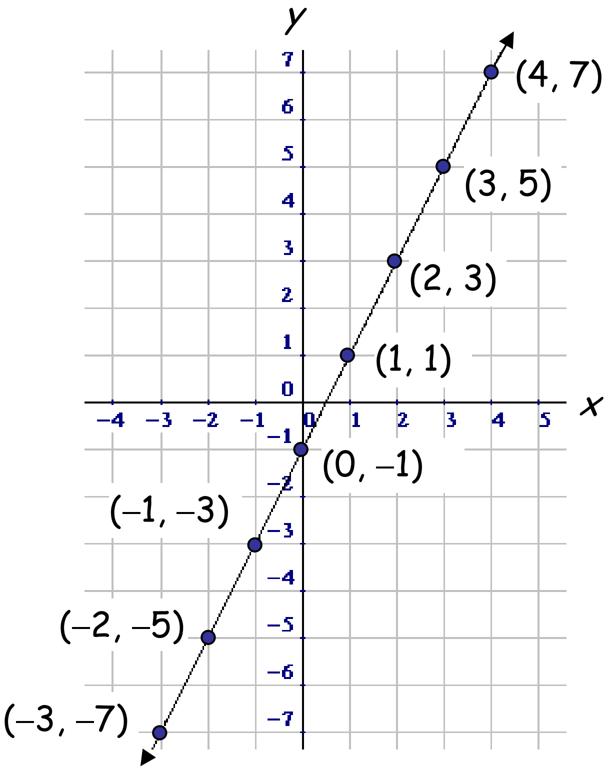
☐ GRAPHING LINES

EXAMPLE 1: **Graph: $y = 2x - 1$**

Solution: The most important aspect of graphing is to learn where the x-values come from. They basically come from your head -- you get to make them up. Later in this course this process will become more sophisticated; but for now, just conjure them up from your imagination. I'm going to choose x-values of $-3, -2, -1, 0, 1, 2, 3$, and 4. For each of these x-values, I will calculate the corresponding y-value using the given formula, $y = 2x - 1$. Organizing all this information in a table is useful:

x	$2x - 1$	(x, y)
–3	2(**–3**) – 1 = –6 – 1 = –7	(–3, –7)
–2	2(**–2**) – 1 = –4 – 1 = –5	(–2, –5)
–1	2(**–1**) – 1 = –2 – 1 = –3	(–1, –3)
0	2(**0**) – 1 = 0 – 1 = –1	(0, –1)
1	2(**1**) – 1 = 2 – 1 = 1	(1, 1)
2	2(**2**) – 1 = 4 – 1 = 3	(2, 3)
3	2(**3**) – 1 = 6 – 1 = 5	(3, 5)
4	2(**4**) – 1 = 8 – 1 = 7	(4, 7)

Now I will plot each of the eight points just calculated (various solutions of the equation) on an x-y coordinate system.



Then the points will be connected with the most reasonable graph, in this case a straight line. Notice that the graph passes through every quadrant except the second; also notice that as we move from left to right (as the x's grow larger) the graph rises.

Final comment: We could let $x = 1{,}000{,}000$ for this equation, in which case y would equal 2(**1,000,000**) – 1 = 1,999,999. That is, the point (1000000, 1999999) is on the line. Can we graph it? Not with the scale we've selected for our graph. But if we traveled along the line, up and up and up, we would eventually run into the point (1000000, 1999999).

We could have used fractions like 2/7 for x. This would have given us the point (2/7, −3/7), which is in Quadrant IV. This point, too, is definitely on the line.

And last, we could have used a number like π for x, in which case we would obtain the point $(\pi, 2\pi - 1)$. This point is impossible to plot precisely, but I guarantee that it's in the first quadrant, and it's on the line we've drawn.

In short, every solution of the equation $y = 2x - 1$ is a point on the line, and every point on the line is a solution of the equation.

Note: It may be true that it takes only two points to completely determine a line. So some students plot exactly two points, connect them with a straight line, and they're done. One warning: You're taking a big gamble when you plot just two points -- if you make an error with either one, you'll get the wrong line, and there might be be no way for you to know you've goofed. Even worse, what if the equation isn't even a line in the first place? Plotting just two points (even if they're both correct) will not suffice to plot a curve that is not a simple straight line.

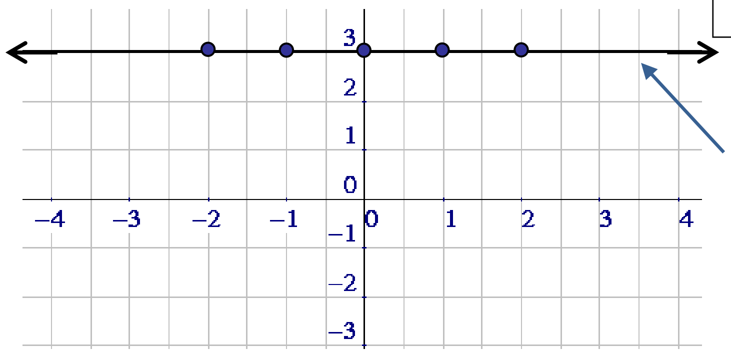
The more points you plot, the better!

EXAMPLE 2: **Graph the line $y = 3$.**

Solution: This strange little equation doesn't even have an x in it. That's fine -- we just think up our favorite x's, and then understand that y is going to be 3 regardless of the x-value we choose. That is, y is a constant -- it doesn't depend on x. Here's a possible table of values for this line. [You are more than welcome to choose x-values different from the ones I've chosen, but it won't make any difference in the final graph.]

x	y
−2	3
−1	3
0	3
1	3
2	3

We therefore have the points (–2, 3), (–1, 3), (0, 3), (1, 3), and (2, 3). Plotting these five points, and then connecting them with a straight line, produces the following **horizontal** line:



The line $y = 3$

Every point on this line has a y-coordinate of 3.

1. Graph each equation:

a. $y = 3x - 1$ b. $y = -2x + 3$ c. $y = -x$ d.

d. $y = x$ e. $y = -x + 3$ f. $y = -2x - 1$

g. $y = 2$ h. $y = 0$ i. $y = -4$

❑ GRAPHING ABSOLUTE VALUES

EXAMPLE 3: **Graph:** $\boldsymbol{y = |x - 3| + 2}$

Solution: Let's plot lots of points, and see what we get. I'll do a couple for you, but be sure you confirm each of the y-values that appears in the table. Recall from Chapter 0 – Prologue that the vertical bars represent **absolute value**. Some examples of absolute value are

$|17| = 17$ $|-44| = 44$ $|0| = 0$

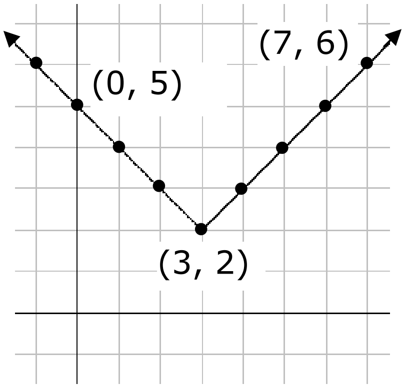
We'll start by letting $x = 6$. Then

$$y = |x-3|+2 = |6-3|+2 = |3|+2 = 3+2 = 5$$

Now choose x to be –1:

$$y = |x-3|+2 = |-1-3|+2 = |-4|+2 = 4+2 = 6$$

x	y
–1	6
0	5
1	4
2	3
3	2
4	3
5	4
6	5
7	6



$y = |x - 3| + 2$

Notice that the graph is in the shape of a "V," with a sharp corner at its bottom point (3, 2).

Homework

2. Graph each equation:

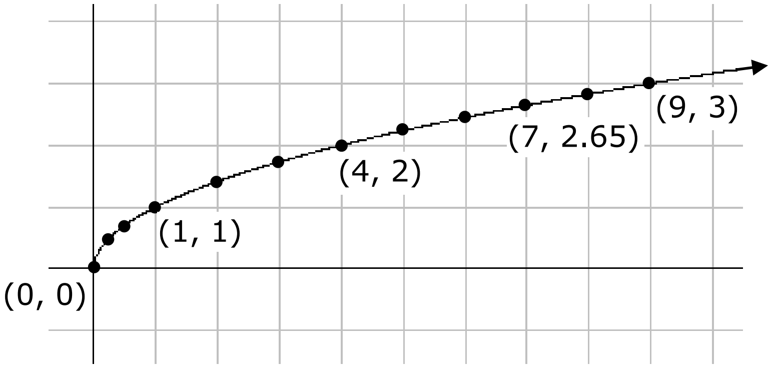
a. $y = |x|$
b. $y = |x+2|$
c. $y = |x|-4$
d. $y = -|x|$
e. $y = |-x|$
f. $y = |x-1|$
g. $y = |x|+3$
h. $y = |x+1|+2$
i. $y = |x+2|-1$
j. $y = -|x+1|$
k. $y = -|x|+3$
l. $y = -|x|-2$

☐ GRAPHING SQUARE ROOTS

EXAMPLE 4: **Graph:** $y = \sqrt{x}$

Solution: In order to take a square root without killing the problem, we recall (from the Prologue) that x must be 0 or bigger (which we write as $x \geq 0$). So we can use 0 or any positive number for x, but we can't use any negative values for x. We'll use a calculator to help us estimate the square roots of numbers that don't have nice square roots.

x	y
0	0
0.25	0.5
0.5	0.707
1	1
2	1.41
3	1.73
4	2
5	2.24
6	2.45
7	2.65
8	2.83
9	3



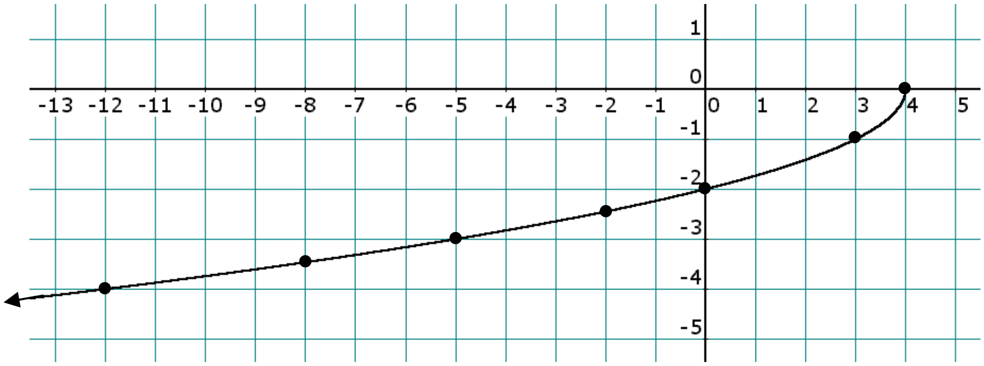
$y = \sqrt{x}$

EXAMPLE 5: **Graph:** $y = -\sqrt{4 - x}$

Solution: We know something's up with that square root in the formula; after all, we know we can't take the square root of a negative number. But it's not obvious what values of x we're allowed to use, so let's not worry about it now; let's just make up our favorite values of x and check them one at a time.

x	y	
0	–2	$y = -\sqrt{4-0} = -\sqrt{4} = -2$
3	–1	$y = -\sqrt{4-3} = -\sqrt{1} = -1$
4	0	$y = -\sqrt{4-4} = -\sqrt{0} = 0$
5	Undefined	$y = -\sqrt{4-5} = -\sqrt{-1} =$ Not a real #
6	Undefined	$y = -\sqrt{4-6} = -\sqrt{-2} =$ Not a real #
–2	–2.45	$y = -\sqrt{4-(-2)} = -\sqrt{6} \approx -2.45$
–5	–3	$y = -\sqrt{4-(-5)} = -\sqrt{9} = -3$
–8	–3.5	$y = -\sqrt{4-(-8)} = -\sqrt{12} \approx -3.5$
–12	–4	$y = -\sqrt{4-(-12)} = -\sqrt{16} = -4$

Plotting these seven points gives us the following graph:



Homework

3. Graph each equation:

a. $y = \sqrt{x+4}$

b. $y=\sqrt{-x}$

c. $y = -\sqrt{x}$

d. $y = \sqrt{x+9}$

e. $y = \sqrt{x}+3$

f. $y = \sqrt{x}-2$

g. $y = -\sqrt{-x}$

h. $y = -\sqrt{x+4}$

i. $y = -\sqrt{x-1}$

Practice Problems

4. Graph: $y = 3x - 4$

5. Graph: $y = \pi$

6. Graph: $y = |x+3| - 2$

7. Graph: $y = \sqrt{x+9} - 5$

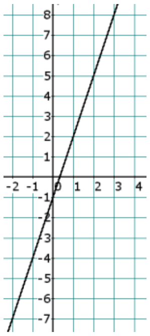
8. Graph: $y = \sqrt{x^2}$

9. Graph: $y = -|x-2| + 4$

10. Graph: $y = -\sqrt{x-3} + 2$

Solutions

1. a.



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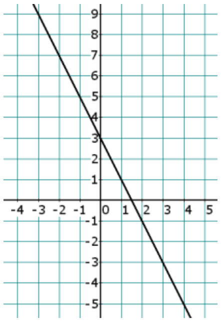

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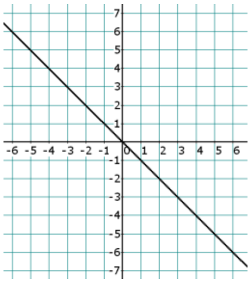

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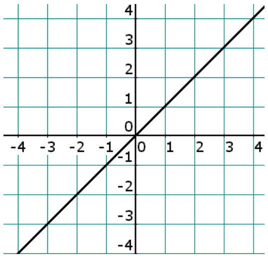

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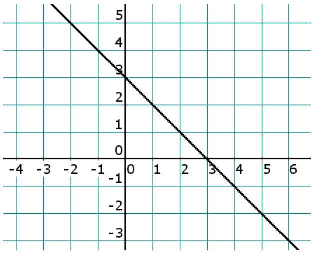

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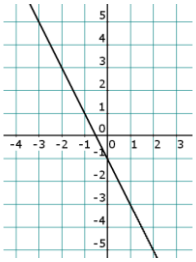

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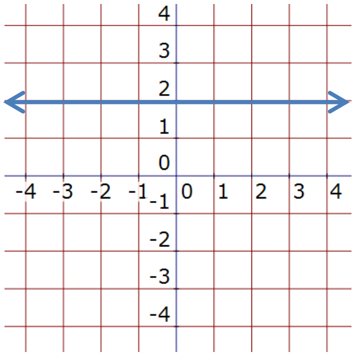

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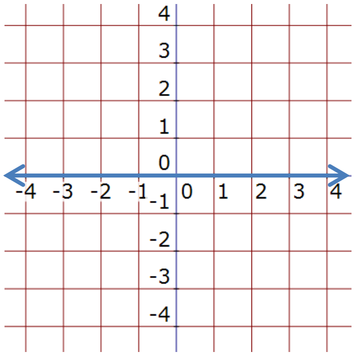

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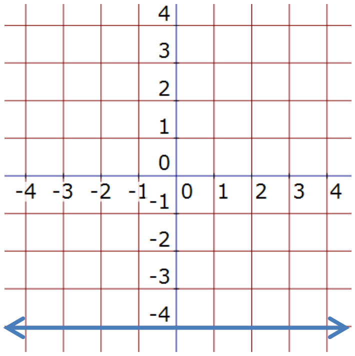

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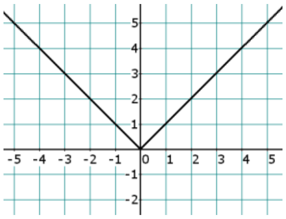

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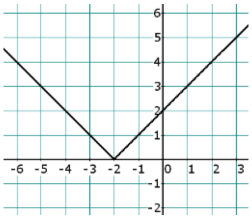

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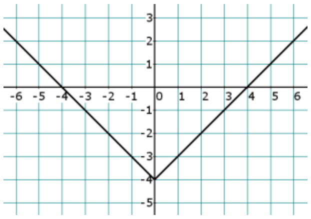

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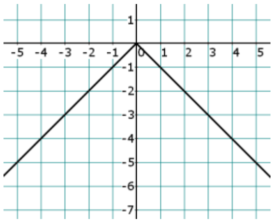

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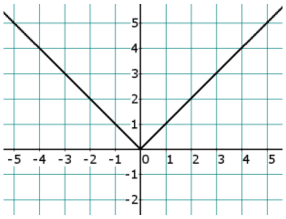

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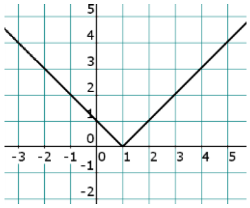

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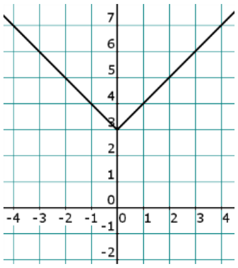

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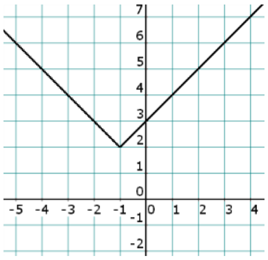

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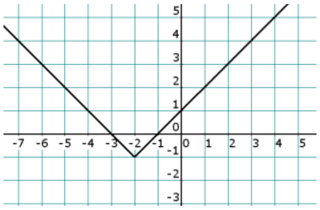

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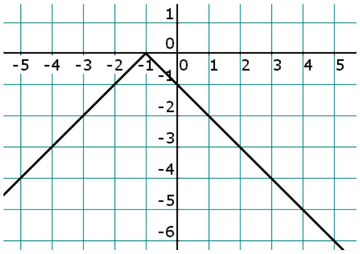

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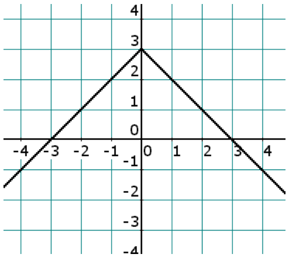

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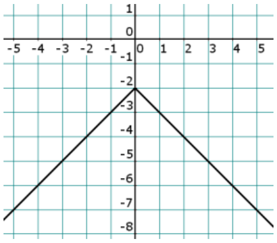

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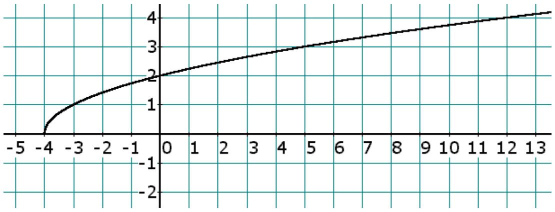

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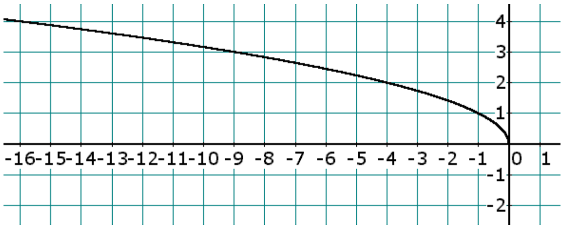

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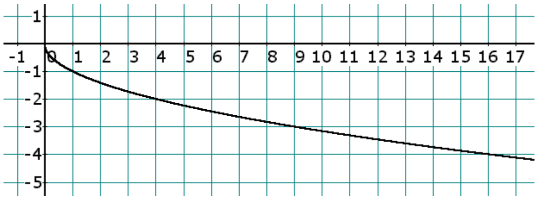

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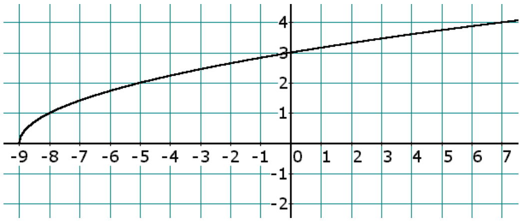

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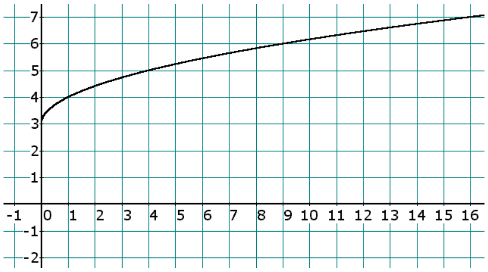

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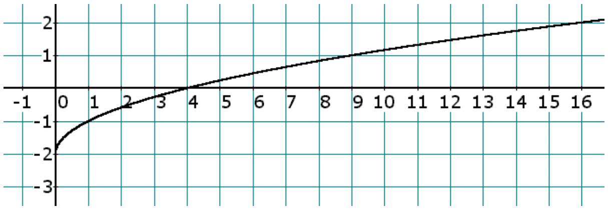

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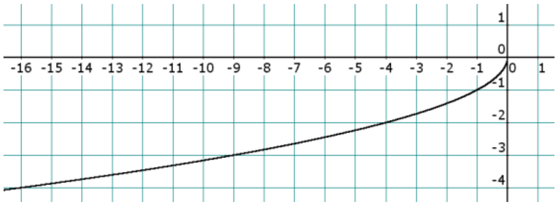

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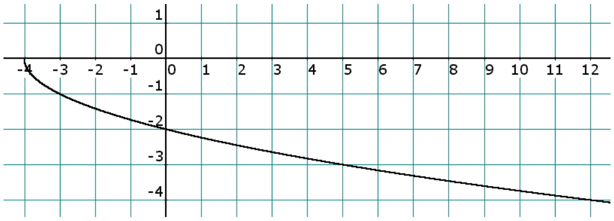

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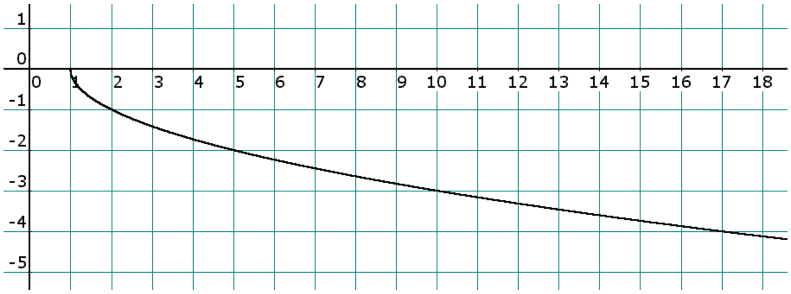
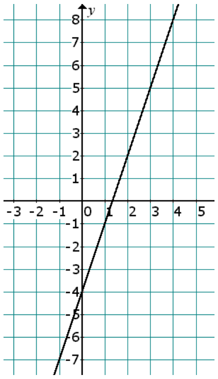
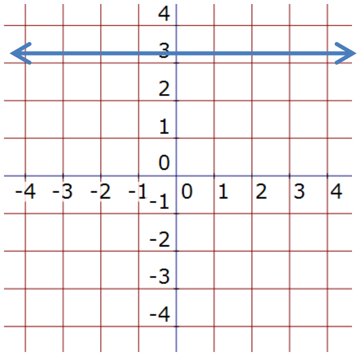

1
0
-1
-2
-3
-4
-5
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

4.


y
8 7 6 5 4 3 2 1
-3 -2 -1 0 1 2 3 4 5
-1 -2 -3 -4 -5 -6 -7

5.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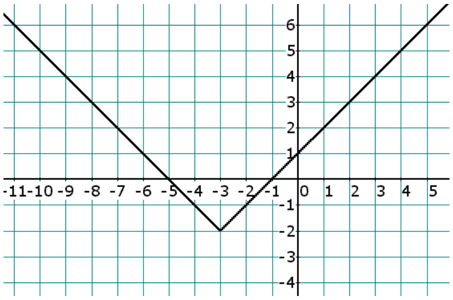
-4 -3 -2 -1 0 1 2 3 4
-1 -2 -3 -4

6.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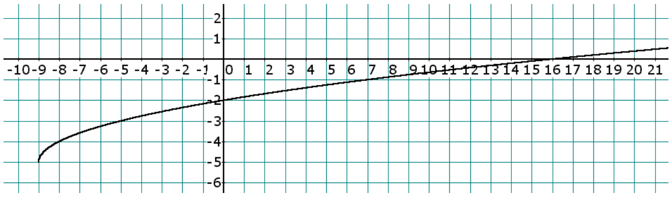
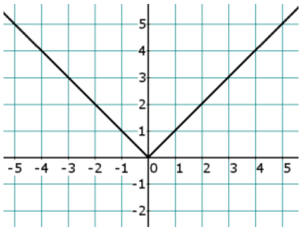
-11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5
-1 -2 -3 -4

7.


2 1
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
-1 -2 -3 -4 -5 -6

8.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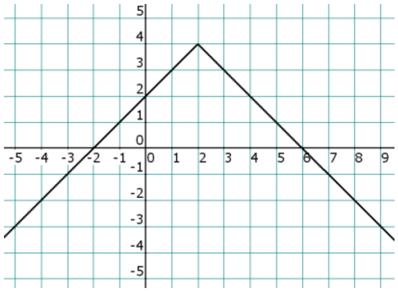
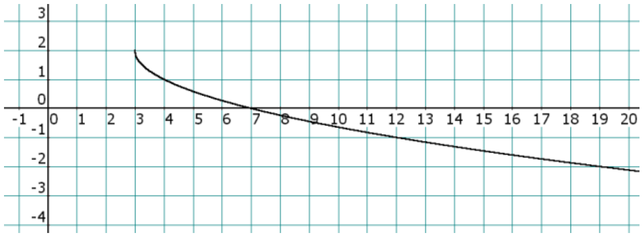
-1 -2

9.


5 4 3 2 1 0
-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
-1 -2 -3 -4 -5

10.


3 2 1 0
-1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
-1 -2 -3 -4

“A good head and good heart are always a formidable combination. But when you add to that a literate tongue or pen, then you have something very special.”

Nelson Mandela
(1918 – 2013)

CH 4 – EQUATIONS AND INEQUALITIES: A GRAPHICAL APPROACH

❑ INTRODUCTION

Some equations, like $3x - 4 = 7x + 9$, are not too hard to solve. Others can be very difficult, if not impossible, to solve using algebra. For example, I have no idea how to solve the equation

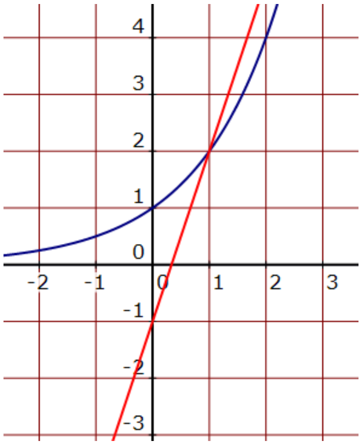
$$2^x = 3x - 1$$

using algebra. But we can find good (if not perfect) solutions using GRAPHING.

❑ OBTAINING SOLUTIONS FROM A GRAPH

EXAMPLE 1: **Solve the equation $2^x = 3x - 1$ graphically.**

Solution: I'm going to give you the graph of each side of the equation on the same grid:



From your knowledge of graphing lines, you should see that the straight line is the graph of $y = 3x - 1$, the right side of the equation. This implies that the curvy graph is the graph of $y = 2^x$, the left side of the equation.

Now, the equation $2^x = 3x - 1$ is a statement of equality: We want to know what values of x will make each side of the equation result in the same number. To do that, we find the any *points of intersection* of the two graphs. That point of intersection seems to the point (1, 2). That means that when $x = 1$, both graphs have a y-value of 2, which also shows that when $x = 1$, both sides of the equation are equal (they're both equal to 2). In short, the solution of the equation $2^x = 3x - 1$ (as far as the graph is concerned) is

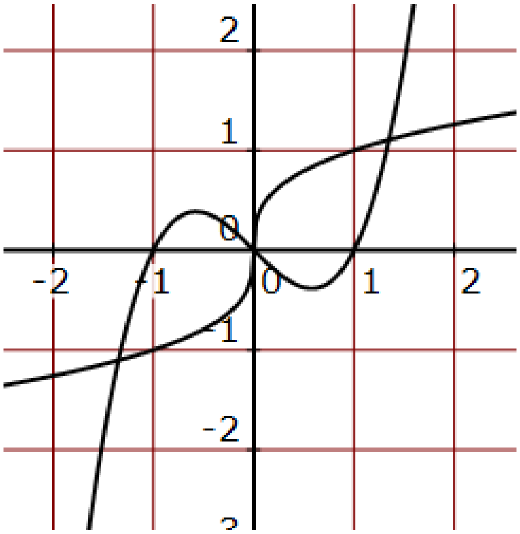
$$\boxed{x = 1}$$

Two Caveats (things to watch out for):

1. How do we know that there's only one solution? Perhaps the graphs cross somewhere way up high in Quadrant I -- using the graph we have, we can never know.
2. It may appear that the point of intersection is (1, 2), but a graph is just a rough picture -- no graph can be perfectly accurate. So maybe the true value of x at the point of intersection is 0.999997, or maybe 1.0000203. If it is one of those, the graph will NEVER tell us that. So the graphing method is not an exact method, but if the equation can't be solved using algebra, a graph may be the best we can get, and may be quite excellent for applications both inside and outside of mathematics.

EXAMPLE 2: **Solve the equation $x^3 - x = \sqrt[3]{x}$ graphically.**

Solution: For this problem, I'm not going to even bother telling you which graph goes with which side of the equation. Just know that each side of the equation has been graphed. Your job is to pick out any points of intersection (which are, ultimately, just approximations). The x-values of those points of intersection will be the solutions of the equation.



First, do you see three points of intersection? There's one in Quadrant I, there's one in Quadrant III, and it also appears that the origin, (0, 0), is a point of intersection. Second comes the guessing part: The x-values of the three points of intersection are approximately –1.3, 0, and 1.3. So these are our best shots at the solutions of the equation:

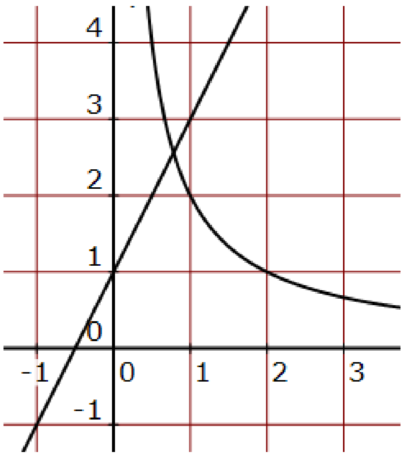
$$\boxed{x = -1.3 \quad x = 0 \quad x = 1.3}$$

For the rest of this chapter (after the following homework section), you'll have to graph the given equation yourself before you find the point(s) of intersection.

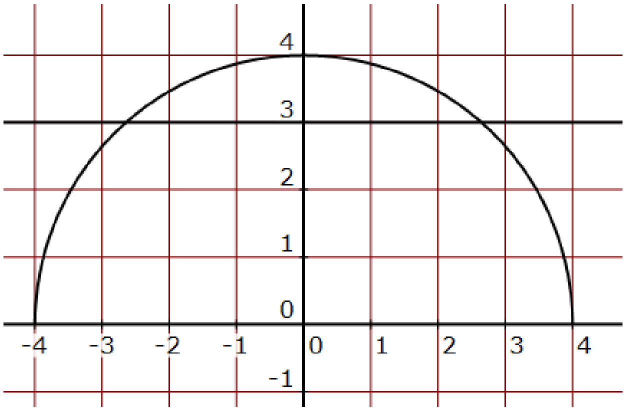
Homework

Solve each equation by using the associated graphs:

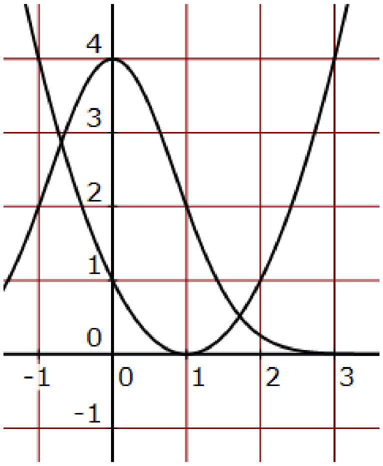
1. $\frac{1}{x} = 2x - 1$



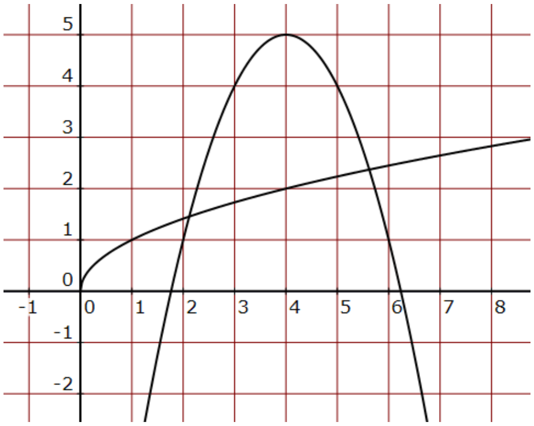
2. $\sqrt{16 - x^2} = 3$



3. $4^{-x^2} = (x-2)^2$



4. $-(x-4)^2 + 5 = \sqrt{x}$



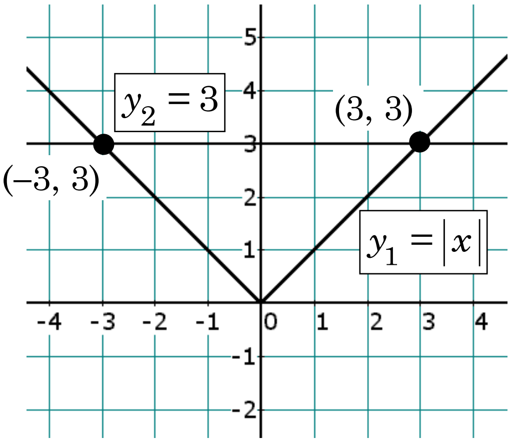
❑ ABSOLUTE VALUE EQUATIONS

<u>EXAMPLE 3:</u> **Solve the equation $|x| = 3$ graphically.**

<u>Solution:</u> We graph each side of the equation separately and look at any points of intersection we might find. To help us keep track, we'll call the left side of the equation y_1 and the right side y_2:

$$y_1 = |x| \qquad y_2 = 3$$

Now graph each equation on the same grid. It appears we have two points of intersection: (–3, 3) and (3, 3). Thus, if $x = -3$ or $x = 3$, the two sides of the equation balance, meaning we have two solutions to our equation:



$$\boxed{x = 3 \text{ or } x = -3}$$

[We could also write $x = \pm 3$]

Check: $x = 3$:

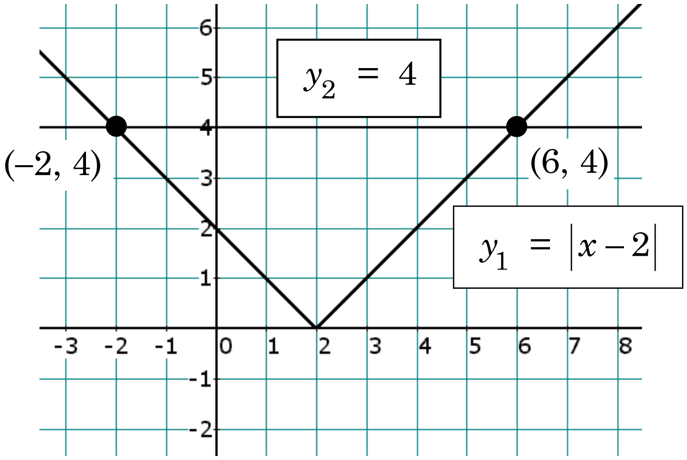
$\|x\|$	3
$\|3\|$	
3	✓

$x = -3$:

$\|x\|$	3
$\|-3\|$	
3	✓

EXAMPLE 4: **Solve the equation $|x-2| = 4$ graphically.**

Solution: Let $y_1 = |x-2|$ and $y_2 = 4$. Graphing y_1 and y_2 on the same grid gives the following graphs:



It looks like y_1 and y_2 intersect at the points (–2, 4) and (6, 4). Therefore, the two solutions of the equation $|x-2| = 4$ are

$$\boxed{x = -2 \text{ or } x = 6}$$

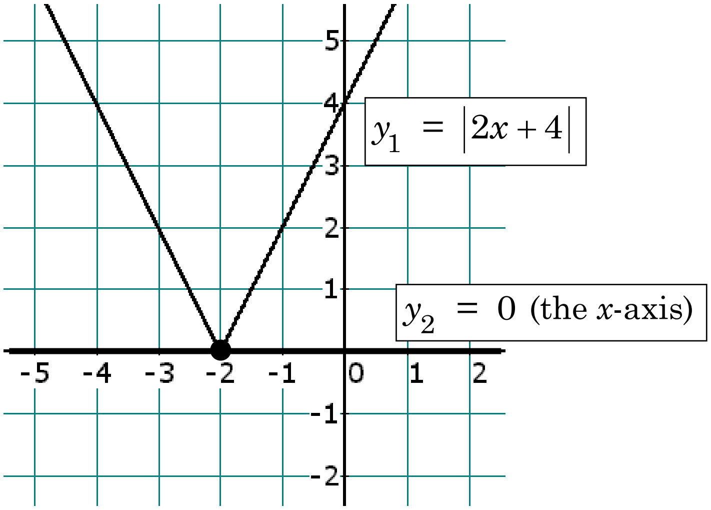
Check:

$x = -2$:	$\lvert x-2\rvert$	4
	$\lvert -2-2\rvert$	
	$\lvert -4\rvert$	
	4	✓

$x = 6$:	$\lvert x-2\rvert$	4
	$\lvert 6-2\rvert$	
	$\lvert 4\rvert$	
	4	✓

EXAMPLE 5: **Solve the equation $|2x+4| = 0$ graphically.**

Solution: Let $y_1 = |2x+4|$ and $y_2 = 0$. Noting that the graph of $y_2 = 0$ is just the x-axis, we graph y_1 and y_2 on the same grid:



The two graphs intersect at one point, (–2, 0). Thus, each formula has the same value (0) when $x = -2$. Thus, the only solution of the given equation is

$$\boxed{x = -2}$$

Check:

$\lvert 2x+4 \rvert$	0
$\lvert 2(-2)+4 \rvert$	
$\lvert -4+4 \rvert$	
$\lvert 0 \rvert$	
0	✓

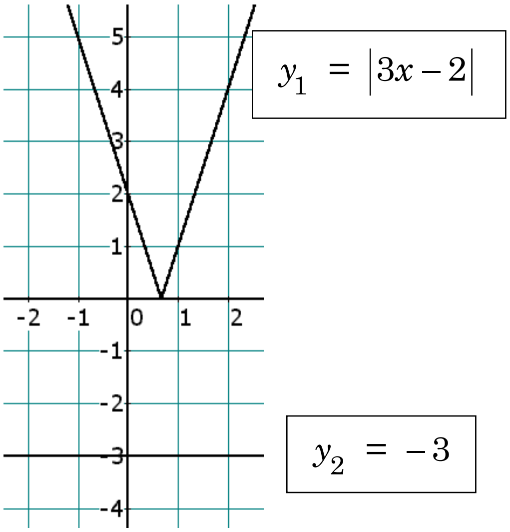
<u>EXAMPLE 6:</u> **Solve the equation $|3x-2| = -3$ graphically.**

<u>Solution:</u> Let's graph $y_1 = 3x-2$ and $y_2 = -3$:

Look at the two graphs. Notice that y_1 and y_2 have <u>no</u> point of intersection. This means that y_1 can <u>never</u> be equal to y_2.

What's our conclusion? The given equation has

No solution



Homework

5. Solve each absolute-value equation graphically:

a. $|x| = 2$ b. $|x| = -3$ c. $|x+1| = 3$

d. $|x-3| = 0$ e. $|2x-2| = 4$ f. $|2x+4| = -1$

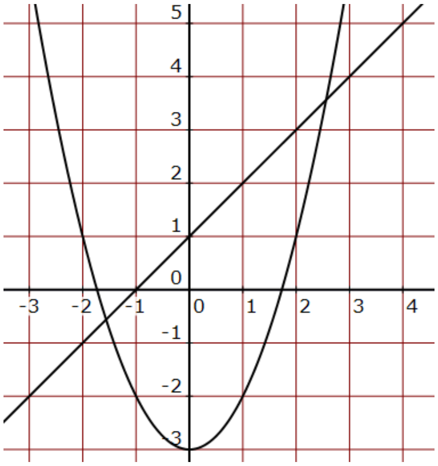
❑ LINES AND CURVES

EXAMPLE 7: **Solve the equation $x^2 - 3 = x + 1$ graphically.**

Solution: Let's graph $y_1 = x^2 - 3$ and $y_2 = x + 1$ on the same grid, and then check out if there are any points of intersection:

The formula $y_2 = x + 1$ is just a straight line, covered in the previous chapter. But the formula $y_1 = x^2 - 3$ might require a little more work. Let's make a table of *x*-*y* values:

x	y
–3	6
–2	1
–1	–2
0	–3
1	–2
2	1
3	6



Now we try to determine where the line and the curve intersection. It appears (although there's no guarantee) that the points of intersection are **(–1.5, –0.5)** and **(2.5, 3.5)**. We can therefore estimate the two solutions of the original equation as

$$\boxed{x = -1.5 \text{ or } x = 2.5}$$

Note: The solutions we obtained in this problem are NOT the actual solutions -- they are simply the best we can get from the picture. Later in the course, we'll have a couple of ways to find the exact solutions.

Homework

Solve each equation by graphing:

6. $x^2 - 2 = x + 4$ 7. $x^2 - 1 = 2x - 2$ 8. $x^2 + 2 = x - 1$

❑ INEQUALITIES

An inequality is quite different from an equation. Whereas an equation usually has one or a couple of solutions, inequalities tend to have infinitely many solutions. Here's an example:

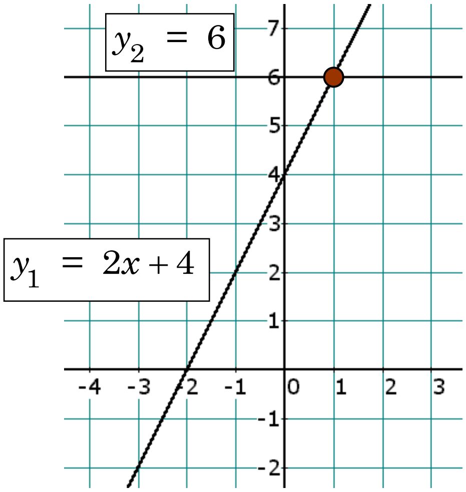
Solve for x: $5x - 3 > 12$

We're seeking any and all values of x which will make the statement true. Suppose $x = 6$. Then $5(\mathbf{6}) - 3 = 30 - 3 = 27$, which is > 12. Thus $x = 6$ is a solution of the inequality; but it's "a" solution, not "the" solution. Why? Because there are others; $x = 5$ will work; $x = 3.7$ will work.

But will $x = 3$ work? Let's see: $5(\mathbf{3}) - 3 = 15 - 3 = 12$, which is <u>not</u> greater than 12. So $x = 3$ fails to satisfy the inequality. What about $x = 0$? $5(\mathbf{0}) - 3 = 0 - 3 = -3$, which is certainly not greater than 12. Let's summarize: 0 and 3 failed, but 3.7, 5 and 6 worked. Our best guess at this point is that <u>any</u> number greater than 3 will work. Our solution is therefore $\boldsymbol{x > 3}$.

<u>EXAMPLE 8:</u> **Solve the inequality $2x + 4 < 6$ graphically.**

<u>Solution:</u> We employ the same technique we used for all the previous equations in this chapter, except we will take into account the "less than" sign after we construct our graphs. Let $y_1 = 2x + 4$ and $y_2 = 6$, both of which are lines.



First we note that the two lines intersect at the point (1, 6). But remember, we're trying to solve the inequality $2x + 4 < 6$, so here's the question: For what values of x is the quantity $2x + 4$ smaller than 6? Equivalently -- and here's the key question -- where on the graph is y_1 below y_2? By looking at the graphs of the lines, we see that y_1 is below y_2 to the left of the point of intersection, (1, 6). In other words, whenever x is smaller than 1, it will follow that $2x + 4$ is less than 6. That is,

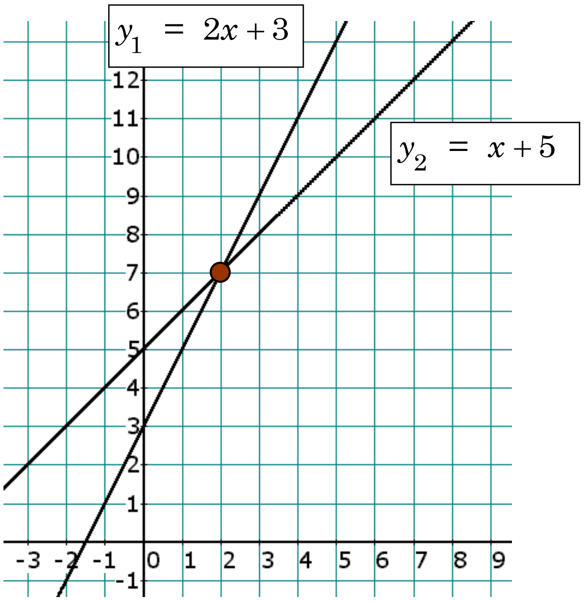
Whenever $x < 1$, we can conclude that $2x + 4 < 6$.

We have found the solution of the inequality $2x + 4 < 6$:

$$\boxed{x < 1}$$

EXAMPLE 9: **Solve the inequality $2x + 3 \geq x + 5$ graphically.**

Solution: Graph $y_1 = 2x+3$ and $y_2 = x+5$ on the same grid.



We're looking for where y_1 is greater than or equal to y_2. Clearly, y_1 and y_2 are equal to each other at the point of intersection, (2, 7). Also, $y_1 > y_2$ wherever the graph of y_1 is above the graph of y_2. This occurs to the right of the point (2, 7); that is, $y_1 > y_2$ whenever

$x > 2$. In short, $y_1 \ge y_2$ whenever $x \ge 2$. Thus, the solution of the inequality $2x + 3 \ge x + 5$ is

$$\boxed{x \ge 2}$$

Homework

9. Solve each inequality graphically:

a. $x + 3 < 6$

b. $x - 1 \ge 5$

c. $2x + 1 \le 5$

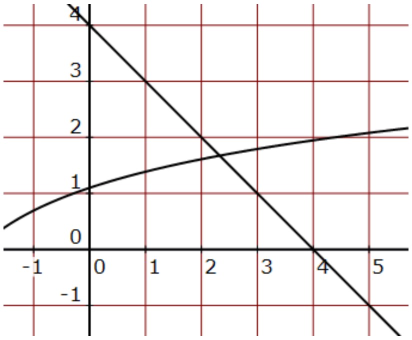
d. $x + 5 \ge 3x - 1$

e. $2x - 1 < x + 2$

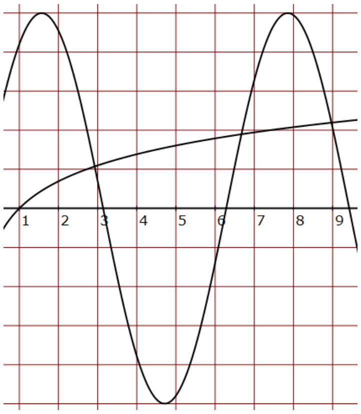
f. $3x + 1 \le 2x - 3$

Practice Problems

10. Solve the equation $\ln(x+3) = -x+4$ by using the associated graph:



11. Solve the equation $5\sin x = \ln x$ by using the associated graph:



12. Solve the equation $|2x-3| = 3$ graphically.

13. Solve the equation $|x+1|+3 = 3$ graphically.

14. Solve the equation $|x-1|+2 = 1$ graphically.

15. Solve the equation $x^2 - 4 = 2x - 3$ graphically.

16. Solve the inequality $2x - 2 \le 4$ graphically.

17. Solve the inequality $2x + 3 > x - 2$ graphically.

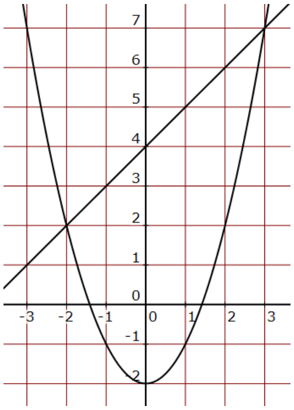
Solutions

Remember that these solutions are approximations; as long as your estimation is close to mine, you have the correct answer.

1. $x = 0.8$ **2**. $x = 2.6, -2.6$ **3**. $x = 1.7, -0.7$ **4**. $x = 2.1, 5.6$

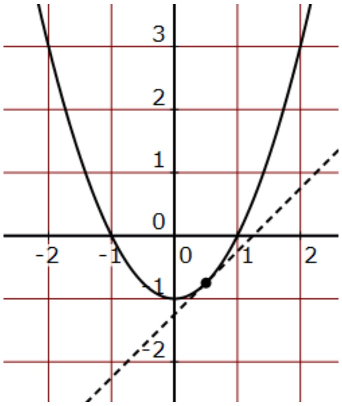
5. a. $x = 2$ or $x = -2$ b. No solution c. $x = 2$ or $x = -4$
d. $x = 3$ e. $x = 3$ or $x = -1$ f. No solution

6.



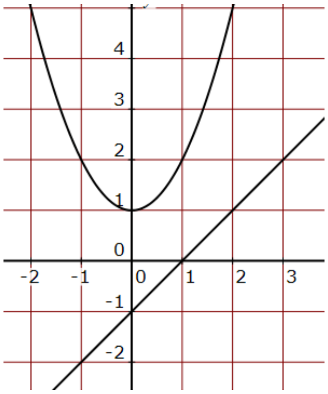
$x = -2, 3$

7.



$x = 0.5$

8.



No Solution

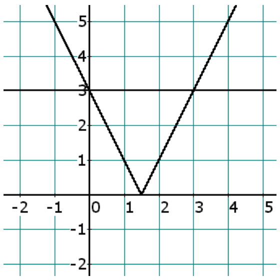
9. a. $x < 3$ b. $x \ge 6$ c. $x \le 2$ d. $x \le 3$

e. $x < 3$ f. $x \le -4$

10. $x = 2.3$ (or anything reasonably close)

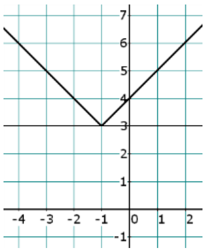
11. There are three points of intersection shown on the graph; the x-values are roughly 2.9, 6.7, and 9.

12.



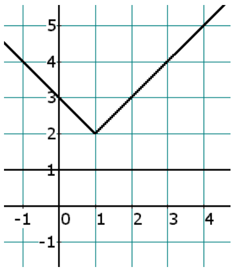
$x = 0, 3$

13.



$x = -1$

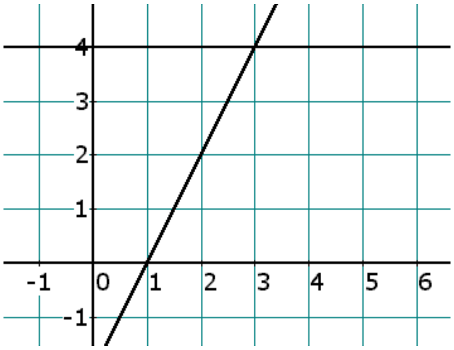
14.



No solution

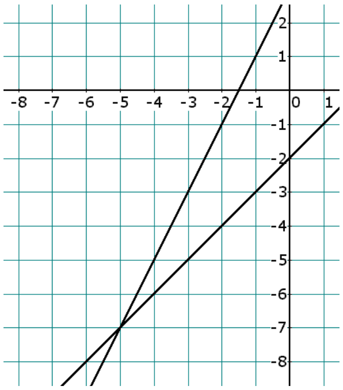
15. $x = -0.4$ or $x = 2.4$ (or anything reasonably close)

16.



$x \le 3$

17.



$x > -5$

"The first step to getting the things you want out of life is this:

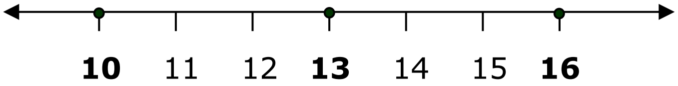
Decide what you want."

\- *Ben Stein*

CH 5 – MIDPOINT, DISTANCE, INTERCEPTS, AND SLOPE

❑ *MIDPOINT ON THE LINE*

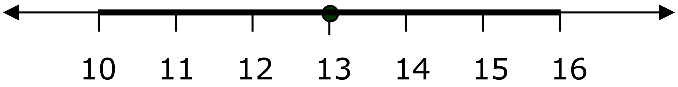
Here's a question for you: What number is *midway* between 10 and 16? You probably know that the number is 13. Why? Because 13 is 3 away from 10 and 13 is also 3 away from 16.



Now we need a simple way to find the number that is midway between any two numbers, even when the numbers are not nice. Notice this: If we take the **average** (officially called the *mean*) of 10 and 16 -- by adding the two numbers and dividing by 2 -- we get

$$\frac{10+16}{2} = \frac{26}{2} = \mathbf{13}\text{, the midway number.}$$

Let's rephrase what we've done with some new terminology. Consider the *line segment* connecting 10 and 16 on the number line:



We can now refer to the 13 as the ***midpoint*** of the line segment connecting 10 and 16.

What is the *midpoint* of the line segment connecting –2.8 and 14.6? Just calculate the average of –2.8 and 14.6:

$$\frac{-2.8+14.6}{2} = \frac{11.8}{2} = \mathbf{5.9}$$

When you see the term ***midpoint***, think *average*!

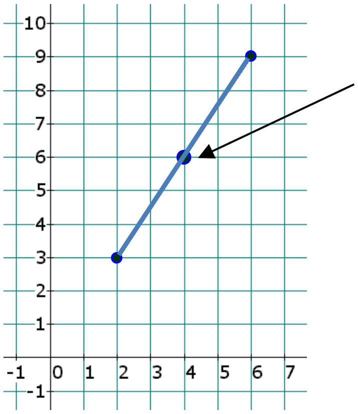
Homework

1. Find the **midpoint** of the line segment connecting the two given numbers on a number line:

a. 10 and 20
b. 13 and 22
c. –8 and –26
d. –3 and 7
e. –7 and 6
f. π and –π
g. –21 and –99
h. 0 and 43
i. –50 and 0
j. –44 and 19
k. –41 and 88
l. 3x and –x

❑ *MIDPOINT IN THE PLANE*

Now for the real question. Consider the two points (2, 3) and (6, 9) in the plane and the line segment that connects them. We need to figure out what point is the *midpoint* of the line segment connecting the two points. Recall the advice given above: When you see midpoint, think *average*. So the x-coordinate of the midpoint is the average of the x-coordinates of the two endpoints:



The **midpoint** is found by averaging the *x*-coordinates and then averaging the *y*-coordinates.

$$x = \frac{2+6}{2} = \frac{8}{2} = 4$$

And the y-coordinate of the midpoint is the average of the y-coordinates of the two endpoints:

$$y = \frac{3+9}{2} = \frac{12}{2} = 6$$

We conclude that the midpoint is **(4, 6)**. That's all there is to it. Now let's do a complete example without plotting any points or drawing any segments.

EXAMPLE 1: **Find the midpoint of the line segment connecting the points (–42, –33) and (90, –10).**

Solution: The x-coordinate of the midpoint is found by averaging the x-coordinates of the two given points:

$$x = \frac{-42+90}{2} = \frac{48}{2} = 24$$

The y-coordinate of the midpoint is found by averaging the y-coordinates of the two given points:

$$y = \frac{-33+(-10)}{2} = \frac{-43}{2} = -\frac{43}{2}$$

The midpoint is therefore the point

$$\boxed{\left(24, -\frac{43}{2}\right)}$$

Homework

2. Find the **midpoint** of the line segment connecting the given pair of points:

a. (–2, 5) and (2, 7)
b. (0, 1) and (0, 6)
c. (–5, 8) and (–5, –8)
d. (–2, 7) and (5, –3)
e. (–9, 2) and (–13, –40)
f. (0, 0) and (–6, –9)
g. (5, 4) and (5, 4)
h. (14, 0) and (0, –9)

i. (8, 8) and (–19, –19)	j. (π, 0) and (–π, 0)
k. $(0, \sqrt{2})$ and $(0, -\sqrt{2})$	l. (a, b) and (c, d)
m. (a, b) and $(a, -b)$	n. $(3a, 3b)$ and $(-3a, b)$

☐ DISTANCE ON THE LINE

Our plan now is to create a formula that will give us the ***distance*** between two points on a line. Consider the two points 10 and 17 on a line. Is it pretty clear that the distance between them is 7? If it's not really obvious, you can simply subtract the smaller number from the larger one:

$$\textit{larger} - \textit{smaller} = 17 - 10 = \mathbf{7}$$

This formula works perfectly for <u>any</u> two numbers on a line:

The distance between –7 and 5 = 5 – (–7) = 5 + 7 = **12**.
[Note that 5 is larger than –7.]

The distance between –9 and –20 = –9 – (–20) = –9 + 20 = **11**.
[Note that –9 is larger than –20.]

Now comes the real problem; remember, this is an Algebra class, so we need a formula for the distance between <u>any</u> two numbers on a line. In other words, we need a formula for the distance between the numbers a and b on a line, when we MAY NOT KNOW which one of them, a or b, is the larger one. Here's the secret: use *absolute value*. This way, if we were to "accidentally" subtract in the wrong direction, and end up with a negative distance (which DOESN'T EXIST), the absolute value will automatically convert the negative number into a positive number.

So, in short, if a and b are any two numbers on the number line, we don't care which one is bigger. We calculate the distance between them by using the formula

$$d = |a-b|$$

Homework

3. Find the **distance** between the given pair of points on the number line:

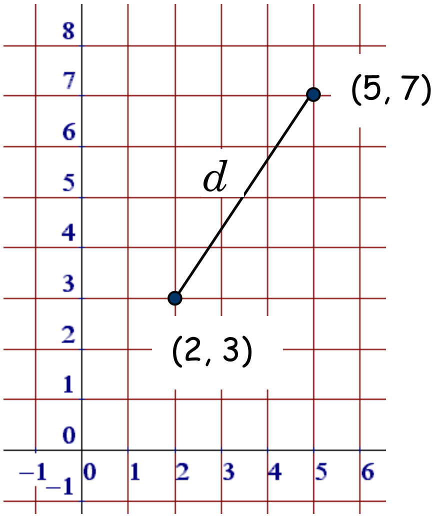
a. 7 and 2	b. –2 and 9	c. –3 and –3
d. 99 and –99	e. –5 and –13	f. –20 and –4

☐ *DISTANCE IN THE PLANE*

Now we add a dimension to the previous section and ask: How do we find the **distance** between two points in the **plane**? If the Earth were flat, it would be like asking how far apart two cities are if we knew the latitude and longitude of each city.

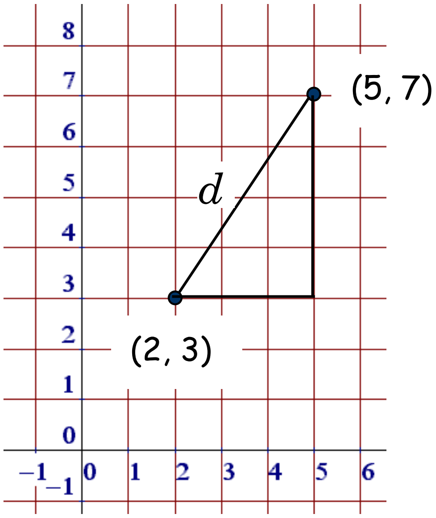
EXAMPLE 2: **Find the distance between the points (2, 3) and (5, 7) in the plane.**

Solution: Let's draw a picture and see what we can see. We'll plot the two given points and connect them with a straight line segment. The distance between the two points, which we'll call d, is simply the length of that line segment.

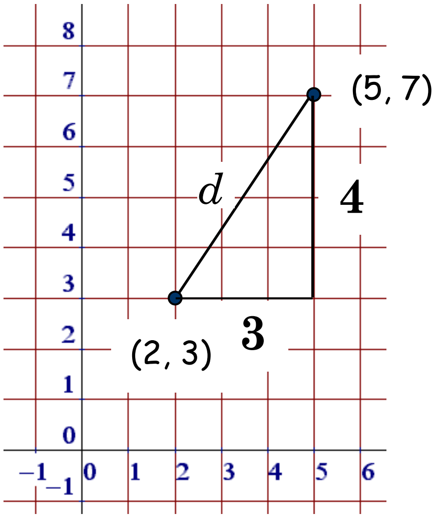


How far is it between the two points (2, 3) and (5, 7)? Equivalently, what is the length of the line segment connecting the two points?

Looks pretty, but now what do we do? Well, here comes the interesting part. If we're creative enough, we can see that the segment connecting the two points could be thought of as the hypotenuse of a right triangle, as long as we sketch in a pair of legs to create that triangle. Let's do that:



Sure enough, we've constructed a right triangle where d is the length of the hypotenuse. If we can determine the lengths of the legs, then we can use the Pythagorean Theorem (see the Prologue) to find the length of the hypotenuse. By counting squares along the base of the triangle, we see that one leg is 3. Similarly, the other leg (the height) is 4.



We've created a right triangle whose legs are 3 and 4, and whose hypotenuse is precisely the distance between the two given points.

Since the square of the hypotenuse is equal to the sum of the squares of the legs, we can write the equation

$$\begin{aligned} & d^2 = 3^2 + 4^2 && \text{(Pythagorean Theorem: hyp}^2 = \text{leg}^2 + \text{leg}^2\text{)} \\ \Rightarrow\quad & d^2 = 9 + 16 && \text{(square the legs)} \\ \Rightarrow\quad & d^2 = 25 && \text{(add)} \\ \Rightarrow\quad & d = 5 && \text{(since } \sqrt{25} = 5\text{)} \end{aligned}$$

Notice that $d = -5$ also satisfies the equation $d^2 = 25$, since $(-5)^2 = 25$. But does a negative value of d make sense? No, since distance can never be negative. We conclude that

$$\boxed{\text{The distance between the two points is 5.}}$$

Homework

4. By plotting the two given points in the plane and using the Pythagorean Theorem, find the **distance** between the given pair of points.

   a. $(2, 3)$ and $(4, 7)$
   b. $(0, 3)$ and $(1, 4)$
   c. $(2, -1)$ and $(-3, 4)$
   d. $(-1, 3)$ and $(-2, 5)$
   e. $(-2, -3)$ and $(-4, -6)$
   f. $(2, 3)$ and $(2, 7)$
   g. $(-1, 5)$ and $(-1, 1)$
   h. $(4, 0)$ and $(8, 0)$
   i. $(4, 6)$ and $(0, 0)$
   j. $(7, 8)$ and $(7, 8)$

☐ *INTERCEPTS*

Consider the line $2x - 3y = 12$. We can find one point on the line very easily by letting $x = 0$. This produces

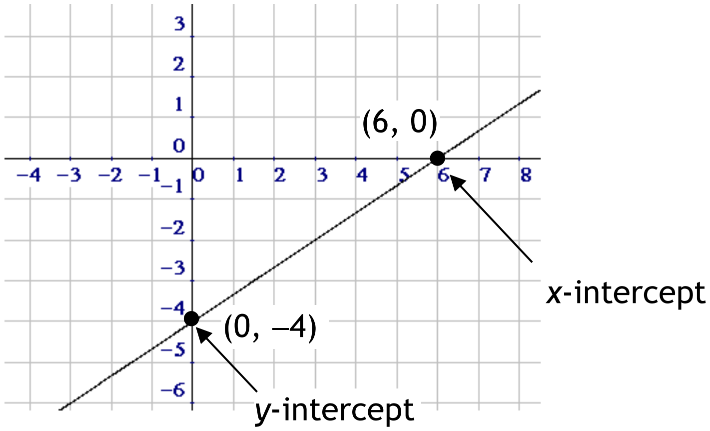
$$\begin{aligned} & 2(\mathbf{0}) - 3y = 12 \\ \Rightarrow \quad & 0 - 3y = 12 \\ \Rightarrow \quad & -3y = 12 \\ \Rightarrow \quad & y = -4 \end{aligned}$$

This shows us that the point **(0, –4)** is on the line.

Now let's set y to 0. We obtain

$$2x - 3(\mathbf{0}) = 12 \;\Rightarrow\; 2x - 0 = 12 \;\Rightarrow\; 2x = 12 \;\Rightarrow\; x = 6$$

We conclude that the point **(6, 0)** is also on the line. Since two points suffice to construct a line (although plotting more than two is an excellent idea), we'll graph our line now using the points $(0, -4)$ and $(6, 0)$:



Notice that the point (6, 0), although certainly on the line $2x - 3y = 12$, also lies on the x-axis. We call the point (6, 0) in this example the ***x-intercept*** of the line. Similarly, we call the point (0, –4) the ***y-intercept*** of the line.

Looking back at the calculations, we see that the $x = 6$ was found by setting y to 0, and that the $y = -4$ was found by setting x to 0. Here's a summary of this easy way to find the intercepts of a line (or <u>any</u> graph):

To find x-intercepts, set $y = 0$.

To find y-intercepts, set $x = 0$.

Of course, a graph like a circle or some other curve may have more than one x-intercept or more than one y-intercept. And when we get to horizontal and vertical lines, you'll see that they may have only one kind of intercept, or they may possibly have an infinite number of intercepts!

EXAMPLE 3: **Find the x-intercept and the y-intercept of the line $3x - 7y = 42$.**

Solution: To find the x-intercept of this line (of any graph, in fact) we set $y = 0$ and solve for x:

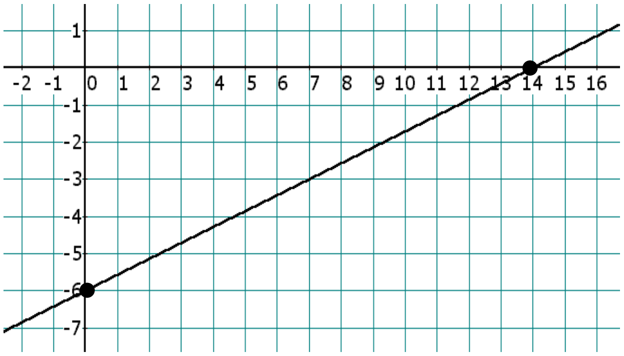
$$\begin{array}{lll} & 3x - 7y = 42 & \text{(the line)} \\ \Rightarrow & 3x - 7(\mathbf{0}) = 42 & \text{(set } y = 0\text{)} \\ \Rightarrow & 3x - 0 = 42 & \text{(anything times 0 is 0)} \\ \Rightarrow & 3x = 42 & \text{(simplify)} \\ \Rightarrow & \dfrac{3x}{3} = \dfrac{42}{3} & \text{(divide each side by 3)} \\ \Rightarrow & x = 14 & \end{array}$$

The x-intercept is therefore $\boxed{(14, 0)}$

Setting $x = 0$ to find the y-intercept gives

$3(\mathbf{0}) - 7y = 42 \Rightarrow -7y = 42 \Rightarrow y = -6$

and so the y-intercept is $\boxed{(0, -6)}$



The x-intercept is (14, 0), the point where the line intercepts the x-axis.

The y-intercept is (0, –6) the point where the line intercepts the y-axis.

Homework

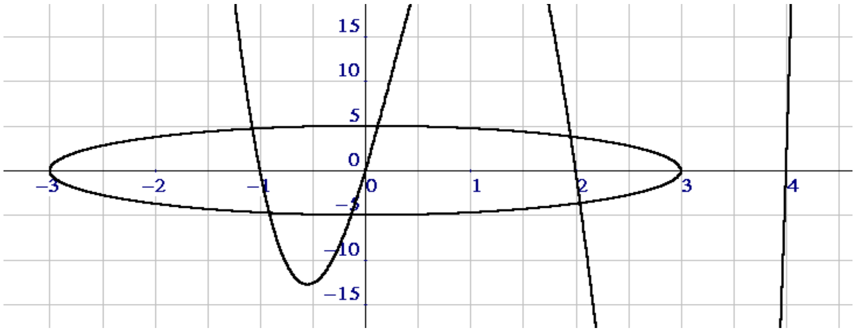
5. Each of the following points (except one) is an intercept. Is it an *x*-intercept or a *y*-intercept?

a. $(0, \pi)$
b. $(-99, 0)$
c. $(0, 0)$
d. $(3\pi, 0)$
e. $(0, -3.7)$
f. $(7, 7)$

6. Find the ***x*-intercept** and the ***y*-intercept** of each line -- be sure that every intercept you write consists of an ordered pair (i.e., two coordinates, one of which must be 0):

a. $2x + y = 12$
b. $3x - 4y = 24$
c. $-4x + 7y = 28$
d. $x - 7y = 7$
e. $y - 3x = 12$
f. $6x + 5y = 60$
g. $4x - 3y = 2$
h. $8y + 3x = 1$
i. $3x - 9y = 0$
j. $y = 7x - 3$
k. $y = -9x + 2$
l. $y = -x - 5$

7. Find <u>all</u> the intercepts of the following graph:



8. ** Find all the intercepts of the graph of $x^2 + y^2 = 25$.

9. ** Find all the intercepts of the graph of $y = x^2 - 9$.

❑ SLOPE

A trucker is keenly aware of the *grade*, or angle, of the road on which a truck travels -- it determines the speed limit and the proper gear that the truck eeds to be in. A roofer is concerned with the *pitch*, or steepness, of a roof. A construction worker needs to make sure that a wheelchair ramp has the right *angle* with the street or the sidewalk. All of these ideas are examples of the concept “steepness.”

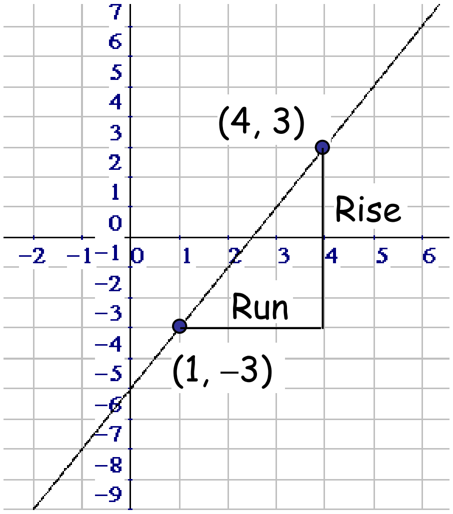
We’ll use the term ***slope*** to represent steepness, and give it the letter *m* (I don’t know why -- maybe *m* for mountain?). Our definition of slope in this course and all future math courses (and chemistry and economics courses) is as follows:

$$\boxed{m = \frac{\text{rise}}{\text{run}}}$$

As we’ll see shortly, a **rise** is a vertical (up/down) change, while a **run** is a horizontal (left/right) change. Slope is defined as the ratio of the rise to the run; we can also say that slope is the quotient of the rise and the run. Let’s do an example where we place a line in the Cartesian coordinate system and analyze the slope of that line.

<u>EXAMPLE 4:</u> **Graph the line $y = 2x - 5$ and determine its slope.**

<u>Solution:</u> If we let $x = 1$, then $y = -3$, so the point $(1, -3)$ is on the line. And if we let $x = 4$, then $y = 3$, giving us the point $(4, 3)$. We could calculate more points for our line, but let’s cut to the chase and graph the line given the two points just computed.



Notice that we've constructed a right triangle using the line segment between the two given points as the hypotenuse. The rise and run are then just the lengths of the legs of the triangle. Counting squares from left to right along the bottom of the triangle, we see that the run is **3**. Counting squares up the side of the triangle yields a rise of **6** . Using the slope formula, we can calculate the slope of the line:

$$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = \boxed{2}$$

Alternate Solution: In the solution above, we essentially started at the point (1, –3), moved 3 units to the right (producing a run of 3), moved 6 units up (producing a rise of 6), finally arriving at the point (4, 3).

Let's reverse our journey; we'll start at the point (4, 3) and travel to the point (1, –3). First we could move 3 units to the left; this produced a run of –3. Then we go down 6 units, giving a rise of –6. This time our calculation of slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{-6}{-3} = \boxed{2}$$

which is, of course, the same slope as before.

Homework

10. For each pair of points, plot them on a grid, find the rise and the run, and then use the formula for slope to calculate the **slope** of the line connecting the two points:

a. (2, 3) and (4, 7)
b. (–3, 0) and (0, 6)
c. (1, –3) and (–2, 5)
d. (2, 2) and (7, 7)
e. (–3, –3) and (0, 0)
f. (–1, –2) and (3, –5)
g. (1, 1) and (–2, 3)
h. (1, 4) and (0, 0)
i. (–3, –2) and (1, –3)
j. (–1, 3) and (1, –3)
k. (–4, 5) and (0, 0)
l. (–1, –1) and (4, –2)

11. Find the **slope** of the given line by graphing the line and then using the rise and run. You may, of course, use any two points on the line to calculate the rise and the run:

a. $y = x + 3$
b. $y = 2x - 1$
c. $y = -2x + 3$
d. $y = 3x + 1$
e. $y = -3x - 2$
f. $y = -x + 2$
g. $x + 2y = 4$
h. $2x - 3y = 1$
i. $3x - y = 3$
j. $-3x + 2y = 6$
k. $2x + 5y = 10$
l. $3x - 4y = -8$

❑ *A New View of Slope*

Finding the slope, $m = \frac{\text{rise}}{\text{run}}$, of a line by plotting two points and counting the squares to determine the rise and the run works fine only when it's convenient to plot the points. Consider the line connecting the points (π, 2000) and (3π, –5000). Certainly these points determine a line, and that line has some sort of slope. But plotting these points is really not feasible -- we need a simpler way to calculate slope.

Notice from Example 4 that the *run* is simply the difference in the x-values and the *rise* is the difference in the y-values. Thus, another way to write the formula for slope is

$$m = \frac{\text{change in } y}{\text{change in } x}$$

But there's even a cooler way to write our formula for m. The natural world is filled with changes. In slope, we've seen changes in x and y in the notions of rise and run. In chemistry, there are changes in the volume and pressure of a gas. In nursing, there are changes in body temperature and blood pressure, and in economics there are changes in supply and demand. This concept occurs so often that there's a special notation for a "change" in something. We use the Greek capital letter delta, Δ, to represent a change in something. A change in volume might be denoted by ΔV and a change in time by Δt.

The Delta Airlines logo

And so now we can redefine ***slope*** as

$$\boxed{m = \frac{\Delta y}{\Delta x}}$$

Slope is the *ratio* of the change in *y* to the change in *x*.

Now we're ready to find the slope using the points mentioned at the beginning of this section: (π, 2000) and (3π, –5000).

EXAMPLE 5: **Find the slope of the line connecting the points (π, 2000) and (3π, –5000).**

Solution: A simple ratio will give us the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{2{,}000-(-5{,}000)}{\pi-3\pi} = \frac{7{,}000}{-2\pi} = \frac{\cancel{2}\cdot 3{,}500}{-\cancel{2}\pi} = -\frac{\mathbf{3{,}500}}{\pi}$$

In the last step of this calculation we used the fact that a positive number divided by a negative number is negative. Also, we could obtain an approximate answer by dividing 3,500 by 3.14 -- then attaching the negative sign -- to get about −1,114.65.

Be careful! You must subtract in the same order. In the problem above, we subtracted from left to right for both the top and the bottom of the slope formula. You could also have subtracted from right to left on the top, as long as you do the subtraction from right to left in the bottom.

Notice that there's no need to plot points and count squares on a grid. We've turned the geometric concept of slope into an arithmetic problem. As we did when we used $\frac{\text{rise}}{\text{run}}$, try reversing the order of the subtractions above to make sure you get the same slope.

Homework

12. Use the formula $m = \frac{\Delta y}{\Delta x}$ to find the **slope** of the line connecting each pair of points:

a. (2, 3) and (4, 7)
b. (–3, 0) and (0, 6)
c. (1, –3) and (–2, 5)
d. (2, 2) and (7, 7)
e. (–3, –3) and (0, 0)
f. (–1, –2) and (3, –5)
g. (1, 1) and (–2, 3)
h. (1, 4) and (0, 0)
i. (–3, –2) and (1, –3)
j. (–1, 3) and (1, –3)
k. (–4, 5) and (0, 0)
l. (–1, –1) and (4, –2)

Practice Problems

13. Find the **midpoint** of the line segment connecting the given pair of points:

a. (1, 0) and (2, 0)
b. (π, –π) and (–π, π)
c. (0, 0) and (–5, 10)
d. (3, –8) and (5, –10)
e. (–2, –4) and (–6, –8)
f. (100, 200) and (50, –2)

14. By plotting the two given points in the plane and using the Pythagorean Theorem, find the **distance** between the given pair of points:

a. (0, 0) and (5, 12)
b. (–1, –1) and (–2, –2)
c. (4, 4) and (4, 0)
d. (π, 3) and (π, 5)
e. (2, 3) and (5, 7)
f. (–1, 2) and (4, –3)

15. Find all the **intercepts** of each line:

a. $y = 7x - 3$	b. $y = -9x + 8$	c. $y = x + 1$
d. $y = -x - 1$	e. $y = \frac{2}{3}x + 5$	f. $y = -\frac{1}{2}x - \frac{4}{5}$
g. $-2x - 7y = 0$	h. $4x + 8y = 6$	i. $18x - 17y = 2$

16. Find the **slope** of the line connecting each pair of points:

a. $(-10, 7)$ and $(-12, -8)$	b. $(12, -10)$ and $(8, -5)$
c. $(12, 3)$ and $(-3, 10)$	d. $(1, 3)$ and $(10, 5)$
e. $(-8, 10)$ and $(12, 8)$	f. $(-9, 1)$ and $(-10, 11)$
g. $(-2, -1)$ and $(1, 5)$	h. $(6, -1)$ and $(-12, -1)$
i. $(4, 6)$ and $(9, -5)$	j. $(3, -3)$ and $(12, 6)$
k. $(3, -12)$ and $(-2, -7)$	l. $(-7, -6)$ and $(-8, 12)$

17. Find the midpoint of the line segment connecting the points $(7, -3)$ and $(14, -9)$.

18. Find the midpoint of the line segment connecting the points (n, w) and $(-2n, 3w)$.

19. Find the distance between the points $(-2, 5)$ and $(1, -4)$.

20. Find the distance between the origin and the point $(7, 24)$.

21. Find all the intercepts of the line $3x - 17y = 102$.

22. Find the slope of the line connecting the points $(2, -7)$ and $(-8, 5)$.

23. Find the slope of the line connecting the points $(2a, b)$ and $(a, -b)$.

Solutions

1. a. 15 b. 17.5 c. –17 d. 2 e. –0.5 f. 0
g. –60 h. 21.5 i. –25 j. –25 k. 11 l. ☺

2. a. (0, 6) b. (0, 7/2) c. (–5, 0) d. (3/2, 2)
e. (–11, –19) f. (–3, –9/2) g. (5, 4) h. (7, –9/2)
i. (–11/2, –11/2) j. (0, 0) k. (0, 0) l. $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$
m. $(a, 0)$ n. $(0, 2b)$

3. a. $|7-2| = |5| = 5$

b. $|-2-9| = |-11| = 11$

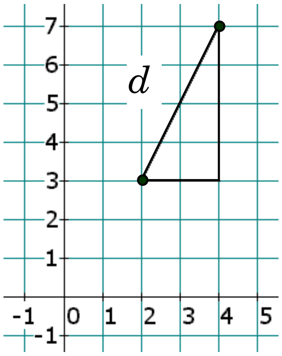
c. $|-3-(-3)| = |-3+3| = |0| = 0$

d. $|99-(-99)| = |99+99| = |198| = 198$

e. 8

f. 16

4. a.



The two points have been plotted in the plane. By counting squares we see the leg at the bottom is 2 while the other leg is 4. The distance between the two given points is simply the length of the hypotenuse. The Pythagorean Theorem gives us

$d^2 = 2^2 + 4^2 \Rightarrow d^2 = 20 \Rightarrow d = 4.472$ (approximately).

(Actually, the equation $d^2 = 20$ has <u>two</u> solutions: ±4.472, but the negative solution is silly since we're looking for distance.)

b. 1.414 c. 7.071 d. 2.236 e. 3.606 f. 4
g. 4 h. 4 i. 7.211 j. 0

Ch 5 – Midpoint, Distance, Intercepts, and Slope

5. a. y-intercept b. x-intercept c. both intercepts
d. x-intercept e. y-intercept f. neither

6. a. (6, 0) (0, 12) b. (8, 0) (0, –6) c. (–7, 0) (0, 4)
d. (7, 0) (0, –1) e. (–4, 0) (0, 12) f. (10, 0) (0, 12)
g. $(\frac{1}{2}, 0)$ $(0, -\frac{2}{3})$ h. $(\frac{1}{3}, 0)$ $(0, \frac{1}{8})$ i. (0, 0)
j. $(\frac{3}{7}, 0)$ (0, –3) k. $(\frac{2}{9}, 0)$ (0, 2) l. (–5, 0) (0, –5)

7. (–3, 0) (–1, 0) (0, 0) (2, 0) (3, 0) (4, 0) (0, –5) (0, 5)

8. The x-intercepts are (5, 0) and (–5, 0). The y-intercepts are (0, 5) and (0, –5).

9. Hint: There are two x-intercepts (and one y-intercept).

10. a. 2 b. 2 c. $-\frac{8}{3}$ d. 1 e. 1 f. $-\frac{3}{4}$
g. $-\frac{2}{3}$ h. 4 i. $-\frac{1}{4}$ j. –3 k. $-\frac{5}{4}$ l. $-\frac{1}{5}$

11. a. 1 b. 2 c. –2 d. 3 e. –3 f. –1
g. $-\frac{1}{2}$ h. $\frac{2}{3}$ i. 3 j. $\frac{3}{2}$ k. $-\frac{2}{5}$ l. $\frac{3}{4}$

12. a. 2 b. 2 c. $-\frac{8}{3}$ d. 1 e. 1 f. $-\frac{3}{4}$
g. $-\frac{2}{3}$ h. 4 i. $-\frac{1}{4}$ j. –3 k. $-\frac{5}{4}$ l. $-\frac{1}{5}$

13. a. (1.5, 0) b. (0, 0) c. (–2.5, 5)
d. (4, –9) e. (–4, –6) f. (75, 99)

14. a. 13 b. 1.414 c. 4 d. 2 e. 5 f. 7.071

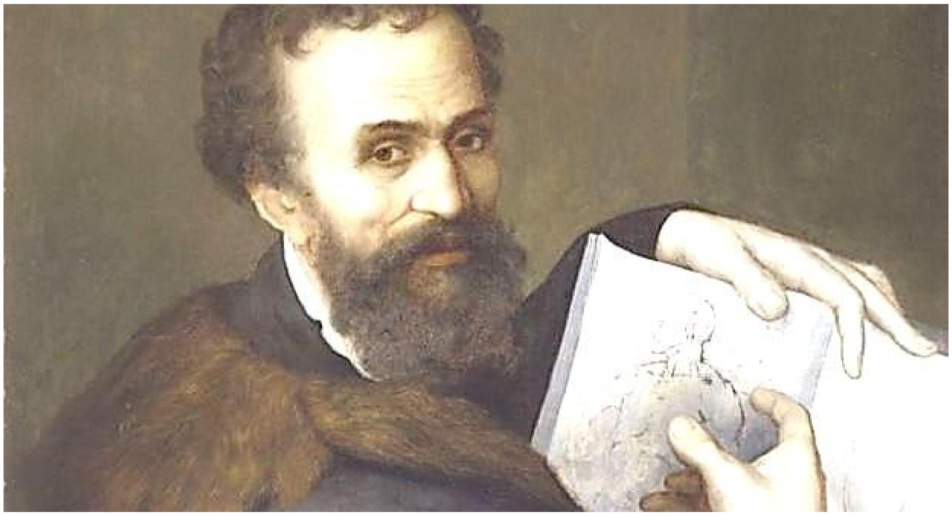
15. a. (0, –3) $(\frac{3}{7}, 0)$ b. (0, 8) $(\frac{8}{9}, 0)$ c. (0, 1) (–1, 0)
d. (0, –1) (–1, 0) e. (0, 5) $(-\frac{15}{2}, 0)$ f. $(0, -\frac{4}{5})$ $(-\frac{8}{5}, 0)$
g. (0, 0) h. $(0, \frac{3}{4})$ $(\frac{3}{2}, 0)$ i. $(0, -\frac{2}{17})$ $(\frac{1}{9}, 0)$

16. a. $\frac{15}{2}$ b. $-\frac{5}{4}$ c. $-\frac{7}{15}$ d. $\frac{2}{9}$ e. $-\frac{1}{10}$ f. -10

g. 2 h. 0 i. $-\frac{11}{5}$ j. 1 k. -1 l. -18

17. $(10.5, -6)$ **18**. $\left(-\frac{n}{2}, 2w\right)$ **19**. 9.49

20. 25 **21**. $(34, 0)$ $(0, -6)$ **22**. $-\frac{6}{5}$ **23**. $\frac{2b}{a}$



“I am still learning.”

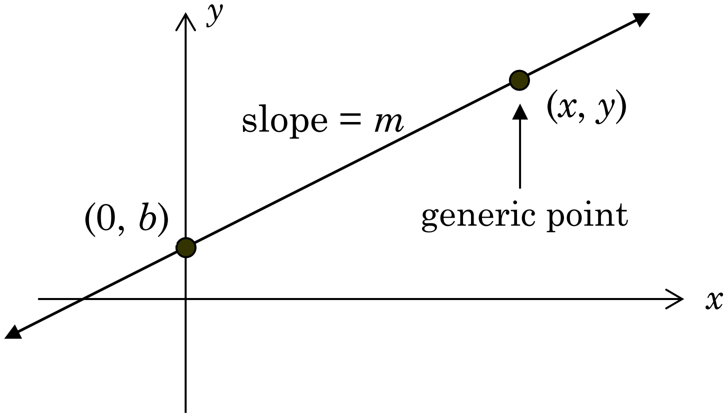
– **Michelangelo, at age 87**

CH 6 – THE EQUATION OF A LINE

❑ *INTRODUCTION*

You might remember $y = mx + b$, the special form of a line where m is the slope and $(0, b)$ is the y-intercept. In Elementary Algebra you probably memorized it and learned how to apply it. In Intermediate Algebra we begin by deriving the equation from scratch. This chapter will also cover parallel and perpendicular lines.

❑ *DERIVING THE SLOPE-INTERCEPT FORM OF A LINE*



We know that the slope of the line is m, and that its y-intercept is $(0, b)$. What is the equation of the line?

Our Goal: Given that a line has slope m and y-intercept $(0, b)$, find the equation of the line.

We've labeled the y-intercept and a generic point (x, y) on the line. On the one hand, the slope of the line is given to be m:

$$slope \;=\; m$$

On the other hand, we can calculate the slope of the line by applying the definition of slope to the two points labeled on the line:

$$slope = \frac{\Delta y}{\Delta x} = \frac{y-b}{x-0} = \frac{y-b}{x}$$

Since the slope of a line is the same everywhere on the line, <u>the two slopes must be equal</u>. That is,

$$\begin{array}{lll} & \dfrac{y-b}{x} = m & \text{(the two expressions for slope are equal)} \\ \Rightarrow & \left(\dfrac{y-b}{x}\right)x = mx & \text{(multiply both sides of the equation by } x\text{)} \\ \Rightarrow & y - b = mx & \text{(cancel the } x\text{'s)} \\ \Rightarrow & y = mx + b & \text{(solve for } y\text{)} \end{array}$$

In summary:

$$y = \underset{\substack{\nearrow \\ \boxed{\text{SLOPE}}}}{m}x + \underset{\substack{\nwarrow \\ \boxed{y\text{-INTERCEPT}}}}{b}$$

To be precise, *b* is not the *y*-intercept; *b* is the *y*-coordinate of the *y*-intercept. The *y*-intercept is properly written (0, *b*).

EXAMPLE 1:

A. Find the slope and y-intercept of the line $y = -\frac{2}{3}x - 5$.
Answer: The slope is $-\frac{2}{3}$ and the y-intercept is (0, –5).

B. Find the equation of the line whose slope is –6 and whose y-intercept is $(0, \frac{4}{5})$. Answer: $y = -6x + \frac{4}{5}$.

EXAMPLE 2: **Find the slope and the y-intercept of the line $3x - 5y + 15 = 0$.**

Solution: The given equation, $3x - 5y + 15 = 0$, doesn't fit the slope-intercept form, $y = mx + b$, of a line. But we can make it fit; we solve the line equation $3x - 5y + 15 = 0$ for y.

$3x - 5y + 15 = 0$ (the original line)

$\Rightarrow$ $3x - 5y = -15$ (subtract 15 from each side)

$\Rightarrow$ $-5y = -3x - 15$ (subtract $3x$ from each side)

$\Rightarrow$ $\frac{-5y}{-5} = \frac{-3x - 15}{-5}$ (divide each side by –5)

$\Rightarrow$ $y = \frac{3}{5}x + 3$ (split the right-hand fraction; see the Prologue)

Now that the line is in the $y = mx + b$ form, we conclude that

The slope is $\frac{3}{5}$ and the y-intercept is (0, 3).

Homework

1. a. Find the slope and y-intercept of the line $y = -17x + 13$.

   b. Find the equation of the line whose slope is $\frac{2}{3}$ and whose y-intercept is $(0, -\pi)$.

   c. Find the slope and y-intercept of the line $y = -\frac{\pi}{2}x + \frac{\sqrt{2}}{\sqrt{3}}$.

   d. Find the equation of the line whose slope is $-\frac{w}{7}$ and whose y-intercept is $\left(0, \frac{a-b}{c}\right)$.

2. Find the slope and y-intercept of each line by converting the line to $y = mx + b$ form (if necessary):

   a. $y = 132x - 1000$
   b. $y = -\frac{8}{7}x - \frac{13}{9}$
   c. $7x - 9y = 10$
   d. $-3x - 5y + 1 = 0$
   e. $2x + 7y = 13$
   f. $-5x + 2y + 3 = 0$
   g. $y = \frac{9x-5}{2}$
   h. $-17x - y = 4$
   i. $2x - 6y = 8$
   j. $-x + 4y - 2 = 0$
   k. $7y - 2x = 0$
   l. $7x + 4y + 5 = 0$

☐ *FINDING THE LINE EQUATION FROM CLUES*

EXAMPLE 3: **Find the equation of the line which has a slope of 7 and which passes through the point (–5, 3).**

Solution: The line equation we are using is $\boldsymbol{y = mx + b}$, where m is the slope and $(0, b)$ is the y-intercept. In this example, we are given the slope of 7. That's good.

And so the line equation $y = mx + b$

becomes $y = 7x + b$ (putting the 7 in for slope)

But we were not given the y-intercept. That's bad. Instead, we are given a point on the line, but it's certainly not the y-intercept (how can you tell?). So now our goal is to find the value of b.

Consider that the problem tells us that the point (–5, 3) is on the line. Therefore, it must work in the equation we just wrote, $y = 7x + b$. So we plug –5 in for x and plug 3 in for y:

$$y = 7x + b \quad \text{(our line with the slope plugged in)}$$

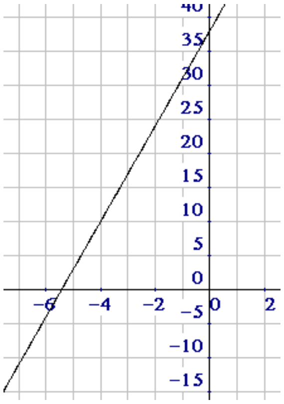
$$\Rightarrow \quad \mathbf{3} = 7(\mathbf{-5}) + b \quad \text{(since (–5, 3) lies on the line)}$$

$$\Rightarrow \quad 3 = -35 + b \quad \text{(multiply)}$$

$$\Rightarrow \quad b = 38 \quad \text{(solve for } b\text{)}$$

Now we put it all together. The value of m is 7 (given in the problem) and the value of b is 38 (we just calculated it). We get our final answer:

$$\boxed{y = 7x + 38}$$



Homework

3. Find the equation of the line with the given slope and passing through the given point:

a. $m = -3$	(8, –16)	b. $m = 4$	(–1, –11)
c. $m = 2$	(0, –10)	d. $m = -3$	(3, –2)
e. $m = 9$	(2, 18)	f. $m = -1$	(–3, 11)
g. $m = 1$	(5, 5)	h. $m = 7$	(0, 10)
i. $m = -8$	(–2, 29)	j. $m = 1$	(10, –89)

EXAMPLE 4: **Find the equation of the line passing through the points (–1, 3) and (8, –15).**

Solution: This problem will really test our deductive skills. Let's begin with the slope-intercept form of a line:

$$y = mx + b$$

Did the problem tell us what the slope is? No. Did the problem give us the *y*-intercept? No. This is not good -- how can we possibly do this problem? Well, even though the slope was not handed to us on a silver platter, we can use the two given points on the line to calculate the slope, using our $m = \frac{\Delta y}{\Delta x}$ formula. So, using the given points (–1, 3) and (8, –15), we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{3-(-15)}{-1-8} = \frac{3+15}{-1-8} = \frac{18}{-9} = \mathbf{-2}$$

We can now write our line as

$$y = -2x + b$$

How do we find b? The same way we did in the previous example: Plug the coordinates of one of the original points in for x and y -- either point will do the job, since each of them lies on the line. Let's use the point (–1, 3).

$$\begin{aligned} & y = -2x + b \\ \Rightarrow\quad & \mathbf{3} = -2(\mathbf{-1}) + b \\ \Rightarrow\quad & 3 = 2 + b \\ \Rightarrow\quad & b = 1 \end{aligned}$$

Putting the values of m and b into the formula $y = mx + b$ gives us the line which passes through the two points:

$$\boxed{y = -2x + 1}$$

Check: Let's check our final answer. If $y = -2x + 1$ is really the line passing through the two given points, then obviously each point should lie on the line. That is, each point should satisfy the equation of the line.

(–1, 3): $\mathbf{3} = -2(\mathbf{-1}) + 1 \Rightarrow 3 = 2 + 1 \Rightarrow 3 = 3$ ✓

(8, –15): $\mathbf{-15} = -2(\mathbf{8}) + 1 \Rightarrow -15 = -16 + 1 \Rightarrow -15 = -15$ ✓

EXAMPLE 5: **Find the equation of the line passing through the points (2, –1) and (5, –6).**

Solution: This problem will be solved in the same manner as the previous example. The equation of the line is

$$y = mx + b,$$

where m and b are to be determined.

$$m = \frac{\Delta y}{\Delta x} = \frac{-6-(-1)}{5-2} = \frac{-6+1}{5-2} = \frac{-5}{3} = -\frac{5}{3}$$

So our line equation is now

$$y = -\frac{5}{3}x + b$$

Placing the first point (either point will work) into this equation gives us

$$-1 = -\frac{5}{3}(2) + b \qquad \text{(since (2, –1) is on the line)}$$

$$\Rightarrow \quad -1 = -\frac{5}{3}\left(\frac{2}{1}\right) + b \qquad \text{(set up for multiply)}$$

$$\Rightarrow \quad -1 = -\frac{10}{3} + b \qquad \text{(multiply the fractions)}$$

$$\Rightarrow \quad -1 + \frac{10}{3} = b \qquad \text{(add } \tfrac{10}{3} \text{ to each side)}$$

$$\Rightarrow \quad -\frac{3}{3} + \frac{10}{3} = b \qquad \text{(common denominator)}$$

$$\Rightarrow \quad \frac{7}{3} = b \qquad \text{(combine the fractions)}$$

And now we have all we need to write the equation of the line:

$$\boxed{y = -\frac{5}{3}x + \frac{7}{3}}$$

Homework

4. Find the equation of the line passing through the two given points. Also be sure you know how to check your answer:

a. (3, 13) and (–1, 5)

b. (1, –9) and (–5, 39)

c. (0, –1) and (2, 9)

d. (1, –11) and (6, –1)

e. (1, –8) and (–2, 31)

f. (0, 0) and (–5, 35)

g. (7, 7) and (–3, –3)

h. (0, 17) and (–17, 0)

i. (–5, –4) and (2, 4)

j. (1, –3) and (–4, –5)

k. (2, –3) and (–1, –10)

l. (–4, 8) and (–1, 6)

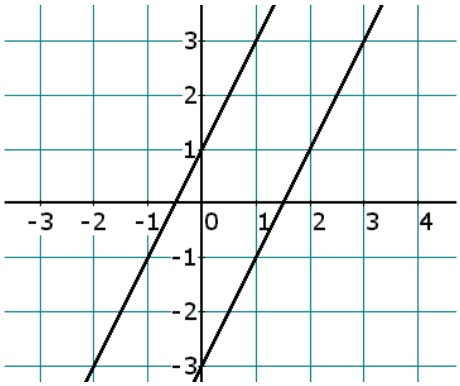
m. (0, –7) and (–2, –4)

n. (3, 7) and (5, –2)

☐ *PARALLEL AND PERPENDICULAR LINES*

Let's begin by assuming that throughout this section we will never be referring to horizontal or vertical lines. These special lines will be covered in detail in Chapter 8.

Parallel Lines



Let's graph the two lines $y = 2x + 1$ and $y = 2x - 3$ on the same grid:

Notice that the two lines are parallel (trust me -- they are). Now, what do the equations of the two lines have in common? The formulas show that each line has a slope of 2. Since slope is a measure of steepness, does it seem reasonable that if two lines are parallel, then they are equally steep, and therefore they must have the same **slope**?

> Parallel lines have the same slope.

For a simple example, suppose Line 1 has a slope of –9, and that Line 2 is parallel to Line 1. We can then deduce that the slope of Line 2 is also –9.

<u>EXAMPLE 6:</u> **Find the slope of <u>any</u> line which is *parallel* to the line $7x - 5y = 2$.**

<u>Solution:</u> Any line which is parallel to the line $7x - 5y = 2$ will have the *same* slope as the line $7x - 5y = 2$. So, if we can compute the slope of this line, we will have the slope of <u>any</u> line <u>parallel</u> to it. The easiest way to find the slope of the line is to convert it to $y = mx + b$ form:

$$7x - 5y = 2 \Rightarrow -5y = -7x + 2 \Rightarrow y = \frac{7}{5}x - \frac{2}{5}$$

The slope of the given line is $\frac{7}{5}$, and so we conclude that any line which is parallel to the line $7x - 5y = 2$ must have a slope of

$$\boxed{\frac{7}{5}}$$

<u>EXAMPLE 7:</u> **Find the equation of the line which is *parallel* to the line $3x + y = 5$, and which passes through the point (6, 2).**

<u>Solution:</u> We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the same as the given line, since the two lines are parallel. Solving the given line for y yields the line $y = -3x + 5$, whose slope is clearly -3. So the slope of our unknown line is also -3. At this point in the problem we can write our line as

$$y = -3x + b$$

Plugging the given point (6, 2) into this equation allows us to find b:

$$\mathbf{2} = -3(\mathbf{6}) + b \;\Rightarrow\; 2 = -18 + b \;\Rightarrow\; 20 = b,$$

and we're done; our line is

$$\boxed{y = -3x + 20}$$

Homework

5. a. A given line has a slope of 7. What is the slope of any line that is parallel to the given line?

   b. A given line has a slope of $-\frac{2}{3}$. What is the slope of any line that is parallel to the given line?

6. a. What is the slope of any line that is parallel to the line $y = \frac{3}{4}x - 9$?

   b. What is the slope of any line that is parallel to the line $5x + 2y = 9$?

7. a. Prove that the lines $2x - 4y = 5$ and $3x - 6y = 1$ <u>are</u> parallel.

   b. Prove that the lines $3x - y = 4$ and $5x + 2y = 10$ are <u>not</u> parallel.

8. Find the equation of the line which is *parallel* to the given line, and which passes through the given point:

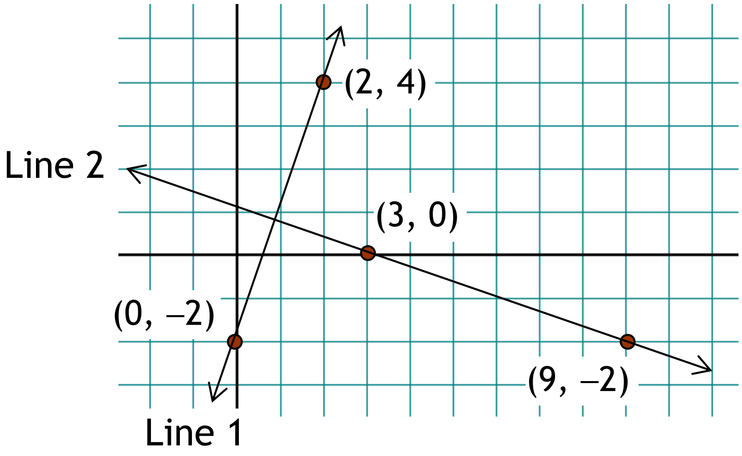
   a. $y = 3x + 4$; $(3, -5)$ b. $y = \frac{1}{2}x - 5$; $(-1, 7)$

   c. $3x + y = 7$; $(1, 9)$ d. $2x - y = 8$; $(-5, -4)$

   e. $2x - 3y = 1$; $(2, -5)$ f. $3x + 4y = 10$; $(8, 0)$

Perpendicular Lines

Parallel lines have the same slope -- certainly *perpendicular* lines do not. But is there a relationship between the slopes of perpendicular lines? Let's see if we can discover one with an example. In the following grid, Line 1 is perpendicular to Line 2, and points have been labeled so that we can easily calculate m_1 and m_2, the slopes of the two lines.



First we compute the slope of Line 1: $m_1 = \dfrac{\Delta y}{\Delta x} = \dfrac{4-(-2)}{2-0} = \dfrac{6}{2} = \mathbf{3}$

Next we compute the slope of Line 2: $m_2 = \dfrac{\Delta y}{\Delta x} = \dfrac{0-(-2)}{3-9} = \dfrac{2}{-6} = -\dfrac{\mathbf{1}}{\mathbf{3}}$

There are two things to note regarding these two slopes of the two perpendicular lines. First, one of the slopes is positive and the other is negative. This makes sense because Line 1 is "rising" while Line 2 is "falling." Second, the slope of Line 1, m_1, is kind of a big number (the line's pretty steep), while the slope of Line 2, m_2, (ignoring the minus sign) is a relatively small number (the line's not very steep).

Specifically, the two slopes have opposite signs, and they are also (ignoring the minus sign) *reciprocals* of each other. In other words, when looking at the slopes of two **perpendicular lines**, each of the slopes is the **opposite reciprocal** of the other.

Perpendicular lines have slopes that are opposite reciprocals of each other.

For example, if a line has a slope of $\frac{7}{4}$, then any perpendicular line must have a slope of $-\frac{4}{7}$.

Consider the line $y = -5x + 1$. Since its slope is -5, it follows that the slope of any perpendicular line must be $\frac{1}{5}$.

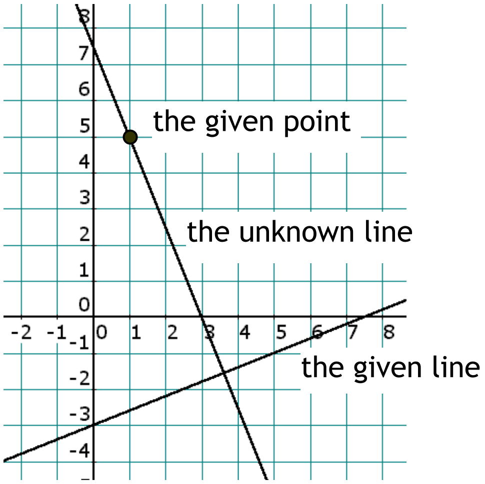
Now for the big example.

EXAMPLE 8: **Find the equation of the line which is *perpendicular* to the line $2x - 5y = 15$, and which passes through the point (1, 5).**

Solution: We're looking for an unknown line

$$y = mx + b$$

The slope of our unknown line was not given to us, but we know that it's the *opposite reciprocal* of the slope of the given line, since the two lines are perpendicular.



To determine the slope of the given line, we solve for y:

$$\begin{aligned} 2x - 5y &= 15 \\ \Rightarrow\quad -5y &= -2x + 15 \\ \Rightarrow\quad \frac{-5y}{-5} &= \frac{-2x+15}{-5} \\ \Rightarrow\quad y &= \frac{2}{5}x - 3, \end{aligned}$$

telling us that the slope of the given line is $\frac{2}{5}$. So the slope of our unknown line is the *opposite reciprocal* of that, which is $-\frac{5}{2}$. At this point in the problem we can write our line as

$$y = -\frac{5}{2}x + b$$

Plugging the given point (1, 5) into this equation allows us to find b:

$$\mathbf{5} = -\frac{5}{2}(\mathbf{1}) + b \quad\Rightarrow\quad 5 + \frac{5}{2} = b \quad\Rightarrow\quad b = \frac{15}{2}$$

and we're done; our line is

$$\boxed{y = -\frac{5}{2}x + \frac{15}{2}}$$

Final Note: We've learned that the slopes of two perpendicular lines are *opposite reciprocals* of each other. But some books say that two lines are perpendicular if *the product of their slopes is* –1. Do both of these rules mean the same thing? Yes -- assume that the product of their slopes is –1:

$$m_1 m_2 = -1$$

Solving for m_1 gives us the equation

$$m_1 = -\frac{1}{m_2},$$

which says that one slope is the opposite reciprocal of the other. So both ways describe the slopes of perpendicular lines.

Homework

9. A given line has a slope of $-\frac{5}{3}$. What is the slope of any line that is perpendicular to the given line?

10. Prove that the lines $7x - 2y = 10$ and $4x + 14y = 23$ are perpendicular.

11. Prove that the lines $3x + 2y = 10$ and $2x + 3y = 9$ are not perpendicular.

12. Prove that the lines $5x - 3y = 10$ and $5x + 3y = 17$ are not perpendicular.

13. Find the slope of any line which is *perpendicular* to the given line:

    a. $y = -7x + 9$
    b. $y = \frac{5}{4}x + 10$
    c. $2x + 7y = 10$
    d. $3x - 2y = 0$

14. Find the equation of the line which is *perpendicular* to the given line, and which passes through the given point:

a. $y = 3x + 4$; (3, –5)	b. $y = \frac{1}{2}x - 5$; (–1, 7)
c. $3x + y = 7$; (1, 9)	d. $2x - y = 8$; (–5, –4)
e. $5x - 3y = 1$; (2, –5)	f. $3x + 4y = 10$; (8, 0)

Practice Problems

15. The slopes of two parallel lines are ______.
16. The slopes of two perpendicular lines are ______.
17. The slope of a line is 5/7. What is the slope of any parallel line?
18. The slope of a line is –4/9. What is the slope of any perpendicular line?
19. T/F: The lines $y = 7x - 3$ and $14x - 2y = 22$ are parallel.
20. T/F: The lines $3x - 7y = 1$ and $7x + 3y = 0$ are perpendicular.
21. Find the equation of the line which is parallel to $3x - 7y = 9$ and passes through the point (–3, 10).
22. Find the equation of the line which is perpendicular to $4x - 9y = 11$ and passes through the point (3, –13).
23. Find the slope and y-intercept of the line $-7x - 3y = 10$.
24. Find the equation of the line with slope –8 and passing through the point (–1, –5).
25. Find the equation of the line passing through the points (–1, 9) and (9, –2).
26. Find the equation of the line which is parallel to the line $3x - 4y = 1$ and which passes through the point (–2, –7).

27. Find the equation of the line which is perpendicular to the line $3x - 4y = 1$ and which passes through the point $(-2, -7)$.

28. Which one of the following lines is parallel to the line $5x - 3y = 7$?

a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

29. Which one of the following lines is perpendicular to the line $5x - 3y = 7$?

a. $y = \frac{3}{5}x - 1$ b. $y = -\frac{3}{5}x + 4$ c. $y = \frac{5}{3}x - 3$

d. $y = -\frac{5}{3}x + 2$ e. $y = \frac{5}{3}$

Solutions

1. a. $m = -17$ y-int = $(0, 13)$ b. $y = \frac{2}{3}x - \pi$

c. $m = -\frac{\pi}{2}$ y-int = $\left(0, \frac{\sqrt{2}}{\sqrt{3}}\right)$ d. $y = -\frac{w}{7}x + \frac{a-b}{c}$

2. a. $m = 132$ y-int = $(0, -1000)$ b. $m = -\frac{8}{7}$ y-int = $(0, -\frac{13}{9})$

c. $m = \frac{7}{9}$ y-int = $(0, -\frac{10}{9})$ d. $m = -\frac{3}{5}$ y-int = $(0, \frac{1}{5})$

e. $m = -\frac{2}{7}$ y-int = $(0, \frac{13}{7})$ f. $m = \frac{5}{2}$ y-int = $(0, -\frac{3}{2})$

g. $m = \frac{9}{2}$ y-int = $(0, -\frac{5}{2})$ h. $m = -17$ y-int = $(0, -4)$

i. $m = \frac{1}{3}$ y-int = $(0, -\frac{4}{3})$ j. $m = \frac{1}{4}$ y-int = $(0, \frac{1}{2})$

k. $m = \frac{2}{7}$ y-int = $(0, 0)$ l. $m = -\frac{7}{4}$ y-int = $(0, -\frac{5}{4})$

3. a. $y = -3x + 8$ b. $y = 4x - 7$ c. $y = 2x - 10$
d. $y = -3x + 7$ e. $y = 9x$ f. $y = -x + 8$
g. $y = x$ h. $y = 7x + 10$ i. $y = -8x + 13$
j. $y = x - 99$

4. a. $y = 2x + 7$ b. $y = -8x - 1$ c. $y = 5x - 1$
d. $y = 2x - 13$ e. $y = -13x + 5$ f. $y = -7x$
g. $y = x$ h. $y = x + 17$ i. $y = \frac{8}{7}x + \frac{12}{7}$
j. $y = \frac{2}{5}x - \frac{17}{5}$ k. $y = \frac{7}{3}x - \frac{23}{3}$ l. $y = -\frac{2}{3}x + \frac{16}{3}$
m. $y = -\frac{3}{2}x - 7$ n. $y = -\frac{9}{2}x + \frac{41}{2}$

5. a. 7 b. $-\frac{2}{3}$

6. a. $\frac{3}{4}$ b. $-\frac{5}{2}$

7. a. Each line has a slope of $\frac{1}{2}$. Same slope $\Rightarrow$ parallel lines.
b. The slopes are 3 and –5/2. Different slopes $\Rightarrow$ non-parallel lines.

8. a. $y = 3x - 14$ b. $y = \frac{1}{2}x + \frac{15}{2}$ c. $y = -3x + 12$
d. $y = 2x + 6$ e. $y = \frac{2}{3}x - \frac{19}{3}$ f. $y = -\frac{3}{4}x + 6$

9. $\frac{3}{5}$

10. The slopes are $\frac{7}{2}$ and $-\frac{2}{7}$, which are opposite reciprocals of each other.

11. The slopes are $-\frac{3}{2}$ and $-\frac{2}{3}$, which are not opposite reciprocals of each other. (They're reciprocals, but not opposites.)

12. The slopes are $\frac{5}{3}$ and $-\frac{5}{3}$, which are not opposite reciprocals of each other. (They're opposites, but not reciprocals.)

13. a. $\frac{1}{7}$ b. $-\frac{4}{5}$ c. $\frac{7}{2}$ d. $-\frac{2}{3}$

14. a. $y = -\frac{1}{3}x - 4$ b. $y = -2x + 5$ c. $y = \frac{1}{3}x + \frac{26}{3}$

d. $y = -\frac{1}{2}x - \frac{13}{2}$ e. $y = -\frac{3}{5}x - \frac{19}{5}$ f. $y = \frac{4}{3}x - \frac{32}{3}$

15. equal **16**. opposite reciprocals

17. 5/7 **18**. 9/4 **19**. T **20**. T

21. $y = \frac{3}{7}x + \frac{79}{7}$ **22**. $y = -\frac{9}{4}x - \frac{25}{4}$

23. $m = -\frac{7}{3}$ $y\text{-int} = \left(0, -\frac{10}{3}\right)$ **24**. $y = -8x - 13$

25. $y = -\frac{11}{10}x + \frac{79}{10}$ **26**. $y = \frac{3}{4}x - \frac{11}{2}$ **27**. $y = -\frac{4}{3}x - \frac{29}{3}$

28. c. **29**. b.

Ben Sweetland:

> “We cannot hold a torch to light another's path without brightening our own.”

CH 7 – BREAK-EVEN POINT, PART I

❑ *INTRODUCTION*

Conducting Business

Certainly the ultimate goal of a business is not merely to "break even"; however, the break-even point is one of the most important concepts in business. It tells the business owner the point (in either production or in time) where losses have probably ended and profits will begin to appear (or, unfortunately, the other way around). Every business which requires funding (investment or borrowing of money) requires a written business plan stating the projected break-even time.

We'll need just a few business terms, with simplified definitions, to help us understand how algebra can be used to represent the real world. We'll use the term ***revenue*** to represent all the money that a company takes in through sales and services. We'll let ***expenses*** represent the money spent by the company to produce those sales and services. And we'll define **profit** to be the difference between revenue and expenses. We're ready to write a formula now:

Profit = Revenue – Expenses, or

$$P = R - E$$

Review of Inequality Symbols

The fact that 3 is **less than** 5 is written

$$3 < 5$$

If we want to say that x is **less than or equal to** 5, we write

$$x \leq 5$$

Similarly, we might write $10 > 8$ to say that 10 is **greater than** 8, and we write $n \geq 15$ to say that n is **greater than or equal to** 15. Notice that $7 \geq 7$ is a true statement because 7 <u>is</u> greater than or equal to 7. (Namely, it's equal to 7.)

Homework

1. a. Solve the profit formula $P = R - E$ for R.

   b. Solve the profit formula $P = R - E$ for E.

2. True/False:

   a. $7 < 9$ b. $-2 < 7$ c. $-3 < -7$ d. $-9 < -1$
   e. $8 \geq 8$ f. $0 > 9$ g. $0 > -9$ h. $-2 \leq -2$
   i. $-2 < -2$ j. $0 < 0$ k. $0 \geq 0$ l. $-12 > -5$
   m. $\pi > \sqrt{2}$ n. $2\pi \geq \pi$ o. $\sqrt{2} < \sqrt{3}$ p. $\sqrt{10} < \pi$

3. Consider the inequality $n \geq 7$. Which of the following values of n would make the statement true?

   a. 15 b. 7.001 c. 6.999 d. 7 e. 2 f. -3 g. -29 h. $\sqrt{40}$

4. Which of the following values of x will satisfy the inequality $x < -3$?

   a. -100 b. -3.001 c. -3 d. -2.999 e. 0 f. 3.14 g. 140

❑ *DEFINITION OF BREAK-EVEN*

There are two ways to define the ***break-even point***. One is to say that break-even occurs when revenues match expenses; that is, when $R = E$. On the other hand, if the revenues and expenses are the same, then there's zero profit. So break-even can also be defined as the point at which the profit is 0.

Either equation can be used to find the ***break-even point***:

$$R = E$$

$$P = 0$$

To prove that the profit must be zero when revenue = expenses, we can use a little algebra. Assume that

$R = E$ (one criterion for break-even)

$\Rightarrow$ $R - E = 0$ (subtract E from each side of the equation)

$\Rightarrow$ $P = 0$ (since $P = R - E$)

❑ USING A TABLE

Assuming w represents the number of widgets a company manufactures and sells, we first make up a formula for revenue:

$$R = 4w$$

And then make up an expense formula:

$$E = 2w + 8$$

Let's create a table with revenue, expenses, and profit for various numbers of widgets sold, according to the given formulas:

w	$R = 4w$	$E = 2w + 8$	$P = R - E$
Widgets	Revenue	Expenses	Profit
0	\$0	\$8	–\$8
1	\$4	\$10	–\$6
2	\$8	\$12	–\$4
3	\$12	\$14	–\$2
4	\$16	\$16	\$0
5	\$20	\$18	\$2
6	\$24	\$20	\$4

Break-Even Point → 4

Now for the important observations:

- ✓ As the number of widgets increases, so do the revenue and the expenses. Also, the profit increases; that is, the numbers –8, –6, –4, –2, 0, 2, and 4 are getting larger.
- ✓ If we produce and sell anywhere from 0 to 3 widgets we incur a loss (a negative profit). That is, when $w \le 3$, $P < 0$.

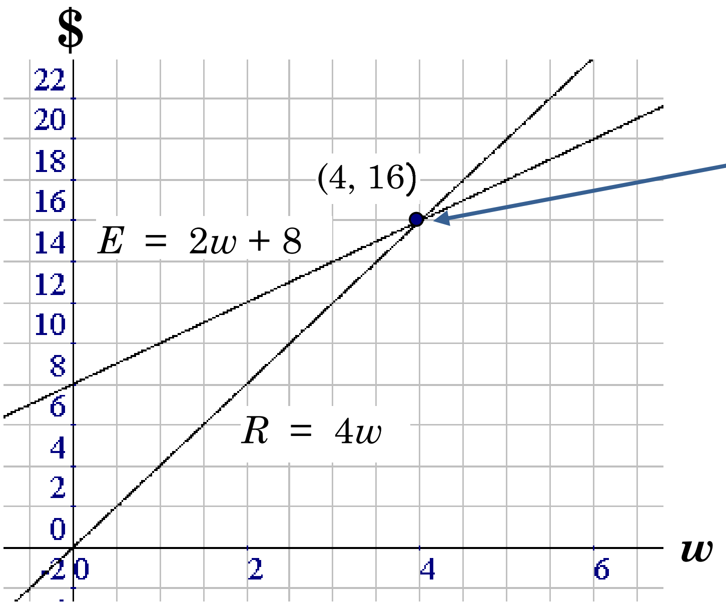
- ✓ At a production level of 4 widgets we break even. This is the point at which the revenue (\$16) equals the expenses (\$16). But also note that this is where the profit is \$0. Either way we look at it, according to the two definitions, $w = 4$ is the break-even point. When $w = 4$, $P = 0$.

- ✓ Any number of widgets beyond 4 (that is, 5 or more) produces a positive profit (we're making money!). Thus, if $w \geq 5$, then $P > 0$.

- ✓ In summary, a **loss** occurs when $w \leq 3$, the **break-even point** occurs when $w = 4$, and a **profit** results when $w \geq 5$.

USING A GRAPH

We're going to make a picture of this situation of revenue, expenses, profit, and the break-even point. Our grid will contain the graphs of both the revenue and expense formulas from the previous example. Note that the horizontal axis will certainly be w, the number of widgets, but the vertical axis will be the generic category money, since both revenue and expenses are in units of money.



The ***break-even point*** occurs when w = 4, where both the revenue and the expenses are \$16.

Fewer than 4 widgets results in a loss -- more than 4 produces a profit.

❑ *USING AN EQUATION*

The table and the graph were quite useful in determining and understanding the relationship among revenue, expenses, profit, and break-even. But these take time to construct and they may give us only approximations. Let's use algebra to solve the same problem for the third time.

EXAMPLE 1: **The revenue formula is $R = 4w$ and the expense formula is $E = 2w + 8$. Find the break-even point.**

Solution: We recall that the break-even point occurs when the revenue equals the expenses. Thus,

$R = E$	(to calculate break-even)
$4w = 2w + 8$	(substituting the given formulas)
$4w - \mathbf{2w} = 2w - \mathbf{2w} + 8$	(subtract $2w$ from each side)
$2w = 8$	(simplify each side)
$\frac{2w}{\mathbf{2}} = \frac{8}{\mathbf{2}}$	(divide each side by 2)
$w = 4$	(simplify)

Thus, the break-even point is $\boxed{\text{4 widgets}}$

just as we saw with the table and the graph.

EXAMPLE 2: **The revenue formula is $R = 12w - 20$ and the expense formula is $E = 2w + 30$.**

a) Calculate the profit formula.

b) Find the break-even point using the profit formula.

Solution: a) The profit formula is $P = R - E$, so we can calculate the profit like this:

$$P = R - E \qquad \text{(the profit formula)}$$

$$\Rightarrow \quad P = (12w - 20) - (2w + 30) \qquad \text{(notice the \underline{parentheses}!)}$$

$$\Rightarrow \quad P = 12w - 20 - 2w - 30 \qquad \text{(distribute)}$$

$$\Rightarrow \quad \boxed{P = 10w - 50} \qquad \text{(combine like terms)}$$

For part b) we find the break-even point by setting the profit formula to 0:

$$P = 0 \qquad \text{(one criterion for break-even)}$$

$$\Rightarrow \quad 10w - 50 = 0 \qquad \text{(substitute the given profit formula)}$$

$$\Rightarrow \quad 10w = 50 \qquad \text{(add 50 to each side of the equation)}$$

$$\Rightarrow \quad \boxed{w = 5} \qquad \text{(divide each side of the equation by 10)}$$

Homework

5. Regarding Example 2, part b), use the profit formula to show that we incur a loss if $w < 5$ and we enjoy a profit if $w > 5$. (A couple of examples will suffice.)

6. Suppose revenue and expense formulas are given by

$$R = 3w + 1 \qquad E = w + 5$$

   a. Construct a table with columns for Widgets, Revenue, Expenses, and Profit. Let w take the values from 0 to 4.

   b. Use the table to determine the break-even point. Explain in two different ways how you arrived at your conclusion.

   c. Graph both formulas on the same grid. Use the graphs to determine the break-even point.

   d. Now find the break-even point by solving the formula equating Revenue and Expenses.

7. Find the **break-even point** for the given revenue and expense formulas by solving an equation:

   a. $R = 10w \qquad E = 7w + 18$

   b. $R = 5w + 1 \qquad E = 4w + 50$

   c. $R = 8w - 9 \qquad E = 3w + 6$

   d. $R = 72w + 12 \qquad E = 50w + 100$

8. Find the **profit** formula for the given revenue and expense formulas:

   a. $R = 30w + 90 \qquad E = 22w - 13$

   b. $R = 22w - 5 \qquad E = 10w + 17$

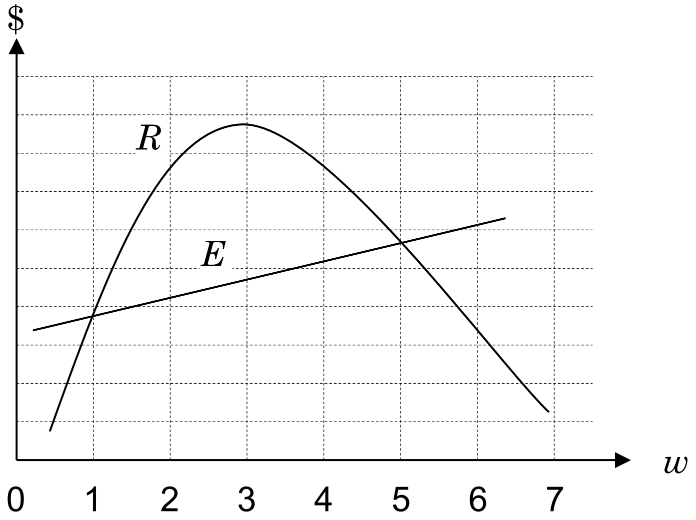
   c. $R = w + 10 \qquad E = 8w - 14$

   d. $R = 13w \qquad E = 2w - 5$

   e. $R = 99w + 17 \qquad E = 9w$

   f. $R = 8w - 1 \qquad E = 7w - 10$

9. Consider the graphs of the revenue and expense formulas:

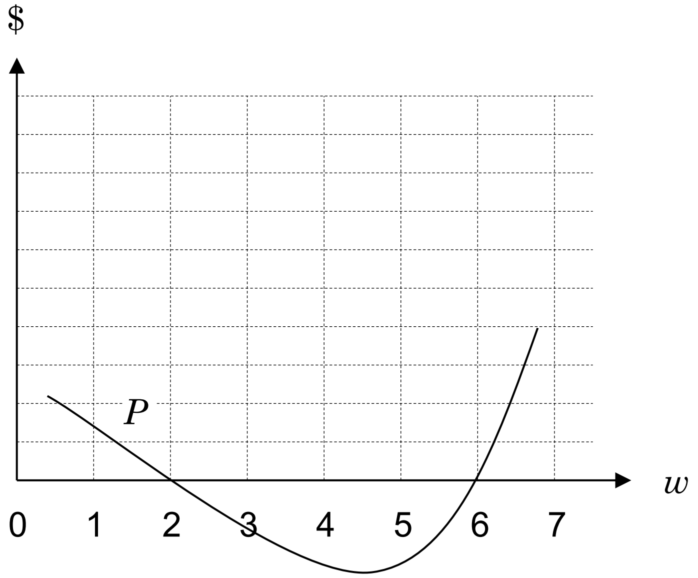


a. Find the two break-even points.

b. Is there a profit or loss when $w = 4$?

c. Is there a profit or loss when $w = 6$?

10. Consider the graph of the **profit** formula:



Find the two break-even points.

11. Find the **break-even point** given each profit formula:

a. $P = 18w - 810$ b. $P = 2.5w - 300$

c. $P = 5.23w - 287.65$ d. $P = -7.6w + 410.4$

Practice Problems

12. Let the revenue and expense formulas be given by $R = 7w + 1$ and $E = 5w + 11$. Construct a table with columns for widgets, revenue, expenses, and profit, and let w take on the values from 3 to 6. Use the table to find the break-even point. Explain in two different ways how you arrived at your conclusion.

13. Use algebra (that is, solve an equation) to find the break-even point if the revenue and expense formulas are given by $R = 15w - 13$ and $E = 7w + 43$.

14. Find the profit formula if the revenue and expense formulas are given by $R = 10w + 13$ and $E = 6w - 5$.

15. Find the break-even point for the profit formula $P = 2.5w - 247.5$.

16. Graph $R = 3w - 4$ and $E = w + 1$ on the same grid, and then estimate the break-even point. How did you find that break-even point?

Solutions

1. a. $R = P + E$ b. $E = R - P$

2.

a.	T	b.	T	c.	F	d.	T	e.	T	f.	F
g.	T	h.	T	i.	F	j.	F	k.	T	l.	F
m.	T	n.	T	o.	T	p.	F				

3. a, b, d **4.** a, b

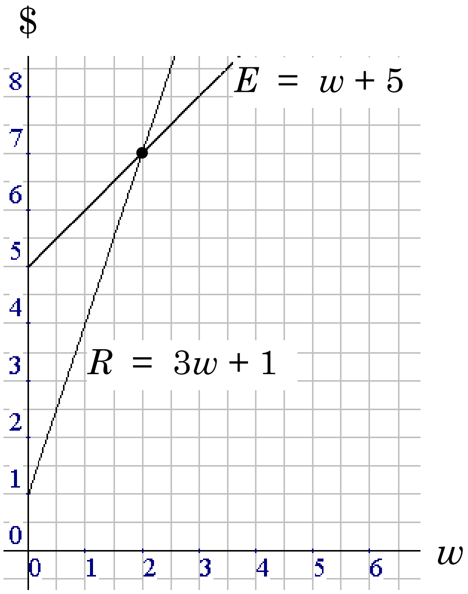
5. Suppose w is 3; then $P = 10(3) - 50 = 30 - 50 = -20$, a loss.
If w is 8, then $P = 10(8) - 50 = 80 - 50 = 30$, a profit.

6. a.

Widgets	Revenue	Expenses	Profit
0	\$1	\$5	–\$4
1	\$4	\$6	–\$2
2	\$7	\$7	\$0
3	\$10	\$8	\$2
4	\$13	\$9	\$4

b. The break-even point is $w = 2$. It's the point where Revenue = Expenses, and it's also the point where Profit = 0.

c.



d. The break-even point can be found by setting Revenue to Expenses:

$R = E$ (to calculate break-even)

$3w + 1 = w + 5$ (substituting the given formulas)

$3w - \boldsymbol{w} + 1 = w - \boldsymbol{w} + 5$ (subtract w from each side)

$2w + 1 = 5$ (simplify)

$2w + 1 - \mathbf{1} = 5 - \mathbf{1}$ (subtract 1 from each side)

$2w = 4$ (simplify)

$\boldsymbol{w = 2}$ (divide each side by 2)

7. a. $w = 6$ b. $w = 49$ c. $w = 3$ d. $w = 4$

8. a. $P = R - E = (30w + 90) - (22w - 13) = 30w + 90 - 22w + 13$
$= 8w + 103$

b. $P = (22w - 5) - (10w + 17) = 22w - 5 - 10w - 17 = 12w - 22$

c. $P = -7w + 24$

d. $P = 11w + 5$

e. $P = 90w + 17$

f. $P = w + 9$

9. a. $w = 1$ and $w = 5$ (found by looking at the w-values where the graphs intersect)

b. profit, since the revenue graph is above the expense graph (i.e., the revenue exceeded the expenses)

c. loss, since the expense graph is above the revenue graph (i.e., the expenses exceeded the revenue)

10. $w = 2$ and $w = 6$ (found by looking where the profit is zero; that is, where the profit graph crosses the w-axis, which is where $P = 0$)

11. To find the break-even points, set each profit formula to zero:

a. $18w - 810 = 0 \Rightarrow 18w = 810 \Rightarrow w = 45$

b. $2.5w - 300 = 0 \Rightarrow 2.5w = 300 \Rightarrow w = 120$

c. $w = 55$

d. $w = 54$

12. $w = 5$. Why is this the break-even point? First, it's where $R = E$; second, it's where $P = 0$.

13. 7 widgets

14. $P = 4w + 18$

15. 99 widgets

16. $w \approx 2.5$; it's the w-coordinate of the point of intersection

“Next in importance
to freedom and justice
is *education*, without
which neither
freedom nor justice
can be permanently
maintained.”

James A. Garfield **(1831 - 1881)**
20th U.S. President

CH 8 – SPECIAL LINES

☐ *INTRODUCTION*

You may have noticed that all the lines we've seen so far in this course have had *slopes* that were either positive or negative. You may also have observed that almost every line had both variables, x and y, in its equation. But there are lines whose slopes are neither positive nor negative, and lines whose equations have only one variable in them. This chapter deals with these special lines.

☐ *GRAPHING*

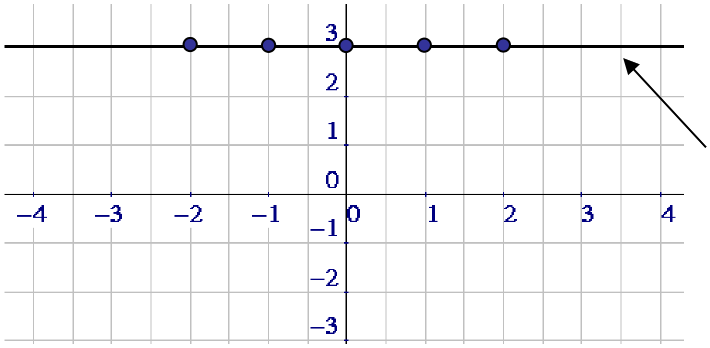
EXAMPLE 1: **Graph the line $y = 3$.**

Solution: This strange little equation doesn't even have an x in it. That's fine -- we just think up our favorite x's, and then understand that y is going to be 3 regardless of the x-value we choose. That is, y is a constant -- it doesn't depend on x. Here's a possible table of values for this line. [You are more than welcome to choose x-values different from the ones I've chosen, but it won't make any difference in the final graph.]

x	y
–2	3
–1	3
0	3
1	3
2	3

We therefore have the points (–2, 3), (–1, 3), (0, 3), (1, 3), and (2, 3). Plotting these five points, and then connecting them with a straight line, produces the following **horizontal** line; notice that the y-intercept of this line is (0, 3), and that there's no x-intercept.

The line $y = 3$



Every point on this line has a y-coordinate of 3.

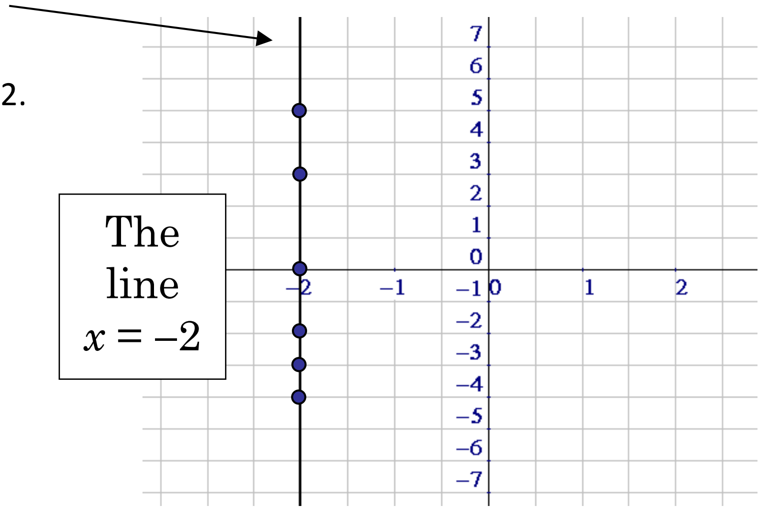
EXAMPLE 2: **Graph the line $x = -2$.**

Solution: This one's as goofy as the previous one -- but this time the y is missing. But more importantly, the equation clearly informs us that x must be -2. Any other choice of x would contradict this requirement. Moreover, since there's no y in the equation, we can choose any number we'd like for y. This leads to a collection of points like this:

$(-2, -4)$ $(-2, -3)$ $(-2, -2)$ $(-2, 0)$ $(-2, 3)$ $(-2, 5)$

When we plot these points and connect them with a straight line, we get the following **vertical** line; note that the x-intercept of this line is $(-2, 0)$, and that there's no y-intercept.

Every point on this line has an x-coordinate of -2.



Homework

1. Graph each line by plotting at least three points:

   a. $y = 4$ b. $y = -3$ c. $x = 5$

   d. $x = -1$ e. $y = 0$ f. $x = 0$

2. a. The horizontal line $y = 0$ is the ________.

   b. The vertical line $x = 0$ is the ________.

3. a. Is the line $x = 1{,}000{,}000$ is horizontal or vertical?

   b. Is the line $y = \sqrt{1679}/\pi$ is horizontal or vertical?

4. At what point do the lines $x = 17$ and $y = -99$ intersect?

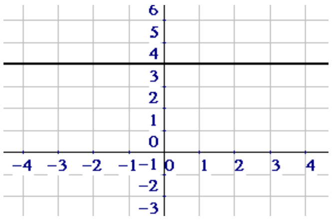
5. Find the intercepts of each line:

   a. $x = 3$ b. $y = -2$ c. $x = 0$ d. $y = 0$

   e. $y = 5$ f. $x = -\pi$ g. $x = \sqrt{2}$ h. $y = x$

❑ THE SLOPE OF A HORIZONTAL LINE

We recall (from Homework 1a) that the graph of the line with the equation $y = 4$ is a horizontal line four units above the x-axis.



Notice that the graph has a y-intercept at (0, 4) but has no x-intercepts. Other points on this horizontal line include (3, 4), (–20, 4), (π, 4), and $(-\sqrt{7}, 4)$. In other words, in the formula $y = 4$, x can be any number, but y must be 4.

Now it's time to calculate the slope of this horizontal line. We need a pair of points on this line -- we'll use (3, 4) and (–20, 4).

$$m = \frac{\Delta y}{\Delta x} = \frac{4-4}{3-(-20)} = \frac{0}{23} = \mathbf{0}$$

Since all horizontal lines ought to have the same slope, we can be confident in drawing the following conclusion:

The slope of any horizontal line is 0.

❑ *THE SLOPE OF A VERTICAL LINE*

Do you remember what the graph of $x = -2$ looks like? Go back to Example 2 and recall that it's a vertical line with x-intercept (–2, 0).

To obtain the slope of this line, we'll use the points (–2, 0) and (–2, 5):

$$m = \frac{\Delta y}{\Delta x} = \frac{5-0}{-2-(-2)} = \frac{5}{-2+2} = \frac{5}{0} = \textbf{Undefined}$$

The conclusion that the slope is "undefined" is based on the fact that division by zero is undefined. We might also observe the "steepness" of the vertical line. It's so steep that no number could possibly measure it, so "undefined" is a good way to describe the slope. Since all vertical lines should have the same slope, we conclude that

The slope of any vertical line is *undefined*.

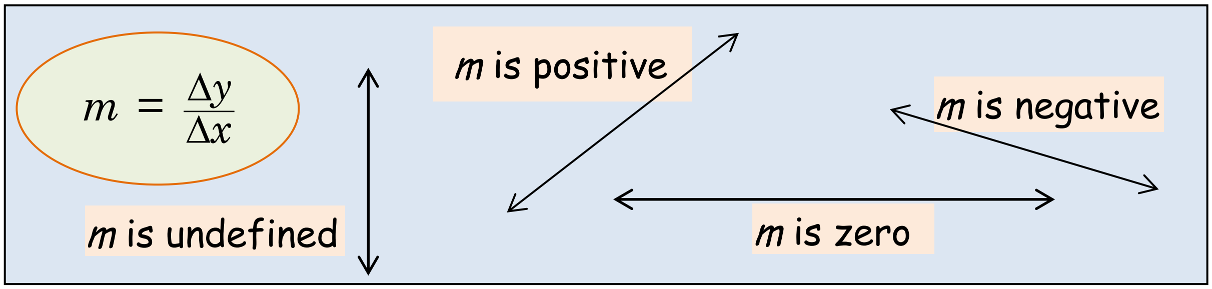
Homework

6. For each line, i) find two points on the line
   ii) use these points and $m = \frac{\Delta y}{\Delta x}$ to find its slope

a. $y = 3$ b. $x = 4$ c. $y = -19$

d. $x = -44$ e. $x = 0$ f. $y = 0$

The following diagram is a summary of our notion of slope:



❑ More Horizontal and Vertical Lines

We know that a horizontal line always has an equation of the form "y = *some number*," while a vertical line always has an equation of the form "x = *some number*." We've also learned that a horizontal line has a slope of 0, while the slope of a vertical line is undefined. We put all this info into a little table to help us see all the important facts about horizontal and vertical lines.

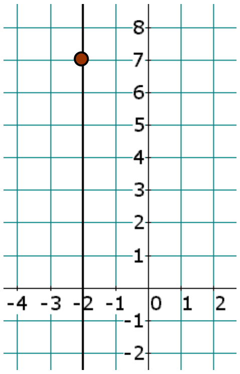
Equation	Type of Line	Slope
y = some number	horizontal	zero
x = some number	vertical	undefined

EXAMPLE 3:

A. **Find the equation of the horizontal line passing through the point (5, 3).**

Solution: A horizontal line has an equation of the form y = *some number*. Since (5, 3) is on the line, the equation of the line must be $\boldsymbol{y = 3}$.

B. **Find the equation of the vertical line passing through the point (–2, 7).**



Solution: A vertical line has an equation of the form x = *some number*. Since (–2, 7) is on the line, the line must have the equation $\boldsymbol{x = -2}$.

C. **Find the equation of the line whose slope is 0 and which passes through the point (–5, 9).**

Solution: If a line has a slope of 0, it must be a horizontal line, whose equation must be of the form y = *some number*. Because (–5, 9) lies on the line, the answer is $\boldsymbol{y = 9}$.

D. **A line has an undefined slope and passes through the point (7, –12). What is the equation of the line?**

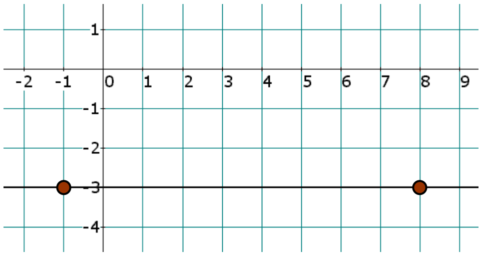
Solution: An undefined slope implies a vertical line, which implies an equation like x = *some number*. Since (7, –12) is on the line, its equation must be $\boldsymbol{x = 7}$.

E. **What is the equation of the line passing through the points (9, 5) and (9, –2)?**

<u>Solution:</u> Plot the two points and you'll notice that (9, 5) is directly above (9, –2), yielding a vertical line. The equation must be $\boldsymbol{x = 9}$.

F. **Find the equation of the line passing through (8, –3) and (–1, –3).**

<u>Solution:</u> A quick sketch shows that (8, –3) lies directly to the right of (–1, –3). This creates a horizontal line whose equation is $\boldsymbol{y = -3}$.



Homework

7. Describe the line $y = 17$.

8. Describe the line $x = -99$.

9. Find the line with a slope of 0 and passing through the point (1, 0).

10. The line $y = 8$ is (horizontal, vertical) and its slope is ______.

11. What is the equation of the line passing through (3, 11) and (–3, 11)?

12. Find the equation of the horizontal line passing through the point (–13, 17).

13. Find the line with a slope of 0 and passing through the point (–1, 7).

14. The line $y = -6$ is (horizontal, vertical) and its slope is ______.

15. What is the equation of the line passing through (–17, –13) and (7, –13)?

16. Find the equation of the vertical line passing through the point (–4, –9).

17. Find the line with a slope of 0 and passing through the point (2, 1).

18. The line $x = -4$ is (horizontal, vertical) and its slope is ______.

19. What is the equation of the line passing through (–1, 9) and (–1, 11)?

20. Find the equation of the horizontal line passing through the point (–1, 0).

21. Find the line with an undefined slope and passing through the point (3, 7).

22. The line $x = -8$ is (horizontal, vertical) and its slope is ______.

23. What is the equation of the line passing through (–5, 10) and (–5, 6)?

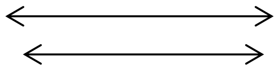
24. Find the equation of the vertical line passing through the point (–18, –11).

25. Find the line with a slope of 0 and passing through the point (–5, –6).

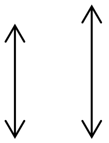
❑ *PARALLEL AND PERPENDICULAR LINES*

Would you agree that a pair of different vertical lines never intersect? When two lines (in the same plane) never intersect, we say that they're ***parallel***. So, for example, the lines $x = 3$ and $x = -4$ and parallel, since each is vertical. Now consider a pair of different horizontal lines. Clearly, they're parallel, too. Thus, for example, the lines $y = 2$ and $y = \pi$ are also parallel.

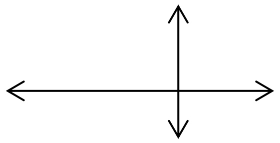
Now consider a vertical line and a horizontal line. They must meet at a 90° angle, and we say that the two lines are ***perpendicular*** (in the same way that the two legs of a right triangle are perpendicular to each other). We can therefore say that the lines $x = 5$ (vertical) and $y = -3$ (horizontal) are perpendicular.



Parallel lines



Parallel lines



Perpendicular Lines

EXAMPLE 4:

A. **Find the equation of the line which is parallel to the line $x = 7$ and which passes through the point (5, 3).**

Solution: The line $x = 7$ is vertical. Any line parallel to this line must also be vertical. What vertical line passes through the point (5, 3)? The line $\boldsymbol{x = 5}$ does.

B. **Find the equation of the line which is perpendicular to the line $x = -5$ and which passes through the point (–2, –9).**

Solution: Since the line $x = -5$ is vertical, the perpendicular line we're seeking has to be horizontal. What is the equation of the horizontal line passing through (–2, –9)? The answer is $\boldsymbol{y = -9}$.

C. **Find the equation of the line which is parallel to the line $y = 17$ and which passes through the point (–1, 3).**

Solution: This time the given line $y = 17$ is horizontal, and since we seek a parallel line, it also must be horizontal. And the horizontal line passing through the point (–1, 3) is certainly $\boldsymbol{y = 3}$.

D. **Find the equation of the line which is perpendicular to the line $y = 11$ and which passes through the point (6, –3).**

<u>Solution</u>: The line $y = 11$ is horizontal, so we need a vertical line passing through (6, –3). That line is $\boldsymbol{x = 6}$.

Homework

26. Fill in each blank with either the word *parallel* or *perpendicular*:

    a. Two different vertical lines are _____.
    b. Two different horizontal lines are _____.
    c. A vertical line and a horizontal line are _____.

27. Fill in each blank with either the word *vertical* or *horizontal*:

    a. A line which is parallel to a vertical line must be ___.
    b. A line which is perpendicular to a horizontal line must be ___.
    c. A line which is parallel to a horizontal line must be ___.
    d. A line which is perpendicular to a vertical line must be ___.

28. a. Are the lines $x = 9$ and $x = -1$ parallel or perpendicular?
    b. Are the lines $y = 7$ and $y = 0$ parallel or perpendicular?
    c. Are the lines $x = -9$ and $y = 7$ parallel or perpendicular?

29. a. Give an example of a line which is parallel to $x = 5$.
    b. Give an example of a line which is parallel to $y = -4$.
    c. Give an example of a line which is perpendicular to $y = -4$.
    d. Give an example of a line which is perpendicular to $x = 8$.

30. Fill in each blank with either the word *vertical* or *horizontal*:

    a. A line which is parallel to the line $y = 7$ must be _____.
    b. A line which is perpendicular to the line $x = 3$ must be _____.
    c. A line which is parallel to the line $x = 8$ must be _____.
    d. A line which is perpendicular to the line $y = -3$ must be _____.

31. a. Find the equation of the line which is parallel to the line $x = 9$ and which passes through the point (1, 7).

    b. Find the equation of the line which is perpendicular to the line $x = -3$ and which passes through the point (–7, 0).

    c. Find the equation of the line which is parallel to the line $y = 10$ and which passes through the point (–5, 8).

    d. Find the equation of the line which is perpendicular to the line $y = -9$ and which passes through the point (7, –2).

Review Problems

32. a. Graph the line $x = -3$ by plotting three points.

    b. Is the line horizontal or vertical?

    c. Find all the intercepts of the line.

    d. Use two of the points and $m = \frac{\Delta y}{\Delta x}$ to calculate the slope.

33. a. Graph the line $y = 2$ by plotting three points.

    b. Is the line horizontal or vertical?

    c. Find all the intercepts of the line.

    d. Use two of the points and $m = \frac{\Delta y}{\Delta x}$ to calculate the slope.

34. a. What is the equation of the x-axis?

    b. What is the equation of the y-axis?

35. The line $y = \sqrt{3}$ is (horizontal, vertical) and its slope is ______.

36. The line $x = \sqrt{2\pi}$ is (horizontal, vertical) and its slope is ______.

37. Graph the line $y = x$. Is it horizontal, vertical, or diagonal? What is its slope?

38. Find the equation of the horizontal line passing through the point (17, 99).

39. Find the equation of the vertical line passing through the point (34, –44).

40. Find the equation of the line with undefined slope passing through the point (2, –π).

41. Find the equation of the line with 0 slope passing through the point (2, –π).

42. What is the equation of the line passing through (2, 7) and (2, 1)?

43. What is the equation of the line passing through (1, 7) and (0, 7)?

44. Find the equation of the line which is parallel to the line $x = 14$ and which passes through the point (–2, –9).

45. Find the equation of the line which is perpendicular to the line $y = -23$ and which passes through the point (π, 0).

46. True/False:

    a. The line $y = \sqrt{2}$ is horizontal.

    b. The line $x = 3$ has an undefined slope.

    c. The line $y = 5$ has exactly one intercept.

    d. The vertical line passing through (2, 7) is $x = 7$.

    e. The equation of the x-axis is $y = 0$.

    f. The line $x = -1$ has infinitely many intercepts.

    g. The point (7, 9) lies on the line $y = 9$.

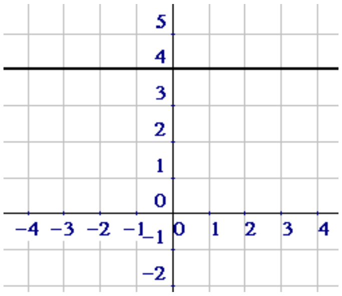
    h. The line $x = -8$ has a negative slope.

    i. The slope of the line $y = 3x + 4$ is 3.

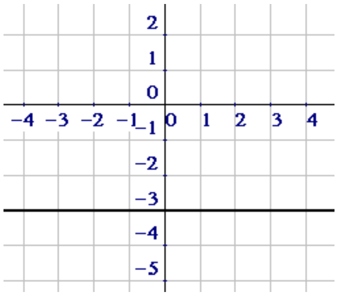
j. The line passing through $(3, \pi)$ and $(3, 1)$ is $x = 3$.
k. The line $y = x$ is horizontal.
l. A line can have two intercepts.
m. The point $(-2, 5)$ lies on the line $x = 5$.
n. The line $y = 7$ has an undefined slope.
o. All horizontal lines have the same slope.
p. The equation of the y-axis is $y = 0$.
q. The line passing through $(1, 2)$ and $(1, 0)$ is $y = 1$.
r. A line can have exactly one intercept.
s. A line can have infinitely many intercepts.
t. The lines $x = 3$ and $y = 4$ are parallel.
u. The slope of the line $y = x$ is 1.

Solutions

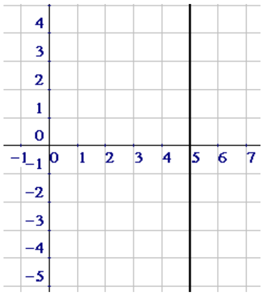
1. a.



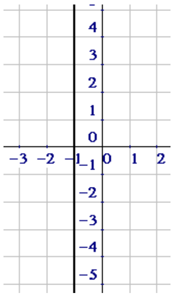
b.



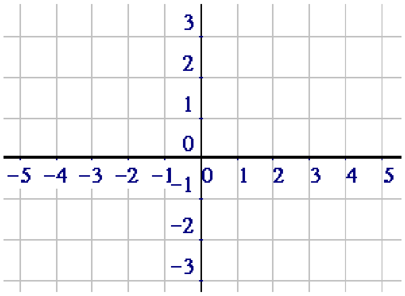
c.



d.



e.



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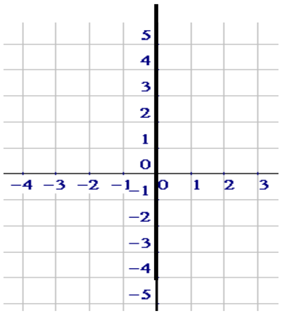


2. a. x-axis b. y-axis **3**. a. vertical b. horizontal **4**. (17, –99)

5. a. x-intercept: (3, 0); No y-intercept
b. No x-intercept; y-intercept: (0, –2)
c. x-int: (0, 0); y-int: all the points on the y-axis are y-intercepts
d. x-int: all the points on the x-axis are x-intercepts; y-int: (0, 0)
e. No x-intercept; y-intercept: (0, 5)
f. x-intercept: $(-\pi, 0)$; No y-intercept
g. x-intercept: $(\sqrt{2}, 0)$; No y-intercept
h. x-int: (0, 0); y-int: (0, 0)

6. a. e.g., (2, 3) and (–4, 3); $m = \frac{3-3}{2-(-4)} = \frac{0}{6} = 0$
b. e.g., (4, 7) and (4, π); $m = \frac{7-\pi}{4-4} = \frac{7-\pi}{0} = \text{Undefined}$
c. $m = 0$ d. m = Undefined
e. m = Undefined f. $m = 0$

7. $y = 17$ is a horizontal line 17 units above the x-axis. Its y-intercept is (0, 17), but it has no x-intercepts; its slope is 0.

8. $x = -99$ is a vertical line 99 units to the left of the y-axis. Its x-intercept is (–99, 0), but it has no y-intercepts; its slope is undefined.

9. $y = 0$ **10**. horizontal; 0 **11**. $y = 11$ **12**. $y = 17$ **13**. $y = 7$
14. horizontal; 0 **15**. $y = -13$ **16**. $x = -4$ **17**. $y = 1$
18. vertical; undefined **19**. $x = -1$ **20**. $y = 0$ **21**. $x = 3$
22. vertical; undefined **23**. $x = -5$ **24**. $x = -18$ **25**. $y = -6$

26. a. parallel b. parallel c. perpendicular

27. a. vertical b. vertical c. horizontal d. horizontal

28. a. parallel b. parallel c. perpendicular

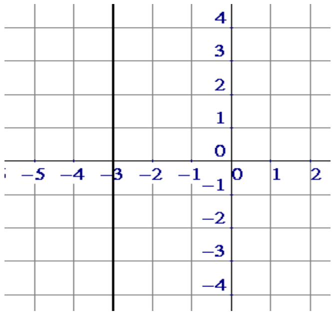
29. a. $x = 23$, for example; any line of the form x = *some number* would work.
b. $y = 9$, for example; any line of the form y = *some number* would work.

c. $x = -\pi$ for example; any line of the form $x =$ *some number* would work.
d. $y = -3$, for example; any line of the form $y =$ *some number* would work.

30. a. horizontal b. horizontal c. vertical d. vertical

31. a. $x = 1$ b. $y = 0$ c. $y = 8$ d. $x = 7$

32. a. For instance, (–3, 0), (–3, –2), and (–3, 4) are three points on the line. Plotting these points and connecting them produces the graph:

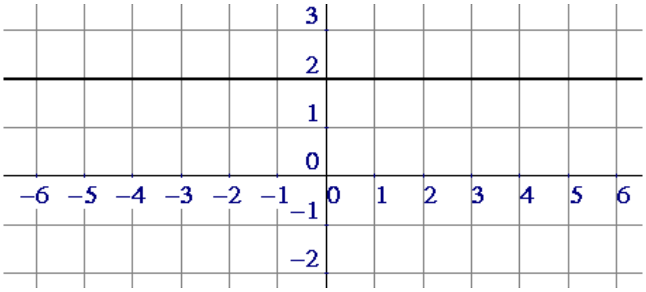


b. The line is vertical.
c. The only intercept of this line is (–3, 0).
d. Using the first two of the three points, we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{0-(-2)}{-3-(-3)} = \frac{0+2}{-3+3} = \frac{2}{0},$$

and therefore <u>the slope is undefined</u>.

33. a. For example, (–1, 2), (3, 2), and (4, 2) are three points on the line. Plotting these points and connecting them produces the graph:



b. The line is horizontal.
c. The only intercept of this line is (0, 2).

d. Using the first two of the three points, we find the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{2-2}{-1-3} = \frac{0}{-4} = 0$$

and therefore <u>the slope is 0</u>.

34. a. $y = 0$ b. $x = 0$

35. horizontal; 0

36. vertical; undefined

37. diagonal; 1

38. $y = 99$

39. $x = 34$

40. $x = 2$

41. $y = -\pi$

42. $x = 2$

43. $y = 7$

44. $x = -2$

45. $x = \pi$

46. a. T b. T c. T d. F e. T f. F g. T h. F i. T
j. T k. F l. T m. F n. F o. T p. F q. F r. T
s. T t. F u. T

"This thing we call 'failure' is not the falling down . . . but the staying down."

Mary Pickford

CH 9 – LINEAR MODELING

INTRODUCTION

Business people are constantly trying to determine their business future. What will interest rates be, what will be our expenses next year, how much could we make if we sell more widgets? All of these and thousands more are questions that may very well determine the success of a company. This chapter will look at problems which involve predicting temperatures, future revenues, and future insurance costs. In business, this predicting is called *projecting*.

CREATING LINEAR MODELS

EXAMPLE 1: **To join the Model Railroad Club, a member must pay an up-front fee of $25 to join, and then pay $10 per month for each month in the club.**

a. Find the total cost for someone to be a club member for 8 months.

b. Find a formula for someone to be a club member for *m* months.

c. Use your formula to calculate the total to be in the club for 36 months.

d. Use your formula to calculate the number of months Sam was in the club if he paid a total of $995.

Solution: Let's begin with a simple table that shows the cost of membership for various months:

months	0	1	2	3	4	5	6	7	8
dues	\$25	\$35	\$45	\$55	\$65	\$75	\$85	\$95	\$105

a. The cost, C, is \$25 plus 8 months at \$10 per month:

$$C = 25 + 10(8) = 25 + 80 = 105$$

Therefore, the cost of 8 months is **\$105**.

b. Just change the 8 in part a. to the variable m:

$$C = 25 + 10m, \text{ or } \mathbf{C = 10m + 25}$$

c. Using the formula in part b., we get a cost of

$$C = 10m + 25 = 10(\mathbf{36}) + 25 = 360 + 25 = \mathbf{\$385}$$

d. $C = 10m + 25 \Rightarrow 995 = 10m + 25 \Rightarrow 970 = 10m$
$\Rightarrow m = 97$. Thus, Sam's been a member for **97 months**.

Homework

1. a. It costs \$2 per mile to take a taxi. If m represents the total miles traveled, write a formula for the total cost, C.

   b. Use your formula to calculate the total cost of a 20-mile trip.

2. a. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. Calculate the total cost of a 21-mile taxi ride.

   b. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

   c. Use your new formula from part b. to calculate the total cost of a 15-mile trip.

d. Using the same formula, assume Joanna paid \$35 for a taxi to the airport. How many miles was the taxi ride?

3. a. It costs \$7 for the first mile of a taxi ride, and \$4 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

   b. Michael paid \$95 for a taxi to the airport. How many miles was the taxi ride?

4. More on the taxi ride:

   Let C = the total cost of the taxi ride

   f = the cost of the first mile

   a = the cost of each mile after the first

   m = the total miles traveled

   Write a formula for the total cost of a taxi ride.

5. a. QRS, Inc. sold 5000 widgets last month at a unit selling price of \$45/widget. Calculate the revenue obtained.

   b. STU, Inc. sold w widgets last month at a unit selling price of \$$p$/widget. Create a formula for the revenue (R) obtained.

6. a. WXY, Inc. produced 600 widgets last quarter, where it cost \$50/widget. In addition, fixed costs (rent, utilities, salaries, etc.) totaled \$2,400 last quarter. Calculate the total cost of producing the 600 widgets.

   b. Let w = the number of widgets produced

   c = unit cost to produce one widget

   f = fixed costs (rent, utilities, salaries, etc.)

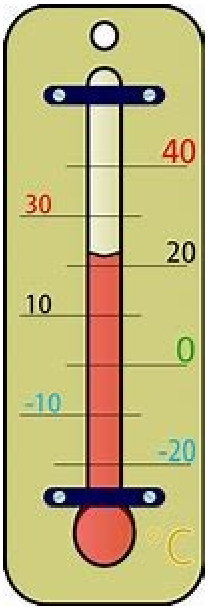
   E = total expense to produce the w widgets

   Create a formula for the total expense of producing w widgets.

7. Ernie earns a base salary of \$2,000/month and a commission of \$50 for each widget he sells. Find Ernie's total salary during a month in which he sold 75 widgets. Construct a formula which

would give Ernie's total salary, S, during a month if he sold w widgets.

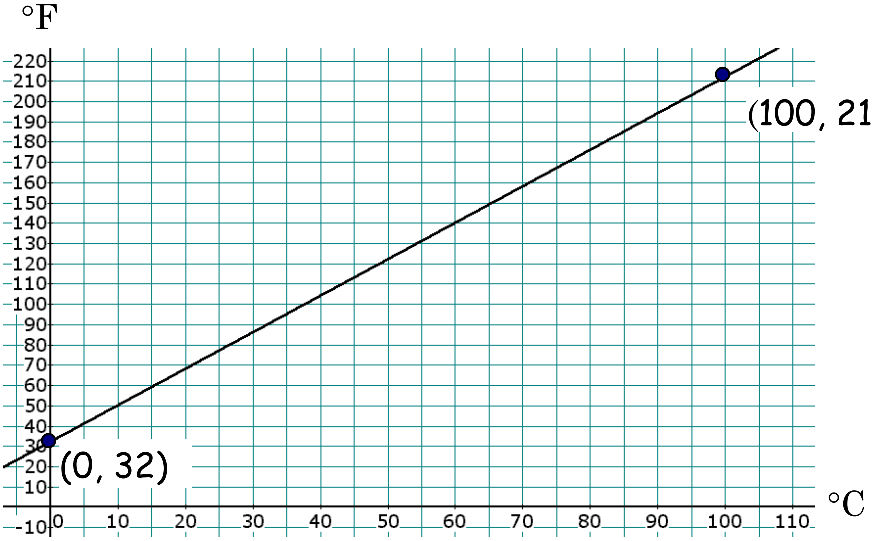
☐ TEMPERATURE SCALES



On the Fahrenheit temperature scale, water freezes at 32°F and boils at 212°F. Later, the Celsius (originally called centigrade) scale was created to make the numbers easier to work with. On this new scale, water freezes at 0°C and boils at 100°C. We conclude that 0°C = 32°F and 100°C = 212°F.

Our goal here is to create, from scratch, a formula which converts from degrees Celsius to degrees Fahrenheit. All we need to accomplish this goal are two facts: 0°C = 32°F, 100°C = 212°F, and one assumption: that the relationship is linear (the graph is a straight line).

If we label the horizontal axis °C and the vertical axis °F, then the temperature facts above translate into the following two points on the line: (0, 32) and (100, 212).



We begin with a linear formula relating °C to °F, where m represents the slope of the line, and b is the "y-intercept."

$F = mC + b$ (note that C is the x-value and F is the y-value)

First, we use the two points on the line to calculate the slope of the line:

$$m = \frac{\Delta F}{\Delta C} = \frac{212-32}{100-0} = \frac{180}{100} = \mathbf{1.8}$$

So now our equation is

$$F = 1.8C + b$$

Second, we use one of the given points -- we'll use (0, 32) -- to find b:

$$32 = 1.8(0) + b \Rightarrow 32 = 0 + b \Rightarrow 32 = b$$

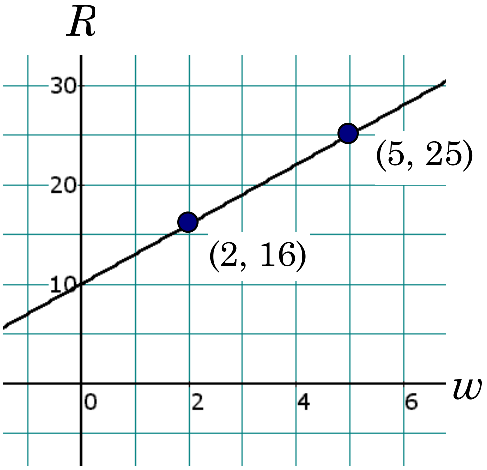
and we have our equation relating the two temperature scales:

$$\boxed{F = 1.8C + 32}$$

☐ *PREDICTING REVENUE*

EXAMPLE 2: **The revenue obtained when 2 widgets were sold was \$16, and the revenue obtained when 5 widgets were sold was \$25. Project (predict) the revenue if 20 widgets are sold.**

Solution: If the unit selling price is the same for each sale of widgets, then it would be an easy arithmetic problem requiring no algebra. Let's see: 2 widgets for \$16 is \$8 per widget, while 5 widgets for \$25 comes to \$5 per widget. They're not the same unit price, so we'll have to assume that some additional source of income is contributing to our revenue amounts. Here's a graph of the given information, where the horizontal w-axis is the number of widgets sold and the vertical R-axis is the revenue obtained:



Let's make the assumption that the appropriate graph through these two points is a straight line. This means we are assuming a linear model ($y = mx + b$ from Chapter 6); that is, we assume that the relationship between revenue and widgets sold is given by a linear formula

$$R = mw + b$$

where R = revenue, m is the slope, w is the number of widgets sold, and b is the R-intercept.

Note: There are an infinite number of possible graphs which pass through the two given points. We might as well assume the simplest one: a straight line.

First we find the slope of the line. Normally, our slope formula would be $m = \frac{\Delta y}{\Delta x}$, but we're not using x and y-axes; we're using w and R. So the slope is calculated using the two given points:

$$m = \frac{\Delta R}{\Delta w} = \frac{25-16}{5-2} = \frac{9}{3} = \mathbf{3}$$

The equation of the line at this point is then

$$R = 3w + b$$

To find b (the R-intercept), we pick either point on the line and place it into the previous equation; let's choose (2, 16):

$$16 = 3(\mathbf{2}) + b \Rightarrow 16 = 6 + b \Rightarrow b = 10$$

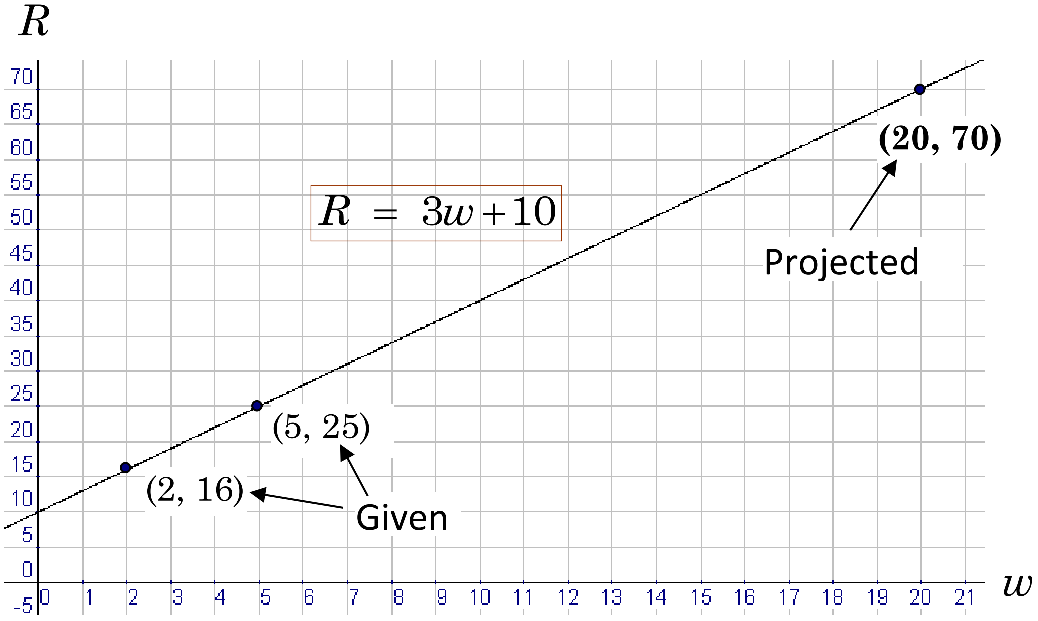
Our line equation is therefore

$$\underline{R = 3w + 10}$$

Now we can project the revenue for any number of widgets. If 20 widgets are sold, the revenue will be $R = 3(\mathbf{20}) + 10 = 60 + 10 =$

$$\boxed{\$70.00}$$

The following graph is a summary of this entire problem.



Homework

8. The revenue obtained when 10 widgets were sold was \$155, and the revenue obtained when 13 widgets were sold was \$176. Project the revenue if 100 widgets are sold.

9. When 7 widgets were sold, \$152 in revenue was obtained, and when 13 widgets were sold, the revenue was \$218. If 85 widgets are sold, project the revenue.

10. The revenue obtained when 100 widgets were sold was \$2,200.75, and the revenue obtained when 120 widgets were sold was \$2,470.75. Project the revenue if 250 widgets are sold.

11. The revenue obtained when 20 widgets were sold was \$360, and the revenue obtained when 36 widgets were sold was \$568. Project the revenue if 100 widgets are sold.

12. When 10 widgets were sold, \$575 in revenue was obtained, and when 21 widgets were sold, the revenue was \$1,015. If 50 widgets are sold, project the revenue.

13. When 50 widgets were sold, \$3000 in revenue was obtained, and when 51 widgets were sold, the revenue was \$3,052. If 200 widgets are sold, project the revenue.

❑ *PREDICTING INSURANCE COSTS*

Management is trying to project the cost of medical insurance premiums for the employees of the company.

EXAMPLE 3: **In 2018, there were 23 employees and the total medical insurance premiums were \$22,050. In 2019, the company had 40 employees and the premiums were \$36,500. Project the medical insurance premiums for the year 2020, when there should be 85 employees.**

Solution: A logical first guess is that the amount paid each year per employee is the same. If this is the case, we'll finish this problem quickly.

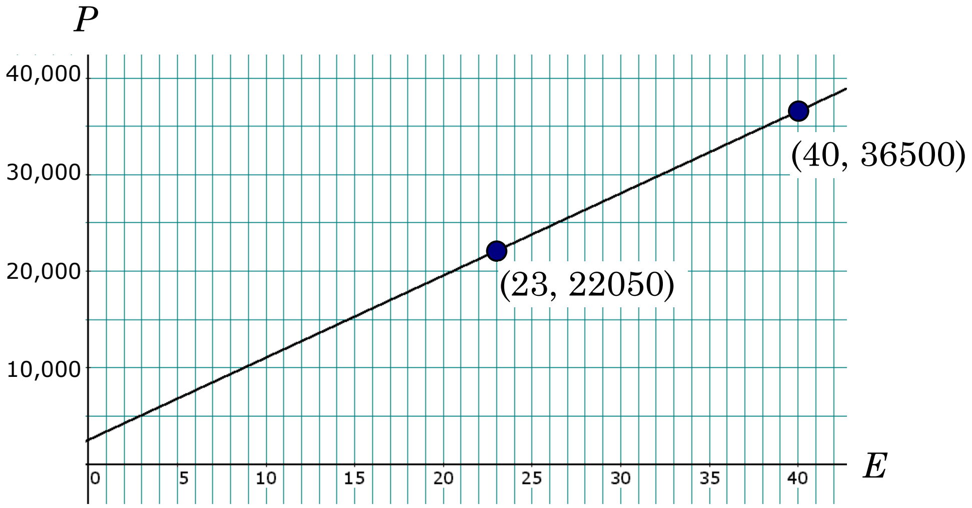
2018: The cost per employee was $\frac{\$22,050}{23 \text{ emp}} = \$958.70/\text{employee}$

2019: The cost per employee was $\frac{\$36,500}{40 \text{ emp}} = \$912.50/\text{employee}$

No such luck -- whatever formula relates the number of employees to the total of medical insurance premiums is more complicated than this (just like the previous example with revenue). So we'll assume the linear formula

$$P = mE + b \qquad \text{(same kind of formula as revenue)}$$

where P = medical premiums, E = number of employees, and m and b are to be determined.



First find the slope of the line connecting the points (23, 22050) and (40, 36500):

$$m = \frac{\Delta P}{\Delta E} = \frac{36500 - 22050}{40 - 23} = \frac{14450}{17} = 850$$

The equation of the line at this point is then

$$P = 850E + b$$

To find b (the P-intercept), we pick either point on the line and place it into the previous equation; let's choose (23, 22050):

$$22050 = 850(\mathbf{23}) + b \Rightarrow 22050 = 19550 + b \Rightarrow b = 2500$$

Our line equation is therefore

$$\underline{P = 850E + 2{,}500}$$

Finally we get to the question at hand -- the projection for the year 2019. Placing $E = 85$ employees into the formula gives

$$P = 850(\mathbf{85}) + 2{,}500 = 72{,}250 + 2{,}500 = 74{,}750$$

Thus, the projected medical insurance premiums for 2020 is

$$\boxed{\$74{,}750}$$

Homework

14. Last year there were 85 employees and the total medical insurance premiums were \$6,525. This year the company has 120 employees and the premiums were \$8,800. Project the medical insurance premiums for next year, when there should be 190 employees.

15. Last year there were 53 employees and the total medical insurance premiums were \$45,710. This year the company has 100 employees and the premiums were \$81,900. Project the medical insurance premiums for next year, when there should be 150 employees.

16. Last year there were 132 employees and the total medical insurance premiums were \$40,204. This year the company has 101 employees and the premiums were \$30,997. Project the medical insurance premiums for next year, when there should be 125 employees.

17. Last year there were 180 employees and the total medical insurance premiums were \$37,200. This year the company has 190 employees and the premiums were \$39,100. Project the medical insurance premiums for next year, when there should be 225 employees.

18. Last year there were 90 employees and the total medical insurance premiums were \$23,300. This year the company has 100 employees and the premiums were \$25,800. Project the medical insurance premiums for next year, when there should be 125 employees.

Practice Problems

19. a. Suppose that the DVC Library currently has 7000 books, and is buying 150 more books per year. How many books will the library have 4 years from now?
    b. Write a formula that will give the number of books the library will have y years from now.
    c. Use your formula to calculate the number of years it will take for the library collection to reach 10,000 books.

20. You must pay \$75 up front to join the Painting Club, and then pay \$35 per month for each month you're a member of the club. Create a formula that will give the total cost, C, to be in the club for m months.

21. The revenue obtained when 2 widgets were sold was \$16, and the revenue obtained when 5 widgets were sold was \$28. Project (predict) the revenue if 50 widgets are sold. Assume the linear model $R = mw + b$, where w = widgets sold and R = revenue.

22. Last year there were 83 employees and the total medical insurance premiums were \$25,599. This year the company has 200 employees and the premiums were \$61,050. Project the medical insurance premiums for next year, when there should be 300 employees. Assume the linear formula $P = mE + b$, where E = number of employees and P = medical premiums.

Solutions

1. a. $C = 2m$ b. $C = 2m = 2(20) =$ **$40**

2. a. $3 + 2(20) = 3 + 40 =$ **$43**
b. $C = 3 + 2(m - 1)$, or $\boldsymbol{C = 2(m - 1) + 3}$
c. $C = 2(m - 1) + 3 = 2(15 - 1) + 3 = 2(14) + 3 = 28 + 3 =$ **$31**
d. $C = \$35$, so we have to solve the equation $35 = 2(m - 1) + 3$ for m.
17 miles

3. a. $C = 4(m - 1) + 7$ b. 23 miles

4. $C = f + a(m - 1)$

5. a. $225,000 b. $R = pw$

6. a. $600(50) + 2,400 =$ **$32,400**
b. $E = cw + f$

7. $2,000 + $50(75) = **$5,750**
In general, $S = 2,000 + 50(w)$, or $\boldsymbol{S = 50w + 2,000}$

8. $785.00 **9**. $1,010.00 **10**. $4,225.75

11. $1,400 **12**. $2,175 **13**. $10,800

14. $13,350.00 **15**. $120,400 **16**. $38,125

17. $45,750 **18**. $32,050

19. a. $7,000 + 4(150) =$ <u>7,600 books</u>
b. $B = 150y + 7,000$
c. $10,000 = 150y + 7,000 \Rightarrow y =$ <u>20 years</u>

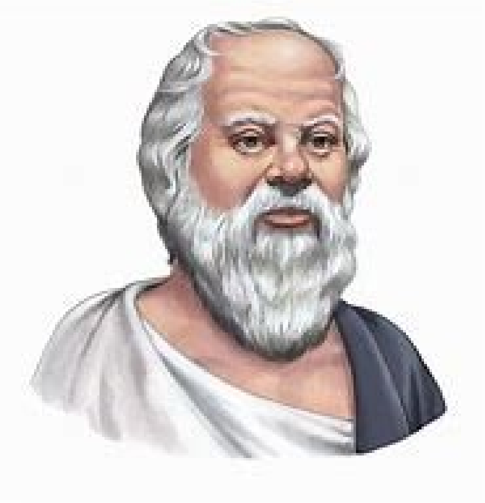
20. The cost would be \$75 plus \$35 for each month in the club.
That's \$75 + \$35 × number of months in the club.
That's $C = 75 + 35m$, which should be written $\underline{C = 35m + 75}$

21. Find the equation of the line connecting (2, 16) and (5, 28), using w and R instead of x and y. You should get $R = 4w + 8$. Now project the revenue if 50 widgets are sold: <u>208 widgets</u>

22. \$91,350

“The only good is knowledge and the only evil ignorance.”

– Σοχρατεσ (Socrates)



CH 10 – MAXIMIZING COMPANY PROFIT

❑ INTRODUCTION

The only good profit is the best profit. Selling just a few widgets clearly does not maximize our profit, but selling too many widgets may also not result in our best profit, due to increased costs. In other words, sometimes we seek an optimal number of widgets to sell. Selling either more or less than this number will result in reduced profit.

❑ ALGEBRA REVIEW

Double Inequalities

Recall that "$x < 8$" means that x is <u>less than</u> 8, and the phrase "$y > 17$" means that y is <u>greater than</u> 17.

If we want to say that x is greater than 9 but less than 15, we write

$$9 < x < 15$$

This can also be read "x is between 9 and 15, but not equal to either of them." If we assume that x is a whole number, then $9 < x < 15$ means that x must be one of the numbers 10, 11, 12, 13, or 14. But if there's no restriction on x (that is, if x can be any real number), then there are infinitely many choices for x, including numbers like 9.001, $10\frac{1}{4}$, 13.83, $14\frac{101}{102}$, and even irrational numbers like 3π and $\sqrt{101}$.

Homework

1. Consider the double inequality $12 < a < 20$, where a is a real number. Which of the following numbers are possible values of a?

11.9 12 12.001 19.9 20 4π

$\sqrt{410}$ $\sqrt{300}$ $\sqrt{140}$ 7π $10\frac{7}{8}$ 0

2. Use the appropriate inequality signs to rewrite each statement:

a. n is less than 7 b. z is larger than 0

c. a is smaller than 12 d. c is greater than -4

e. x is between 5 and 7, and is equal to neither of them

f. y is between -3 and 3, and is equal to neither of them

3. a. $13^2 =$ b. $(-9)^2 =$ c. $-3^2 =$ d. $-10^2 =$

e. $(-12)^2 =$ f. $-12^2 =$ g. $(-7)^2 =$ h. $-7^2 =$

4. a. $(-5)^2 - 2(3) + 1 =$ b. $-5^2 + 2(-3) - 1 =$

c. $-6^2 - 5(-6) + 3 =$ d. $-(-4)^2 - 3(-1) + 10 =$

e. $-3^2 - 2(3) - 3 =$ f. $-(-1)^2 + 4(7) + 1 =$

g. $-10^2 - 4(-3) - (-6) =$ h. $-(-2)^2 - 3(4) + 12 =$

i. $(-7)^2 - 7(-7) + 2(-1) =$ j. $4^2 -(9)(-1) + (-3) =$

5. Evaluate the given expression for the given value of x:

a. $x^2 + 7x + 1$ $x = 5$ b. $2x^2 - x - 3$ $x = -3$

c. $3x^2 - 8x$ $x = 0$ d. $9x^2 + 7$ $x = 0$

e. $-2x^2 + 3x - 4$ $x = -3$ f. $-3x^2 - 2x - 1$ $x = -2$

g. $-x^2 + 2x - 1$	$x = 4$	h. $-x^2 - 4x + 5$	$x = -10$
i. $-4x^2 - x + 3$	$x = -5$	j. $-x^2 + 7x - 12$	$x = 13$

☐ PROFIT AND BREAK-EVEN POINTS REVISITED

EXAMPLE 1: **If the revenue is given by the formula $R = 2w^2 - 6w + 9$ and the expense formula is given by $E = w^2 + 7w - 10$, find the profit formula in simplest form.**

Solution: We have our basic formula for profit which says that profit is the difference between revenue and expenses:

$$P = R - E \qquad \text{(the profit formula)}$$

$$\Rightarrow \quad P = (2w^2 - 6w + 9) - (w^2 + 7w - 10) \qquad \text{(given formulas)}$$

$$\Rightarrow \quad P = 1(2w^2 - 6w + 9) - 1(w^2 + 7w - 10) \qquad \text{(this step is \textit{optional})}$$

$$\Rightarrow \quad P = 2w^2 - 6w + 9 - w^2 - 7w + 10 \qquad \text{(distribute)}$$

$$\Rightarrow \quad \boxed{P = w^2 - 13w + 19} \qquad \text{(combine like terms)}$$

Homework

6. For the given revenue and expense formulas, calculate the **profit** formula in simplest form:

a. $R = 3w^2 - 6w + 1$ $\quad E = 2w^2 + 6w - 31$

b. $R = 5w^2 + 8w - 10$ $\quad E = 4w^2 + 8w + 90$

c. $R = 8w^2 - 55$ $\quad E = 6w^2 + 22w - 1$

d. $R = 2w^2 + 3w + 3$ $\quad E = w^2 + 2w$

EXAMPLE 2: **Consider the profit formula $P = w^2 - 9$. Graph this formula and determine the break-even point.**

Solution: Let's calculate the values of the profit formula for a few values of *w*. We'll include some negative values of *w* so that we get a better picture of the graph, but note that a break-even point can never be negative, since we can't actually produce a negative number of widgets.

$w = 0 \Rightarrow P = \mathbf{0}^2 - 9 = -9$ (a loss)

$w = 1 \Rightarrow P = \mathbf{1}^2 - 9 = 1 - 9 = -8$ (a loss)

$w = 3 \Rightarrow P = \mathbf{3}^2 - 9 = 9 - 9 = 0$ (a *break-even point*)

$w = 4 \Rightarrow P = \mathbf{4}^2 - 9 = 16 - 9 = 7$ (a real profit!)

$w = -2 \Rightarrow P = (\mathbf{-2})^2 - 9 = 4 - 9 = -5$

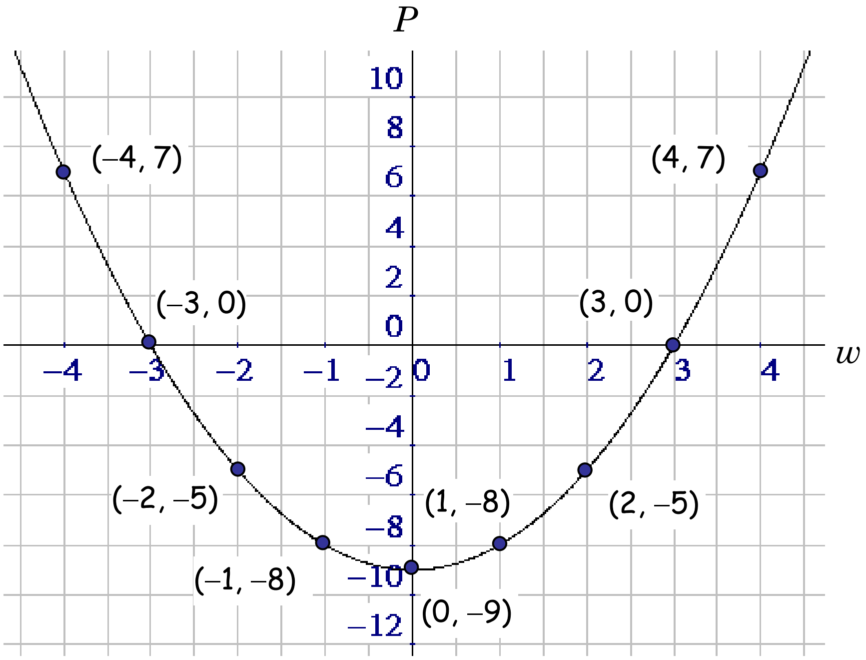
$w = -4 \Rightarrow P = (\mathbf{-4})^2 - 9 = 16 - 9 = 7$

These calculations, along with a few others for you to confirm, give us the following table:

w	–4	–3	–2	–1	0	1	2	**3**	4
P	7	0	–5	–8	–9	–8	–5	**0**	7

The break-even point is 3 widgets, which produces a profit of $0.

We can make some deductions now. (Remember, we're looking at positive values of *w* only.) If we sell 2 or fewer widgets, we incur a loss. At 3 widgets our profit is 0, and thus $w = 3$ is the break-even point. Four widgets and up, we make a profit. Now it's time for our graph. By plotting some of the points in the table above, we get our picture:



$$P = w^2 - 9$$

Notice that this graph is not a line. The exponent of 2 in the formula makes it a curvy graph called a ***parabola*** (puh-RAB-oh-luh).

What can we tell from the right-half of the graph? When $w < 3$, we're running a loss; when $w = 3$, the profit is zero, and so this is the break-even point; and when $w > 3$, the profit is positive, and we're finally making money. It also appears from our graph that as w grows larger, so does our profit.

EXAMPLE 3: **Graph the parabola $y = x^2 - 4$.**

a) Where does the graph cross the x-axis?

b) Where does the graph cross the y-axis?

c) Find the lowest point on the parabola.

Solution: Here are a few calculations:

$x = -3 \;\Rightarrow\; y = (\mathbf{-3})^2 - 4 = 9 - 4 = \underline{5} \;\Rightarrow\; (-3, 5)$

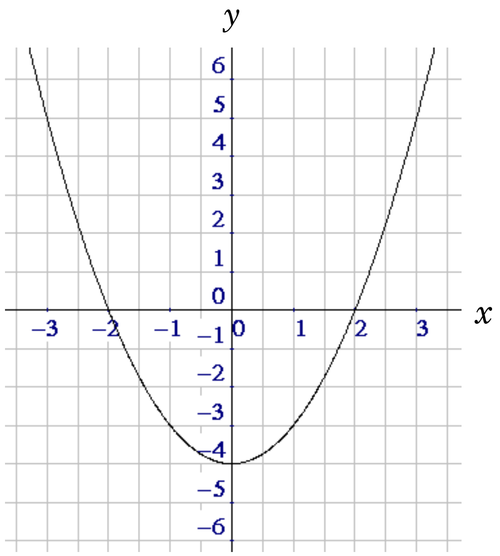
$x = -2 \;\Rightarrow\; y = (\mathbf{-2})^2 - 4 = 4 - 4 = \underline{0} \;\Rightarrow\; (-2, 0)$

$x = 0 \;\Rightarrow\; y = \mathbf{0}^2 - 4 = 0 - 4 = \underline{-4} \;\Rightarrow\; (0, -4)$

$x = 3 \;\Rightarrow\; y = \mathbf{3}^2 - 4 = 9 - 4 = \underline{5} \;\Rightarrow\; (3, 5)$

You should now check all the values in the following table:

x	–3	–2	–1	0	1	2	3
y	5	0	–3	–4	–3	0	5



a) The graph crosses the *x*-axis at the points (2, 0) and (–2, 0).

b) The graph crosses the *y*-axis at the point (0, –4).

c) The lowest point on the graph is the point (0, –4).

<u>EXAMPLE 4:</u> **Graph the parabola $z = -u^2 - 4u + 5$.**

a) Where does the graph cross the *u*-axis?

b) Where does the graph cross the *z*-axis?

c) Find the highest point on the parabola.

<u>Solution:</u> There's no way to tell for sure which variable, *u* or *z*, goes on which axis. But the given formula indicates that the value of *z* depends on the value of *u*, and this means that *u* should go on the horizontal axis and *z* on the vertical axis. Here are some calculations. As before, you should verify these calculations and those in the table which follows.

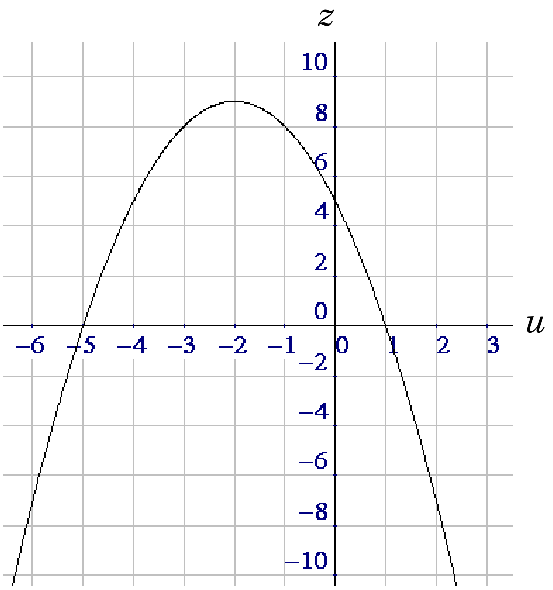
$u = -5 \Rightarrow z = -(\mathbf{-5})^2 - 4(\mathbf{-5}) + 5 = -25 + 20 + 5 = \underline{0}$

which gives the point (–5, 0).

$u = 2 \Rightarrow z = -(\mathbf{2})^2 - 4(\mathbf{2}) + 5 = -4 - 8 + 5 = \underline{-7}$

which gives the point (2, –7).

u	–6	–5	–4	–3	–2	–1	0	1	2
z	–7	0	5	8	9	8	5	0	–7



a) The graph crosses the *u*-axis at the points (–5, 0) and (1, 0).

b) The graph crosses the *z*-axis at the point (0, 5).

c) The highest point on the graph is the point (–2, 9).

EXAMPLE 5: **Consider the profit formula**

$$\boldsymbol{P = -w^2 + 10w - 21}$$

Determine the two break-even points and also determine the selling level (the value of *w*) which will produce the maximum (highest) profit.

Solution: We start with a table and try to answer the questions. Then in the homework you'll graph the formula and try to answer the questions again.

$$w = 1 \;\Rightarrow\; P = -(\mathbf{1})^2 + 10(\mathbf{1}) - 21 = -1 + 10 - 21 = \underline{-12}$$

$$w = 3 \;\Rightarrow\; P = -(\mathbf{3})^2 + 10(\mathbf{3}) - 21 = -9 + 30 - 21 = \underline{0}$$

$$w = 6 \;\Rightarrow\; P = -(\mathbf{6})^2 + 10(\mathbf{6}) - 21 = -36 + 60 - 21 = \underline{3}$$

You do the calculations to fill in the rest of the table:

w	1	2	3	4	5	6	7	8
P	–12	–5	0	3	4	3	0	–5

Since a break-even point occurs when the profit is zero, we see that there are actually two break-even points, $w = 3$ and $w = 7$. We can also see from the table that we have a loss when $w < 3$, we have a profit when $3 < w < 7$, and we incur a loss if $w > 7$.

How can selling more than 7 widgets cause a loss in the business? Perhaps that much selling requires an extra shift at the factory to make the widgets, or perhaps we need a new warehouse to store all the widgets. Whatever the reason, many businesses have gone under trying to pursue that "one extra dollar."

Now for the second part of the question. Finding the value of w which produces the maximum profit means we should look for the highest number in the second row of the table, the profit row. Clearly, the maximum profit is $P = 4$, and this occurs when $w = 5$.

Thus, we say that

> The two break-even points are $w = 3$ and $w = 7$. And, a selling level of 5 widgets produces the maximum profit of $4.

Homework

7. For each profit formula, use a table to find the **break-even points**.

   a. $P = w^2 - 16$
   b. $P = -w^2 + 4$
   c. $P = 25 - w^2$
   d. $P = w^2 - 8w + 12$
   e. $P = -w^2 + 6w - 5$
   f. $P = 2w^2 - 5w - 3$

8. Graph the profit formula in Example 5, and verify the final results of the example. By the way, the very highest (or lowest) point on a parabola is called its ***vertex***.

9. For each profit formula, use a table -- but verify with a graph -- to find the value of *w* which will produce the **maximum profit**:

   a. $P = -w^2 + 6w - 5$
   b. $P = -w^2 + 10w$
   c. $P = -w^2 + 8w + 7$
   d. $P = -2w^2 + 8w + 1$

10. Graph each parabola:

a. $y = x^2 + 2x$

b. $y = x^2 - 4x$

c. $y = x^2 - 1$

d. $y = -x^2 + 4$

e. $y = x^2 + 6x + 9$

f. $y = x^2 - 4x + 4$

g. $y = -x^2 + 10x - 25$

h. $y = -x^2 + 6x - 5$

Practice Problems

11. If the revenue is given by the formula $R = 2w^2 - 6w + 9$ and the expense formula is given by $E = w^2 + 7w - 10$, find the profit formula in simplest form.

12. Consider the profit formula $P = -w^2 + 8w - 12$.

a. Graph the profit formula.
b. Find the two break-even points.
c. Find the number of widgets which will produce the maximum profit.
d. What is the maximum profit?

13. Graph the parabola $y = x^2 + 4x + 4$. Find all the intercepts and the vertex.

14. If $P = w^2 - 12w + 20$ is the profit formula, find the break-even points.

15. Graph the parabola $y = 0.2x^2 - 7$. Use the graph to estimate the x-intercepts.

Solutions

1. 12.001   19.9 4π $\sqrt{300}$

2. a. $n < 7$ b. $z > 0$ c. $a < 12$
d. $c > -4$ e. $5 < x < 7$ f. $-3 < y < 3$

3. a. 169   b. 81   c. –9   d. –100
e. 144   f. –144   g. 49   h. –49

4. a. 20   b. –32   c. –3   d. –3   e. –18   f. 28
g. –82   h. –4   i. 96   j. 22

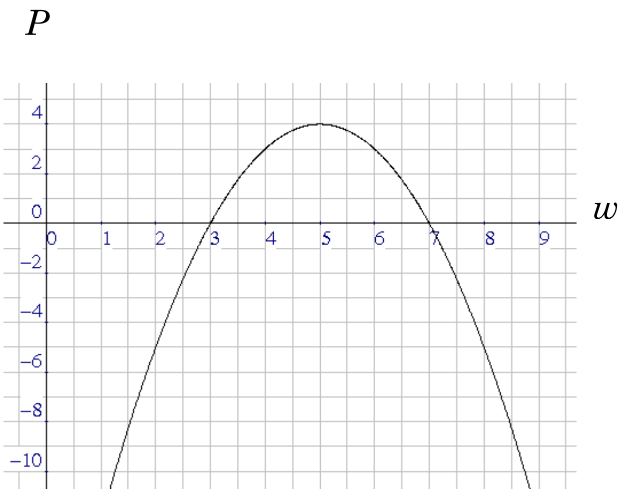
5. a. 61   b. 18   c. 0   d. 7   e. –31   f. –9
g. –9   h. –55   i. –92   j. –90

6. a. $P = R - E = (3w^2 - 6w + 1) - (2w^2 + 6w - 31)$
$= 3w^2 - 6w + 1 - 2w^2 - 6w + 31 = w^2 - 12w + 32$

b. $P = w^2 - 100$ c. $P = 2w^2 - 22w - 54$ d. $P = w^2 + w + 3$

7. a. $w = 4$ b. $w = 2$ c. $w = 5$
d. $w = 2$ and $w = 6$ e. $w = 1$ and $w = 5$ f. $w = 3$

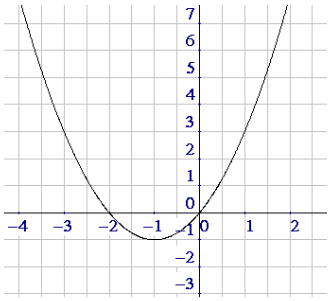
8.



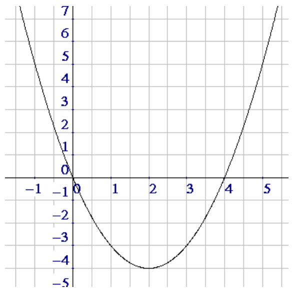
The break-even points are the w values where the profit is zero. This occurs on the w-axis, where the graph crosses at (3, 0) and (7, 0). Thus, the break-even points are $w = 3$ and $w = 7$. As for $w = 5$ being the optimal number of widgets to sell, this is indicated by the fact that the vertex of the parabola is at (5, 4). This means that 5 widgets will produce the most profit, namely $4.

9. a. $w = 3$ b. $w = 5$ c. $w = 4$ d. $w = 2$

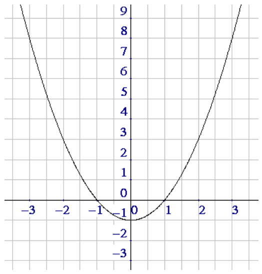
10. a.



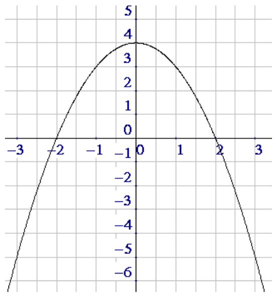
b.



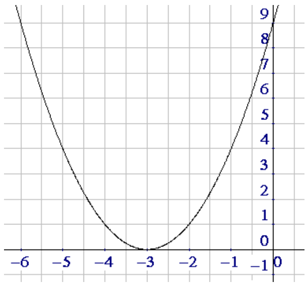
c.



d.



e.



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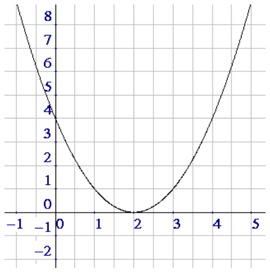
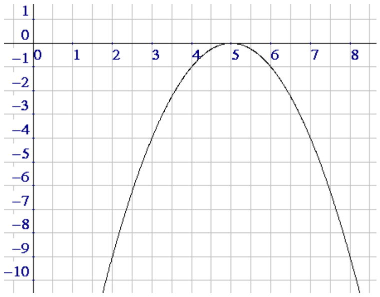
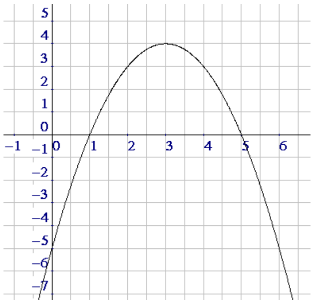


g.

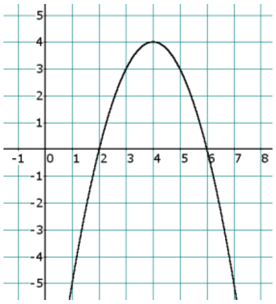


h.



11. $P = w^2 - 13w + 19$

12. a.



b. $w = 2$, $w = 6$

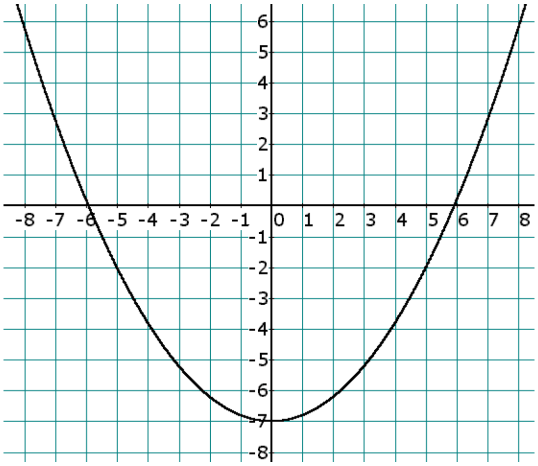
c. $w = 4$

d. $4

13. x: (–2, 0) y: (0, 4) V(–2, 0)

14. $w = 2$, $w = 10$

15.



The x-intercepts are approximately (5.9, 0) and (–5.9, 0)

“It is in fact a part of
the function of education
to help us escape, not
from our own time – for
we are bound by that –
but from the intellectual
and emotional limitations
of our time.”

– T.S. Eliot

CH 11 – POLYNOMIALS

INTRODUCTION

It's very difficult to define what a **polynomial** is at this point in your algebra studies, because we haven't come across many things that aren't polynomials. Suffice it to say that a typical polynomial looks like

$$3x^5 - \pi x^3 + x^2 - 9x + \frac{4}{5}$$

The main theme of a polynomial is that all of the exponents on the x (or whatever variable) must be one of the whole numbers 0, 1, 2, 3, ... The following are <u>not</u> polynomials: $8x^{-2}$ and $3x^{1/2}$, because the exponents –2 and 1/2 are not whole numbers.

WORKING WITH POLYNOMIALS

A polynomial with one term is called a ***monomial***. The expressions $7n$ and $10x^2$ are monomials. The key to multiplying monomials is that each monomial is a single term whose <u>final operation is multiplication</u>.

For example, to find the product $(7x)(9x)$, we proceed the long way -- you don't ever have to do it this way, but it's important to see.

$$\begin{aligned}
& (7x)(9x) && \text{(the original expression)} \\
= \quad & 7 \cdot 9 \times x \cdot x && \text{(it's all multiplication)} \\
= \quad & (7 \cdot 9) \times (x \cdot x) && \text{(regroup the factors)} \\
= \quad & 63 \times x^2 && \text{(something times itself is squaring)} \\
= \quad & 63x^2 && \text{(no need for the multiplication sign)}
\end{aligned}$$

Another example is $3(-10n) = (3 \cdot -10)n = -30n$.

But don't forget that adding and subtracting don't follow the same rules as multiplication. Two monomials can be added or subtracted only if they're **like terms**. See if the homework sorts all of this out.

Homework

1. Simplify each expression:

a. $3(7L)$
b. $-5(2x)$
c. $-6(-2T)$
d. $20(-3w)$
e. $3 + 7L$
f. $-5 + 2x$
g. $-6 - 2T$
h. $20 - 3w$
i. $(7y)5$
j. $(-2p)(-5)$
k. $(-3a)(10)$
l. $(5n)(-2)$
m. $7y + 5$
n. $(4x)(3x)$
o. $4x + 3x$
p. $(2n)(-3n)$
q. $2n - 3n$
r. $(-8x)(-7x)$
s. $(7u)(-u)$
t. $(-4c)(4c)$
u. $-4c + 4c$
v. $(7m)(6n)$
w. $7m - 6n$
x. $(13k)(-13k)$
y. $13k - 13k$
z. $-14x + 20x$

Homework

2. Suppose a friend believed that $4n^2$ and $7n$ were like terms, and that their sum should be $11n^3$. Prove your friend wrong by letting $n = 2$, and then showing that

$$4n^2 + 7n \neq 11n^3$$

3. Simplify each expression by combining like terms:

a. $3x^2 - 7x + 5x^2 + 9$
b. $n^2 - 9 + 9 - n^2$
c. $1 - 3u - u^2 - 3u^2 + 7u - 1$
d. $7a^2 - 8a + 7 - 9a^2 + 7a - 7$

e. $x^2 - 3x - 1 + 7x^2 - 3x + 1$ f. $3y^2 - 2 + 3y^2 - 2$

g. $1 - 3x - x^2 + 5 - 7x + x^2$ h. $-5w^2 + 2 - 3w + 8w - 2 - w^2$

4. Simplify each expression by distributing and then combining like terms:

   a. $(3c^2 - 2c - 1) + 2(c^2 + 5c - 7)$
   b. $3(x^2 - 8x + 1) - 5(2x^2 + 7x - 1)$
   c. $-(a^2 - a - 1) + 3(-a^2 + a)$
   d. $7w^2 - 13w + 8 - (5w^2 - 3w - 2)$
   e. $-(7u^2 - 7u - 6) - (-6u^2 + 3u + 5)$
   f. $(3x^2 - x - 1) - (3x^2 - x - 1)$
   g. $-2(x^2 - 3x + 7) - (3x^2 + 10x - 1)$
   h. $-(3n^2 + 8n - 1) - 3(n^2 + 2n - 1)$

❑ *THE DOUBLE DISTRIBUTIVE PROPERTY*

As stated before, a polynomial with one term is called a ***monomial***; just as a bicycle has two wheels, a polynomial with two terms is called a ***binomial***. A problem where we must multiply a monomial by a binomial is the following:

$3x(2x + 10)$. ($3x$ is the monomial and $2x + 10$ is the binomial)

Finding the product of these two polynomials is pretty easy -- just distribute the $3x$ to the $2x$ and to the 10:

$(3x)(2x) + 3x(10)$

$= 6x^2 + 30x$, and it's done.

What we need now is a way to multiply two binomials together. For example, how do we simplify the product $(x + 7)(x + 5)$? The **Double Distributive Property** says, in a nutshell,

Multiply each term in the first binomial
by each term in the second binomial.

EXAMPLE 2: **Multiply out (simplify): $(x + 7)(x + 5)$**

Solution: Multiply each term in the first binomial by each term in the second binomial:

i) Multiply the first x by the second x: x^2
ii) Multiply the first x by the 5: $5x$
iii) Multiply the 7 by the second x: $7x$
iv) Multiply the 7 by the 5: 35

Add the four terms together: $x^2 + 5x + 7x + 35$, and then combine like terms

$$\boxed{x^2 + 12x + 35}$$

EXAMPLE 3: **Simplify the given expression:**

A. $(2n + 1)(n - 8)$

$= 2n^2 - 16n + n - 8$ (double distribute)

$= \mathbf{2n^2 - 15n - 8}$ (combine like terms)

B. $(7a - 3)(4a - 5)$

$= 28a^2 - 35a - 12a + 15$ (double distribute)

$= \mathbf{28a^2 - 47a + 15}$ (combine like terms)

C. $(6k - 7)(6k + 7)$

$= 36k^2 + 42k - 42k - 49$ (double distribute)

$= \mathbf{36k^2 - 49}$ (combine like terms)

D. $(10 + y)(10 - y)$

$= 100 - 10y + 10y - y^2$ (double distribute)

$= \mathbf{100 - y^2}$ (combine like terms)

E. $(2x + 9)^2$ The square of a quantity is the product of the quantity with itself:

$(2x + 9)^2$

$= (2x + 9)(2x + 9)$ (since $N^2 = N \cdot N$)

$= 4x^2 + 18x + 18x + 81$ (double distribute)

$= \mathbf{4x^2 + 36x + 81}$ (combine like terms)

EXAMPLE 4: **Simplify: $(2x + 1)(x - 5) - (x - 4)^2$**

Solution: The Order of Operations tells us to multiply and square first, and subtract last:

$(2x + 1)(x - 5) - (x - 4)^2$

$= (2x^2 - 10x + x - 5) - \mathbf{(}x^2 - 4x - 4x + 16\mathbf{)}$ (multiply and square)

[Notice how parentheses still enclose the result of the squaring.]

$= (2x^2 - 9x - 5) - (x^2 - 8x + 16)$ (combine like terms)

$= 2x^2 - 9x - 5 - x^2 + 8x - 16$ (distribute the –1)

$= \boxed{x^2 - x - 21}$ (combine like terms)

Homework

5. Simplify each expression by double distributing:

a. $(x + y)(w + z)$
b. $(c + d)(a - b)$
c. $(x + 2)(y + 3)$
d. $(x + 3)(x + 4)$
e. $(n - 4)(n - 1)$
f. $(a + 3)(a - 7)$
g. $(y + 9)(y - 9)$
h. $(u - 3)(u + 3)$
i. $(t - 20)(t - 19)$
j. $(z + 3)(z + 3)$
k. $(v - 4)(v - 4)$
l. $(N + 1)(N - 1)$

6. Simplify each expression by double distributing:

a. $(3a + 7)(a - 9)$
b. $(2n - 3)(n + 4)$
c. $(3n - 8)(n - 1)$
d. $(5x + 7)(5x + 6)$
e. $(7w + 2)(7w - 2)$
f. $(x + 12)(x - 12)$
g. $(2y + 1)(2y + 1)$
h. $(7x + 3)(6x - 7)$
i. $(q + 7)(3q - 7)$
j. $(3n + 1)(3n + 1)$
k. $(3x - 7)(6x + 5)$
l. $(u - 7)(u - 7)$

7. Square and simplify each expression:

a. $(y + 4)^2$
b. $(z - 9)^2$
c. $(3x + 5)^2$
d. $(2a - 1)^2$
e. $(n + 12)^2$
f. $(6t - 7)^2$
g. $(q - 15)^2$
h. $(5b + 3)^2$
i. $(7u - 1)^2$
j. $(2x + 1)^2$
k. $(3h - 12)^2$
l. $(5y - 5)^2$

8. Simplify each expression:

a. $(a + b)(c - d)$
b. $(2x - 3)(2x + 3)$
c. $(3n - 1)^2$
d. $(3t + 1)(2t - 3)$
e. $(2x + 4)(3x - 6)$
f. $(n + 1)(n - 1)$
g. $(7a - 10)(6a - 10)$
h. $(10c + 7)^2$
i. $(L + 4)^2$
j. $(7x - 3)(3x + 7)$
k. $(13n - 7)(13n + 7)$
l. $(12d - 20)^2$

9. Simplify each expression:

   a. $(2n + 1)(n + 1) + (n - 1)(n + 1)$

   b. $(x + 1)^2 + (x + 2)^2$

   c. $(3a + 2)(a - 1) - (a + 1)(a + 2)$

   d. $(4w + 1)^2 - (w - 1)(w - 3)$

   e. $(y + 2)(y - 3) - (2y - 1)^2$

   f. $(2y + 1)^2 - (2y - 1)^2$

10. Prove that $(a + b)^2 \neq a^2 + b^2$ in two ways:
    i) Plug in numbers.
    ii) Simplify $(a + b)^2$ the correct way.

11. Use numbers to prove that $(x + y)^3 \neq x^3 + y^3$

☐ *TRINOMIALS*

Just as a trio consists of three musicians, a ***trinomial*** is a polynomial consisting of three terms. Here are a couple of problems where we subtract some trinomials and multiply with a trinomial.

<u>EXAMPLE 5:</u> **Simplify each expression:**

A. $(2x^2 - x + 1) - (x^2 - 7x + 2)$ (difference of 2 **trinomials**)

$= 2x^2 - x + 1 - x^2 + 7x - 2$ (distribute the minus sign)

$= 2x^2 - x^2 - x + 7x + 1 - 2$ (rearrange the terms)

$= \mathbf{x^2 + 6x - 1}$ (combine like terms)

B. $(a - 3)(a^2 + 2a - 5)$ (the product of a **binomial** and a **trinomial**)

The secret here is to multiply each of the terms in the binomial by each of the terms in the trinomial:

Multiply a by all three terms: $a^3 + 2a^2 - 5a$

Multiply -3 by all three terms: $-3a^2 - 6a + 15$

Now combine like terms: $\boldsymbol{a^3 - a^2 - 11a + 15}$

Homework

12. Simplify each expression:

a. $(3n^2 - 14n + 2) + (2n^2 + 2n - 1)$

b. $(4x^2 - x - 1) - (x^2 - 1)$

c. $(x + 2)(x^2 + 3x + 4)$

d. $(y - 1)(y^2 - 1)$

e. $(z + 3)(2z^2 - z - 1)$

f. $(2x - 5)(x^2 - 5x + 5)$

g. $(4w^2 - 3w - 1)(2w + 5)$

h. $(x + 3)(x^2 - 3x + 9)$

i. $(x^2 + 1)(x^2 + 2)$

j. $(2a + 1)(a^2 + 1)$

k. $(x - 3)(x^2 + 7x - 1)$

l. $(3t^2 - 5t + 3)(2t - 3)$

☐ *CUBING A BINOMIAL*

EXAMPLE 6: **Cube the binomial $2x + 5$. That is, simplify the expression $(2x + 5)^3$.**

Solution: The cube of anything is found by multiplying three of those anythings together: $A^3 = A \times A \times A$. Therefore, the expression

$$(2x + 5)^3$$

can be expanded to get

$$(2x+5)(2x+5)(2x+5)$$

We know that one way to multiply three things together is to multiply the first two of them together, and then multiply that result by the 3rd thing. (For example, (2)(3)(4) = (6)(4) = 24.) Multiplying the first two factors together gives

$$(4x^2 + 10x + 10x + 25)(2x + 5) \qquad \text{(double distribute)}$$

$$= \quad (4x^2 + 20x + 25)(2x + 5) \qquad \text{(combine like terms)}$$

We now have a trinomial times a binomial. What do we do? Most students find that reversing the trinomial and the binomial makes things a little easier to keep track of, so let's do it.

$$= \quad (2x + 5)(4x^2 + 20x + 25) \qquad \text{(commutative property)}$$

We multiply each term in the binomial by each term in the trinomial:

$$= \quad \mathbf{2x}(4x^2) + \mathbf{2x}(20x) + \mathbf{2x}(25) + \underline{5}(4x^2) + \underline{5}(20x) + \underline{5}(25)$$

$$= \quad 8x^3 + 40x^2 + 50x + 20x^2 + 100x + 125$$

$$= \quad \boxed{8x^3 + 60x^2 + 150x + 125}$$

Homework

13. Simplify each expression:

    a. $(x + 3)^3$ b. $(y + 1)^3$ c. $(n - 5)^3$

    d. $(2a + 4)^3$ e. $(3m - 2)^3$ f. $(5q + 3)^3$

14. Prove that $(x + y)^3 \neq x^3 + y^3$ by cubing the binomial.

❑ PREVIEW OF A FUTURE CHAPTER

Consider simplifying (expanding) the expression $(a + b)^9$. You should realize that the answer is not $a^9 + b^9$.

First of all, earlier examples have shown us that $(a + b)^2$ is not equal to $a^2 + b^2$. And the previous example showed us that $(2x + 5)^3$ is not equal to $(2x)^3 + 5^3$. It therefore seems reasonable that $(a + b)^9$ would not be equal to $a^9 + b^9$.

Second, watch what happens when we test the *conjecture* that $(a + b)^9 = a^9 + b^9$. Let a and b both take on the value 1. Then

$$(a + b)^9 = (1 + 1)^9 = 2^9 = \mathbf{512};$$

$$\text{but, } \quad a^9 + b^9 = 1^9 + 1^9 = 1 + 1 = \mathbf{2} \text{ -- not even close!!}$$

We therefore conclude that $(a + b)^9 \neq a^9 + b^9$. So how do we raise the sum of a and b to the 9th power? Here's the hard way:

$$(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)(a + b)$$

Start with the first two binomials; multiply that result by the third binomial, and so on and so on. You'd be done in a few hours (most likely with errors), but there's a much quicker way that we'll learn about at the very end of this book.

❑ DIVIDING A POLYNOMIAL BY A MONOMIAL

Just as $\frac{1}{7} + \frac{3}{7} = \frac{4}{7}$, we can do the problem $\frac{a}{b} + \frac{c}{b}$ by adding the numerators, and placing that sum over the common denominator b:

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

By reversing this reasoning we can take the fraction $\frac{a+c}{b}$ and, if we like, split it into the sum of two fractions:

$$\frac{a+c}{b} = \frac{a}{b}+\frac{c}{b}$$

This is the trick we need to divide a polynomial by a monomial.

EXAMPLE 7: **Divide:** $\dfrac{\mathbf{12x^2y^3 - 8x^3y^2 + 7xy^3}}{\mathbf{2x^2y}}$

Solution: Split the fraction into three separate fractions:

$$\frac{12x^2y^3}{2x^2y} - \frac{8x^3y^2}{2x^2y} + \frac{7xy^3}{2x^2y},$$

and then simplify (reduce) each fraction:

$$\boxed{6y^2 - 4xy + \frac{7y^2}{2x}}$$

Homework

15. Perform each division problem, where the divisor is a monomial:

a. $\dfrac{x^3 - x^2 + x}{x}$ b. $\dfrac{14xy + 21x^2y - 28xy^2}{7xy}$ c. $\dfrac{x^2 + 3x + 1}{x}$

d. $\dfrac{a+b}{b}$ e. $\dfrac{x-y}{y}$ f. $\dfrac{ax+bx}{x}$

☐ DIVIDING A POLYNOMIAL BY A POLYNOMIAL

First we need the right terminology. When written as a fraction, a division problem has two parts:

$$\frac{\text{dividend}}{\text{divisor}}$$

When written in the standard "long division" format, we write

$$\text{divisor}\overline{\smash{)}\text{dividend}}$$

The result of dividing is called the ***quotient***, and the leftover is called the ***remainder***. For example,

$$\begin{array}{r} 5 \\ 3\overline{\smash{)}17} \\ 15 \\ \hline 2 \end{array}$$

dividend = 17

divisor = 3

quotient = 5

remainder = 2

We can then write the answer as $5+\frac{2}{3}$ $\left(\text{dividend}+\frac{\text{remainder}}{\text{divisor}}\right)$, which is written as the mixed number $5\frac{2}{3}$ when we're dealing with numbers.

Think back when you were a kid and learned long division of numbers. Though I've seen different ways of doing this, the standard method boils down to a 4-step process, a process that is repeated until the problem is finished:

1. Divide the divisor into the first part of the dividend
2. Multiply the part of the quotient calculated in step 1 by the divisor
3. Subtract
4. Bring down the next digit

And then repeat steps 1 – 4 as many times as necessary.

We use the same process for polynomial long division in algebra.

EXAMPLE 8: **Perform the long division:** $\boldsymbol{\dfrac{3x^3-5x-2}{x-1}}$

Solution: The first step is to fill in the missing term in the dividend. Since there is no x^2 term, we put in the "place-holder" $0x^2$ between the cubic term and the linear term, giving us a dividend of $3x^3 + 0x^2 - 5x - 2$. So our long division problem is

$$\begin{array}{r} x-1\,\big)\overline{3x^3+0x^2-5x-2} \end{array}$$

1. Divide x into $3x^3$; it goes in $3x^2$ times (since $3x^2 \cdot x = 3x^3$):

$$\begin{array}{rl} & \boldsymbol{3x^2} \\ \boldsymbol{x}-1\,\big) & \overline{\boldsymbol{3x^3}+0x^2-5x-2} \end{array}$$

2. Multiply $3x^2$ by the divisor, $x - 1$:

$$\begin{array}{rl} & \boldsymbol{3x^2} \\ \boldsymbol{x-1}\,\big) & \overline{3x^3+0x^2-5x-2} \\ & \underline{\boldsymbol{3x^3-3x^2}} \end{array}$$

3. Subtract; $3x^3 - 3x^3 = 0$; $0x^2 - (-3x^2) = 3x^2$:

$$\begin{array}{rl} & 3x^2 \\ x-1\,\big) & \overline{\boldsymbol{3x^3}+\boldsymbol{0x^2}-5x-2} \\ & \underline{-\,(\boldsymbol{3x^3-3x^2})} \\ & \boldsymbol{0}\quad+\boldsymbol{3x^2} \end{array}$$

4. Bring down the next term, $-5x$:

$$\begin{array}{rl} & 3x^2 \\ x-1\,\big) & \overline{3x^3+0x^2-\boldsymbol{5x}-2} \\ & \underline{-\,(3x^3-3x^2)}\;\downarrow \\ & 0\quad+3x^2-\boldsymbol{5x} \end{array}$$

1. And repeat: Divide x into $3x^2$:

$$\begin{array}{r l}
 & \quad 3x^2 \; + \mathbf{3x} \\ \cline{2-2}
\boldsymbol{x} - 1 & \big)\, 3x^3 + 0x^2 - 5x - 2 \\
 & -(3x^3 - 3x^2) \\ \cline{2-2}
 & \quad 0 \; + \mathbf{3x^2} - 5x
\end{array}$$

2. Multiply $3x$ by $x - 1$, the divisor:

$$\begin{array}{r l}
 & \quad 3x^2 \; + \mathbf{3x} \\ \cline{2-2}
\boldsymbol{x} - \mathbf{1} & \big)\, 3x^3 + 0x^2 - 5x - 2 \\
 & -(3x^3 - 3x^2) \\ \cline{2-2}
 & \quad 0 \; + 3x^2 - 5x \\
 & \qquad\; \mathbf{3x^2 - 3x}
\end{array}$$

3. Subtract; $3x^2 - 3x^2 = 0$; $-5x - (-3x) = -2x$:

$$\begin{array}{r l}
 & \quad 3x^2 \; + 3x \\ \cline{2-2}
x - 1 & \big)\, 3x^3 + 0x^2 - 5x - 2 \\
 & -(3x^3 - 3x^2) \\ \cline{2-2}
 & \quad 0 \; + \mathbf{3x^2 - 5x} \\
 & \qquad -(\mathbf{3x^2 - 3x}) \\ \cline{2-2}
 & \qquad\quad \mathbf{0} \; - \mathbf{2x}
\end{array}$$

4. Bring down the next (and last) term, -2:

$$\begin{array}{r l}
 & \quad 3x^2 \; + 3x \\ \cline{2-2}
x - 1 & \big)\, 3x^3 + 0x^2 - 5x - \mathbf{2} \\
 & -(3x^3 - 3x^2) \qquad\quad \big\downarrow \\ \cline{2-2}
 & \quad 0 \; + 3x^2 - 5x \\
 & \qquad -(3x^2 - 3x) \\ \cline{2-2}
 & \qquad\quad 0 \; - 2x - \mathbf{2}
\end{array}$$

1. Divide x into $-2x$:

$$\begin{array}{r} 3x^2 \ + 3x - \mathbf{2} \\ \mathbf{x} - 1 \overline{)\, 3x^3 + 0x^2 - 5x - 2} \\ \underline{-(3x^3 - 3x^2)} \qquad\qquad \\ 0 \ + 3x^2 - 5x \qquad \\ \underline{-(3x^2 - 3x)} \quad \\ 0 \ - \mathbf{2x} - 2 \end{array}$$

2. Multiply -2 by $x - 1$:

$$\begin{array}{r} 3x^2 \ + 3x - \mathbf{2} \\ \mathbf{x} - \mathbf{1} \overline{)\, 3x^3 + 0x^2 - 5x - 2} \\ \underline{-(3x^3 - 3x^2)} \qquad\qquad \\ 0 \ + 3x^2 - 5x \qquad \\ \underline{-(3x^2 - 3x)} \quad \\ 0 \ - 2x - 2 \\ -\mathbf{2x} + \mathbf{2} \end{array}$$

3. Subtract; $-2x - (-2x) = 0$; $-2 - (+2) = -4$

$$\begin{array}{r} 3x^2 \ + 3x - 2 \\ x - 1 \overline{)\, 3x^3 + 0x^2 - 5x - 2} \\ \underline{-(3x^3 - 3x^2)} \qquad\qquad \\ 0 \ + 3x^2 - 5x \qquad \\ \underline{-(3x^2 - 3x)} \quad \\ 0 \ - \mathbf{2x} - \mathbf{2} \\ \underline{-(-\mathbf{2x} + \mathbf{2})} \\ \mathbf{0} \ \ -\mathbf{4} \end{array}$$

There are no terms left to bring down in the dividend, so we write the remainder (the -4) over the divisor and add it to the quotient. The final answer to the long division problem is

$$\boxed{3x^2 + 3x - 2 + \frac{-4}{x - 1}}$$

Homework

16. Perform each polynomial long division problem, expressing any remainder as a fraction added to the quotient:

a. $\frac{x^2+5x+6}{x+3}$ b. $\frac{x^2-9}{x-3}$ c. $\frac{x^2+2x+1}{x+1}$

d. $\frac{n^2+n-4}{n+5}$ e. $\frac{2a^2-5a+2}{a+3}$ f. $\frac{3w^2+10}{w+5}$

g. $\frac{6b^2+b-15}{2b+3}$ h. $\frac{3y^2-9}{y+5}$ i. $\frac{10x^2+3x-7}{2x-1}$

j. $\frac{x^3+1}{x+1}$ Hint: $x^3 + 1 = x^3 + 0x^2 + 0x + 1$

k. $\frac{n^3-8}{n-2}$ l. $\frac{a^3+27}{a^2-3a+9}$

17. Perform each polynomial long division problem (Hint: there is no remainder):

a. $\frac{40x^3+97x^2+60x+27}{5x+9}$ b. $\frac{8w^3+22w^2+13w+2}{2w^2+5w+2}$

c. $\frac{40r^3-4r^2-7r-3}{8r^2+4r+1}$ d. $\frac{63m^3+43m^2+13m+1}{7m^2+4m+1}$

Practice Problems

18. Simplify each expression:

a. $7x^2 - 3x + 7 - 7x^2 - 3x - 7$

b. $-8(3y^2 - 4y - 1)$

c. $2(a^2 - 8) - (a^2 - 2a - 1)$

d. $-(4n^2 - 4n) - (4n - 4n^2)$

e. $3(4g^2 - g + 3) - 2(6g^2 + g - 1)$

f. $(x + y)(w + z)$

g. $(3x)(-4x)$

h. $10(3y)$

i. $-3n + 4n$

j. $-2(x^2 - 3x - 1)$

k. $3(x^2 - x - 2) - (2x^2 + 7x + 8)$

l. $10x^2 + 29x$

19. Simplify each expression:

a. $(x + 9)(x + 8)$

b. $(y - 1)(y - 8)$

c. $(2z + 5)(2z - 5)$

d. $(N + 10)(N - 10)$

e. $(x - 9)^2$

f. $(a + 5)^2$

g. $(t + 9)(t - 5)$

h. $(a - 22)(a + 1)$

i. $(a - 11)(a + 2)$

j. $(2x + 1)(x - 5)$

k. $(3x + 8)(2x - 5)$

l. $(6x + 5)(x - 3)$

m. $(6a + 17)(a - 1)$

n. $(R + 12)(R - 12)$

o. $(5n - 3)^2$

p. $(1 - a)(2 - a)$

q. $(7w + 5)^2$

r. $(3a - 1)(3a - 2)$

s. $(9a - 1)(a - 2)$

t. $(x + 18)(x - 2)$

u. $(x + 36)(x + 1)$

v. $(5c - 1)(6c - 1)$

w. $(8a + 1)(2a - 1)$

x. $(6q + 5)^2$

y. $(3 + n)(3 - n)$

z. $(16n - 9)(2n - 3)$

20. Prove that $(u + w)^4 \neq u^4 + w^4$. [Letting both u and w equal 1 will do the trick.]

21. Simplify each expression:

a. $(2n - 5)(3n - 1)$

b. $(8x + 3)(8x - 3)$

c. $(7z - 5)^2$

d. $(8 - 7a)(8 + 7a)$

e. $(2x - 1)(3x + 4) - (4x - 1)^2$

22. Simplify each expression:

a. $(w-5)(3w^2-2w-1)$ b. $(2x-5)^3$

23. True/False, and prove your answer:

a. $(a-b)^2 = a^2+b^2$ b. $(x-y)^3 = x^3-y^3$

24. Prove that $(a+b)^5 \neq a^5+b^5$

25. Divide: $\dfrac{4x^3-8x^2+6x-10}{4x^2}$

26. Divide: $\dfrac{x^2+9}{x-5}$

27. Divide: $\dfrac{x^3-3x+8}{x+3}$

28. Divide: $\dfrac{x^4-1}{x+1}$

29. Divide: $\dfrac{n^3+8}{n+2}$

Solutions

1. a. $21L$ b. $-10x$ c. $12T$ d. $-60w$ e. As is f. As is
g. As is h. As is i. $35y$ j. $10p$ k. $-30a$ l. $-10n$
m. As is n. $12x^2$ o. $7x$ p. $-6n^2$ q. $-n$ r. $56x^2$
s. $-7u^2$ t. $-16c^2$ u. 0 v. $42mn$ w. As is x. $-169k^2$
y. 0 z. $6x$

2. $4n^2+7n = 4(\mathbf{2})^2+7(\mathbf{2}) = 4(4)+7(2) = 16+14 = 30$,
whereas $11n^3 = 11(\mathbf{2})^3 = 11(8) = 88$
Therefore, $4n^2+7n \neq 11n^3$

3. a. $8x^2-7x+9$ b. 0 c. $-4u^2+4u$ d. $-2a^2-a$
e. $8x^2-6x$ f. $6y^2-4$ g. $-10x+6$ h. $-6w^2+5w$

4. a. $5c^2 + 8c - 15$ b. $-7x^2 - 59x + 8$ c. $-4a^2 + 4a + 1$
d. $2w^2 - 10w + 10$ e. $-u^2 + 4u + 1$ f. 0
g. $-5x^2 - 4x - 13$ h. $-6n^2 - 14n + 4$

5. a. $xw + xz + wy + yz$ b. $ac - bc + ad - bd$ c. $xy + 3x + 2y + 6$
d. $x^2 + 7x + 12$ e. $n^2 - 5n + 4$ f. $a^2 - 4a - 21$
g. $y^2 - 81$ h. $u^2 - 9$ i. $t^2 - 39t + 380$
j. $z^2 + 6z + 9$ k. $v^2 - 8v + 1 \quad 6$ l . $N^2 - 1$

6. a. $3a^2 - 20a - 63$ b. $2n^2 + 5n - 12$ c. $3n^2 - 11n + 8$
d. $25x^2 + 65x + 42$ e. $49w^2 - 4$ f. $x^2 - 144$
g. $4y^2 + 4y + 1$ h. $42x^2 - 31x - 21$ i. $3q^2 + 14q - 49$
j. $9n^2 + 6n + 1$ k. $18x^2 - 27x - 35$ l. $u^2 - 14u + 49$

7. a. $y^2 + 8y + 16$ b. $z^2 - 18z + 81$ c. $9x^2 + 30x + 25$
d. $4a^2 - 4a + 1$ e. $n^2 + 24n + 144$ f. $36t^2 - 84t + 49$
g. $q^2 - 30q + 225$ h. $25b^2 + 30b + 9$ i. $49u^2 - 14u + 1$
j. $4x^2 + 4x + 1$ k. $9h^2 - 72h + 144$ l. $25y^2 - 50y + 25$

8. a. $ac - ad + bc - bd$ b. $4x^2 - 9$ c. $9n^2 - 6n + 1$
d. $6t^2 - 7t - 3$ e. $6x^2 - 24$ f. $n^2 - 1$
g. $42a^2 - 130a + 100$ h. $100c^2 + 140c + 49$ i. $L^2 + 8L + 16$
j. $21x^2 + 40x - 21$ k. $169n^2 - 49$ l. $144d^2 - 480d + 400$

9. a. $3n^2 + 3n$ b. $2x^2 + 6x + 5$ c. $2a^2 - 4a - 4$
d. $15w^2 + 12w - 2$ e. $-3y^2 + 3y - 7$ f. $8y$

10. i) By letting $a = 3$ and $b = 4$, for instance, we get:

$(a + b)^2 = (\mathbf{3} + \mathbf{4})^2 = 7^2 = 49$, whereas

$a^2 + b^2 = \mathbf{3}^2 + \mathbf{4}^2 = 9 + 16 = 25$.

Clearly, $(a + b)^2 \neq a^2 + b^2$

ii) $(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

11. Choosing, for example, $x = 1$ and $y = 2$, we would get the following results:

$$(x + y)^3 = (\mathbf{1} + \mathbf{2})^3 = 3^3 = 27;$$

on the other hand, $x^3 + y^3 = \mathbf{1}^3 + \mathbf{2}^3 = 1 + 8 = 9$.

12. a. $5n^2 - 12n + 1$ b. $3x^2 - x$
c. $x^3 + 5x^2 + 10x + 8$ d. $y^3 - y^2 - y + 1$
e. $2z^3 + 5z^2 - 4z - 3$ f. $2x^3 - 15x^2 + 35x - 25$
g. $8w^3 + 14w^2 - 17w - 5$ h. $x^3 + 27$
i. $x^4 + 3x^2 + 2$ j. $2a^3 + a^2 + 2a + 1$
k. $x^3 + 4x^2 - 22x + 3$ l. $6t^3 - 19t^2 + 21t - 9$

13. a. $x^3 + 9x^2 + 27x + 27$ b. $y^3 + 3y^2 + 3y + 1$
c. $n^3 - 15n^2 + 75n - 125$ d. $8a^3 + 48a^2 + 96a + 64$
e. $27m^3 - 54m^2 + 36m - 8$ f. $125q^3 + 225q^2 + 135q + 27$

14. $(x + y)^3 = (x + y)(x + y)(x + y) = (x + y)(x^2 + 2xy + y^2)$
$= x^3 + 3x^2y + 3xy^2 + y^3$,
which is most likely <u>not</u> equal to $x^3 + y^3$ for all values of x and y.

15. a. $x^2 - x + 1$ b. $2 + 3x - 4y$ c. $x + 3 + \frac{1}{x}$
d. $\frac{a}{b} + 1$ e. $\frac{x}{y} - 1$ f. $a + b$

16. a. $x + 2$ b. $x + 3$ c. $x + 1$
d. $n - 4 + \frac{16}{n+5}$ e. $2a - 11 + \frac{35}{a+3}$ f. $3w - 15 + \frac{85}{w+5}$
g. $3b - 4 + \frac{-3}{2b+3}$ h. $3y - 15 + \frac{66}{y+5}$ i. $5x + 4 + \frac{-3}{2x-1}$
j. $x^2 - x + 1$ k. $n^2 + 2n + 4$ l. $a + 3$

17. a. $8x^2 + 5x + 3$ b. $4w + 1$ c. $5r - 3$
d. $9m + 1$

18. a. $-6x$ b. $-24y^2 + 32y + 8$ c. $a^2 + 2a - 15$
d. 0 e. $-5g + 11$ f. $xw + xz + wy + yz$
g. $-12x^2$ h. $30y$ i. n
j. $-2x^2 + 6x + 2$ k. $x^2 - 10x - 14$ l. As is

19. a. $x^2 + 17x + 72$ b. $y^2 - 9y + 8$ c. $4z^2 - 25$
d. $N^2 - 100$ e. $x^2 - 18x + 81$ f. $a^2 + 10a + 25$
g. $t^2 + 4t - 45$ h. $a^2 - 21a - 22$ i. $a^2 - 9a - 22$
j. $2x^2 - 9x - 5$ k. $6x^2 + x - 40$ l. $6x^2 - 13x - 15$
m. $6a^2 + 11a - 17$ n. $R^2 - 144$ o. $25n^2 - 30n + 9$
p. $a^2 - 3a + 2$ q. $49w^2 + 70w + 25$ r. $9a^2 - 9a + 2$
s. $9a^2 - 19a + 2$ t. $x^2 + 16x - 36$ u. $x^2 + 37x + 36$
v. $30c^2 - 11c + 1$ w. $16a^2 - 6a - 1$ x. $36q^2 + 60q + 25$
y. $9 - n^2$, or $-n^2 + 9$ z. $32n^2 - 66n + 27$

20. $(1 + 1)^4 = 2^4 = 16$; whereas $1^4 + 1^4 = 1 + 1 = 2$.

21. a. $6n^2 - 17n + 5$ b. $64x^2 - 9$ c. $49z^2 - 70z + 25$
d. $64 - 49\text{a}^2$ e. $-10x^2 + 13x - 5$

22. a. $3w^3 - 17w^2 + 9w + 5$ b. $8x^3 - 60x^2 + 150x - 125$

23. a. False; let $a = 5$ and $b = 2$:

$$(a - b)^2 = (5 - 2)^2 = 3^2 = 9$$
$$a^2 + b^2 = 5^2 + 2^2 = 25 + 4 = 29$$

b. False; let $a = 2$ and $b = 3$

$(x-y)^3 = (5-4)^3 = 1^3 = 1$

$x^3 - y^3 = 5^3 - 4^3 = 125 - 64 = 61$

24. Let $a = 2$ and $b = 3$

$(2+3)^5 = 5^5 = 3125$

$2^5 + 3^5 = 32 + 243 = 275$

25. $x - 2 + \frac{3}{2x} - \frac{5}{2x^2}$

26. $x + 5 + \frac{34}{x-5}$

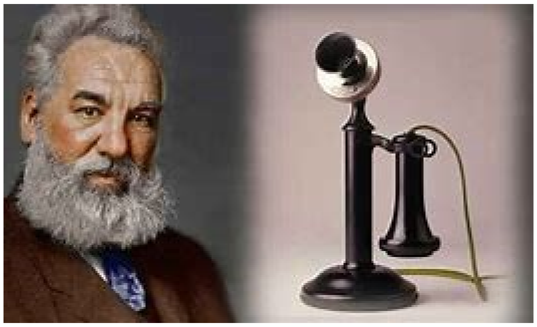
27. $x^2 - 3x + 6 + \frac{-10}{x+3}$

28. $x^3 - x^2 + x - 1$

29. $n^2 - 2n + 4$

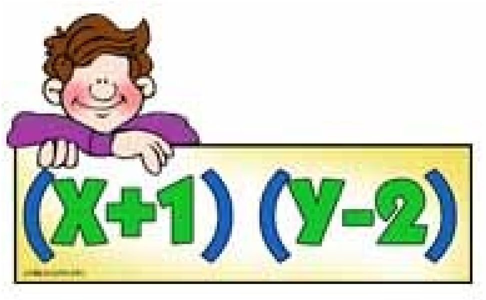
"When one door closes, another opens; but we often look so long and so regretfully upon the closed door that we do not see the one which has opened for us."

- *Alexander Graham Bell*



CH 12 – FACTORING, PART I

❑ *INTRODUCTION*



In the next chapter we will continue our discussion of finding *break-even points* by looking at quadratic equations (where the variable is squared), but instead of using a table or a graph, we will solve *quadratic equations* to find the break-even points.

One method of solving a quadratic equation is called the **factoring method**. Therefore, in this chapter, we have to become very skilled at factoring quadratic expressions.

❑ *A DIFFERENT VIEW OF THE DISTRIBUTIVE PROPERTY*

We've generally viewed the distributive property in a form like

$$A(B + C) = AB + AC, \qquad \textbf{DISTRIBUTING}$$

and saw the power of such a property in simplifying expressions and solving complicated equations. But the distributive property is a statement of equality -- we might find it useful to flip it around the equals sign and write it as

$$AB + AC = A(B + C) \qquad \textbf{FACTORING}$$

This provides a whole new perspective. It allows us to take a pair of terms, the sum $AB + AC$, find the **common factor** A (it's in both terms), and "pull" the A out in front, and write the sum $AB + AC$ as the product $A(B + C)$.

This use of the distributive property in reverse is called ***factoring***. Notice that using the distributive property in reverse converts two terms into one term.

For example, suppose we want to factor $9a + 9b$; that is, we want to convert $9a + 9b$ from a sum to a product. First we notice that 9 is a common factor of both terms. We pull the 9 away from both terms, and put it out in front to get $9(a + b)$, and we're done factoring:

$9a + 9b$ factors into $9(a + b)$

To check this answer, distribute $9(a + b)$ and you'll get the original $9a + 9b$.

RECAP:

$9(a + b)$	Distribute	$9a + 9b$	**One term to two terms**
$9a + 9b$	Factor	$9(a + b)$	**Two terms to one term**

☐ FACTORING OUT THE GCF

EXAMPLE 1: **Factor each expression:**

A. $7x + 7y = 7(x + y)$

B. $3x + 12 = 3(x + 4)$

C. $ax + bx = x(a + b)$

D. $Rw - Ew = w(R - E)$

E. $9z + 9 = 9(z + 1)$

F. $mn - m = m(n - 1)$

G. $-6R + 8 = -2(3R - 4)$

Alternatively, we could pull out a positive 2, yielding $2(-3R + 4)$, but it's customary to pull out the leading negative sign.

H. $-ax - at = -a(x + t)$

I. $-x + 5 = -(x - 5)$

J. $-n - 9 = -(n + 9)$

K. $6r + 8s - 10t = 2(3r + 4s - 5t)$

Note that every problem in the preceding example can be checked by distributing the answer. Our next example shows how we can factor out a variable.

EXAMPLE 2: **Factor each expression:**

A. $x^2 + 3x = x(x + 3)$

B. $n^2 - 7n = n(n - 7)$

C. $t^2 + t = t(t + 1)$

D. $y^2 - y = y(y - 1)$

E. $m^2 - 10m = m(m - 10)$

F. $a^2 + 40a = a(a + 40)$

EXAMPLE 3: **Factor: $2a^2 - 8a$**

Solution: What common factor can be pulled out in front? Since 2 is a factor of both terms, it can be pulled out. But a is also a common factor, so it needs to come out in front, also. In other words, the quantity $2a$ is common to both terms (and it's the largest quantity that is common to both terms). So we factor it out and leave in the parentheses what must be left.

$2a^2 - 8a = 2a(a - 4)$ (check by distributing)

New Terminology: Look at the previous example. We decided that 2 was a common factor of the two given terms. We also realized that a was another common factor of the two terms. But the quantity $2a$, which is, of course, common to both terms, is called the greatest common factor. This is because $2a$ is certainly a factor of both terms, and it is the <u>biggest</u> such factor -- nothing bigger than $2a$ is common to both terms. We call $2a$ the ***greatest common factor***, or **GCF**, of the expression $2a^2 - 8a$.

Homework

1. How would you convince your buddy that factoring $20x + 30y$ produces a result of $10(2x + 3y)$?

2. Your friend adamantly believes that $6w + 9z$ factors to $6(w + 3z)$. Prove her wrong.

3. Finish the factorization of each expression:

a. $wx + wz = w(\qquad)$

b. $4P - 4Q = 4(\qquad)$

c. $9x - 36 = 9(\qquad)$

d. $8y - 12t = 4(\qquad)$

e. $7u + 7 = 7(\qquad)$

f. $-2n + 8 = -2(\qquad)$

g. $-a + b = -(\qquad)$

h. $-c - d = -(\qquad)$

i. $2x + 4y - 8z = 2(\qquad)$

j. $aw - au + az = a(\qquad)$

k. $14x^2 - 21x = 7x(\qquad)$

l. $20a^2 + 30a - 40 = 10(\qquad)$

4. Factor each expression:

a. $3P + 3Q$

b. $9n - 27$

c. $cn + dn$

d. $wx - xy$

e. $7t - 7$

f. $x + xy$

g. $-8L + 10$

h. $-ab - bc$

i. $-u - 5$

j. $-z - x + 10$

k. $2x + 2y + 2z$

l. $5a - 10b + 15c$

5. Finish the factorization of each expression:

a. $4a + 8b = 4(\qquad)$

b. $9u^2 - 3u = 3u(\qquad)$

c. $15Q - 45R = 15(\qquad)$

d. $18x^2 + 12x = 6x(\qquad)$

e. $10y^2 - 20y = 10y(\quad)$

f. $50a + 75b = 25(\quad)$

g. $7t^2 + 28t = 7t(\quad)$

h. $48w - 64z = 16(\quad)$

i. $100a^2 - 80a = 20a(\quad)$

j. $47y^2 + 47y = 47y(\quad)$

❑ *Using the GCF to Solve Formulas*

Do you remember how, in the Prologue, we solved for x in the formula (literal equation)

$$wx + y = A \text{ ?}$$

We subtracted y from each side of the equation:

$$wx = A - y$$

and then we divided each side of the equation by w:

$$x = \frac{A - y}{w}$$ and we've isolated the x.

This is all fine and dandy when the unknown, in this case the x, occurs only once in the formula. But how do we isolate something that occurs more than once in an equation? For example, how do we solve for x in the formula

$$ax - c = bx \text{ ?}$$

There's an x-term on each side of the equation. This x is going to be tough to isolate. What would you do if, instead of the symbols a, b, and c in the equation, they had been numbers?

> For example, suppose the equation had been
>
> $$7x - 10 = 4x$$
>
> We subtract $4x$ from each side: $7x - 4x - 10 = 0$
>
> Then combine like terms: $3x - 10 = 0$
>
> Now add 10 to each side: $3x = 10$
>
> And lastly, divide each side by 3: $\boldsymbol{x = \frac{10}{3}}$

We follow the same procedure for solving the formula $ax - c = bx$ for x.

The original formula: $ax - c = bx$

Subtract bx from each side: $ax - bx - c = 0$

Add c to each side: $ax - bx = c$

How do we "combine the like terms" ax and $-bx$? Here's where <u>factoring</u> comes to the rescue -- by factoring x out of $ax - bx$, we get $x(a - b)$: $\Big\downarrow$ Factoring out the GCF $x(a - b) = c$

Divide each side by $a - b$, and we're done: $\boldsymbol{x = \dfrac{c}{a-b}}$

Note: There are no x's on the right side of the answer. Can you explain why this fact is so important?

<u>EXAMPLE 4:</u> **Solve for n: $Qn - n + P = R$**

<u>Solution:</u> $Qn - n + P = R$ (the original formula)

$\Rightarrow$ $Qn - n = R - P$ (subtract P from each side)

$\Rightarrow$ $n(Q - 1) = R - P$ (factor out the n)

$\Rightarrow$ $\dfrac{n\cancel{(Q-1)}}{\cancel{Q-1}} = \dfrac{R-P}{Q-1}$ (divide each side by $Q - 1$)

$\Rightarrow$ $\boxed{n = \dfrac{R-P}{Q-1}}$ (simplify)

<u>EXAMPLE 5:</u> **Solve for a: $c(a - d) + 3 = 5(e - a)$**

<u>Solution:</u> $c(a - d) + 3 = 5(e - a)$ (the original formula)

$\Rightarrow$ $ac - cd + 3 = 5e - 5a$ (distribute)

$\Rightarrow \quad ac + 5a - cd + 3 = 5e$ (add $5a$ to each side)

$\Rightarrow \quad ac + 5a - cd = 5e - 3$ (subtract 3 from each side)

$\Rightarrow \quad ac + 5a = 5e - 3 + cd$ (add cd to each side)

Note: These steps were designed to get the variable a on one side of the equation, and the rest of the things on the other side.

$\Rightarrow \quad a(c + 5) = 5e - 3 + cd$ (factor out the a)

$\Rightarrow \quad \dfrac{a(c+5)}{\mathbf{c+5}} = \dfrac{5e-3+cd}{\mathbf{c+5}}$ (divide each side by $c + 5$)

$\Rightarrow \quad \boxed{a = \dfrac{5e-3+cd}{c+5}}$ (simplify)

Be sure you understand thoroughly why there are no a's on the right side of the answer. And can you see why c cannot be equal to -5 in this answer?

Homework

6. Solve each formula for n:

a. $cn + dn = 3$

b. $an - cn = d$

c. $Ln + n = M$

d. $tn = c - sn$

e. $rn = 3 + tn$

f. $m(n + 1) - Qn - R = 0$

g. $a(n + 3) + b(n + c) = R$

h. $an + bn + cn = d$

i. $an - n - a = 0$

j. $2(n + 1) + an = c$

7. Solve each formula for x:

a. $cx + 7x = 14$

b. $rx - ux = w + v$

c. $wx - x = w$

d. $ax - b = c - dx$

e. $u(x-a)+x = w$ f. $a(x+1)+b(x-1) = 0$

g. $mx-3(x-w) = z+u$ h. $p(x-3) = q(x+2)$

i. $c(x+a)-a(x-1) = a-b$ j. $a(b-x)+c(2-x) = R-Q$

❑ *A Quick Review of Double Distributing*

To set the stage for factoring, we recall from the previous chapter the concept we called "double distributing" to multiply two binomials:

$$(2x+3)(x+5)$$

i) Multiply the FIRST terms in each set of parentheses: $2x$ and x. The product is $\underline{2x^2}$.

ii) Multiply the OUTER terms: $2x$ and 5. The product is $\underline{10x}$.

iii) Multiply the INNER terms: 3 and x. The product is $\underline{3x}$.

iv) Multiply the LAST terms in each set of parentheses: 3 and 5. The product is $\underline{15}$.

v) Writing out these four products,

$$= \underset{\substack{\uparrow \\ \text{product of} \\ \text{First terms}}}{2x^2} + \underset{\substack{\uparrow \\ \text{product of} \\ \text{Outer terms}}}{10x} + \underset{\substack{\uparrow \\ \text{product of} \\ \text{Inner terms}}}{3x} + \underset{\substack{\uparrow \\ \text{product of} \\ \text{Last terms}}}{15}$$

$$= \mathbf{2x^2 + 13x + 15}$$

The key idea to absorb here is that the $2x^2$ in the answer is the product of the first terms, while the 15 is the product of the last terms. Also, the middle term in the answer, $13x$, is the sum of the outer and inner products.

Homework

8. Find the following products -- do all the work in your head:

a. $(2x - 1)(x + 4)$

b. $(3n - 3)(2n - 5)$

c. $(a + 9)(3a - 1)$

d. $(6y - 1)(2y + 7)$

e. $(2m - 7)(2m + 7)$

f. $(4w + 5)(4w + 5)$

g. $(5x + 1)(5x - 1)$

h. $(2n + 3)(7n - 10)$

i. $(7u - 3)(7u - 3)$

j. $(12a + 13)(12a - 13)$

☐ *Reverse Double Distributing*

We know that

$$(2x + 3)(x + 5) \;=\; 2x^2 + 13x + 15,$$

and we call this **double distributing**.

But as described at the beginning of this chapter, we can turn the equality around,

$$2x^2 + 13x + 15 \;=\; (2x + 3)(x + 5)$$

and call it **factoring**.

Homework

9. True/False:

a. $x^2 + 5x + 6$ factors into $(x + 3)(x + 2)$.
b. $y^2 - 16$ factors into $(y + 4)(y - 4)$.
c. $x^2 + 9x + 15$ factors into $(x + 3)(x + 5)$.
d. $n^2 - 6n + 9$ factors into $(n - 3)(n - 3)$.
e. $2a^2 - 11a - 6$ factors into $(2a + 1)(a - 6)$.
f. $u^2 + 25$ factors into $(u + 5)(u + 5)$.
g. $a^2 + 10a + 25$ factors into $(a + 5)^2$.
h. $y^2 - 2y + 4$ factors into $(y - 2)^2$.

10. Matching:

a. ___ $x^2 + 4x + 3$
b. ___ $x^2 - 9$
c. ___ $x^2 + 10x + 25$
d. ___ $x^2 + 16$
e. ___ $x^2 + 4x + 4$
f. ___ $2x^2 - 11x - 6$
g. ___ $x^2 - 25$
h. ___ $2x^2 + x - 6$

1. $(2x + 1)(x - 6)$
2. $(x + 5)(x - 5)$
3. $(x + 3)(x + 1)$
4. $(x + 3)(x - 3)$
5. $(2x - 3)(x + 2)$
6. $(x + 5)(x + 5)$
7. $(x + 2)(x + 2)$
8. Not factorable

11. Finish the factorization of each expression:

a. $x^2 - 10x + 16 = (x - 8)(\qquad)$

b. $n^2 + 5n - 14 = (n + 7)\ (\qquad)$

c. $a^2 - 17a + 72 = (a - 9)\ (\qquad)$

d. $q^2 - 49 = (q + 7)\ (\qquad)$

e. $c^2 + 6c + 9 = (c + 3)\ (\qquad)$

12. Finish the factorization of each expression:

a. $12n^2 + 8n - 15 = (6n - \quad)(2n + \quad)$

b. $16x^2 - 9 = (4x + \quad)(4x - \quad)$

c. $21z^2 - 4z - 1 = (7z + \quad)(3z - \quad)$

d. $9a^2 + 24a + 16 = (3a + \quad)(3a + \quad)$

e. $6x^2 - 23x + 7 = (2x - \quad)(3x - \quad)$

f. $25t^2 - 49 = (5t + \quad)(5t - \quad)$

g. $9w^2 - 225 = (3w + \quad)(3w - \quad)$

h. $16c^2 - 24c + 9 = (4c - \quad)(4c - \quad)$

i. $14x^2 - 58x + 8 = (7x - \quad)(2x - \quad)$

☐ *THE PROCESS OF FACTORING*

Hopefully, we now understand the concept of factoring. We now present a series of examples which try to turn the random method into something a little more methodical; but no matter what method is used, it's essentially a matter of *trial-and-error*.

EXAMPLE 6: **Factor: $6x^2 - 7x - 5$**

Solution: Factoring a trinomial like $6x^2 - 7x - 5$ can be viewed as a 3-step process:

1) Split the $6x^2$ into two factors (by guessing)
2) Split the -5 into two factors (by guessing)
3) See if we guessed right by double distributing

Step 1: $6x^2 - 7x - 5$

$$(3x \qquad)(2x \qquad)$$

Split $6x^2$ into 2 factors and place these factors in the front of each binomial.

Step 2: $6x^2 - 7x - 5$

$$(3x - 5)(2x + 1)$$

Split -5 into 2 factors and place these factors in the back of each binomial.

Step 3: See if we guessed right:

$$(3x - 5)(2x + 1) = 6x^3 + 3x - 10x - 5 = 6x^2 - 7x - 5 \ ✓$$

We got lucky on the first try -- the factorization of $6x^2 - 7x - 5$ is

$$\boxed{(3x - 5)(2x + 1)}$$

EXAMPLE 7: **Factor: $n^2 - 5n + 6$**

Solution: As before, we focus on the first and last terms.

Step 1: Split the n^2; there's only one way to do this:

$$(n \quad)(n \quad)$$

Step 2: Split the 6; let's try 6 and 1:

$$(n + 6)(n + 1)$$

Step 3: See if we guessed right:

$$(n + 6)(n + 1) = n^2 + n + 6n + 6 = n^2 + 7n + 6 \ ☹$$

Let's redo Step 2: Split the 6 into 3 and 2:

$$(n + 3)(n + 2) = n^2 + 2n + 3n + 6 = n^2 + 5n + 6 \ ☹$$

Notice that the $n^2 + 5n + 6$ we obtained is almost the original expression that we're trying to factor; it differs only in the sign of the middle term. Let's redo Step 2 again: Split the 6 into –3 and –2:

$$(n - 3)(n - 2) = n^2 - 2n - 3n + 6 = n^2 - 5n + 6$$ ☺

and thus our final factorization is

$$\boxed{(n - 3)(n - 2)}$$

EXAMPLE 8: **Factor: $y^2 + 7y + 14$**

Solution: Looking at the y^2 first, we see that the only way to split it up is $y \cdot y$. Since all the terms of the trinomial are positive, a couple of ways to split up the 14 is $14 \cdot 1$ and $7 \cdot 2$. Let's give it a try:

$$(y + 14)(y + 1) = y^2 + y + 14y + 14 = y^2 + 15y + 14$$ ☹

$$(y + 7)(y + 2) = y^2 + 2y + 7y + 14 = y^2 + 9y + 14$$ ☹

No luck yet; let's reverse the order of the factors:

$$(y + 1)(y + 14) = y^2 + 14y + y + 14 = y^2 + 15y + 14$$ ☹

$$(y + 2)(y + 7) = y^2 + 7y + 2y + 14 = y^2 + 9y + 14$$ ☹

Nothing but sad faces, and we've tried every possible arrangement. This can mean only one thing -- that there's no way to factor the expression $y^2 + 7y + 14$ -- the expression is

$$\boxed{\text{Not factorable}}$$

EXAMPLE 9: **Factor: $w^2 - 25$**

Solution: This quadratic expression has only two terms, but which one is missing? When a quadratic starts with w^2, we expect the next term to contain a w (which is w^1). So it's the middle term that is missing. But the middle term is not the one we focus on anyway, so let's begin the usual process. Notice that the fact that the last term is negative means that it must split into one positive factor and one negative factor.

$(w + 25)(w - 1) = w^2 - w + 25w - 25 = w^2 + 24w - 25$ ☹

$(w - 25)(w + 1) = w^2 + w - 25w - 25 = w^2 - 24w - 25$ ☹

$(w + 1)(w - 25) = w^2 - 25w + w - 25 = w^2 - 24w - 25$ ☹

$(w - 1)(w + 25) = w^2 + 25w - w - 25 = w^2 + 24w - 25$ ☹

$(w + 5)(w - 5) = w^2 - 5w + 5w - 25 = w^2 - 25$ ☺

Therefore, the expression $w^2 - 25$ factors into

$$\boxed{(w + 5)(w - 5)}$$

EXAMPLE 10: **Factor: $9u^2 + 12u + 4$**

Solution: This is quite a problem. We need to notice that $9u^2$ can be split in two ways: $(9u)(u)$ and $(3u)(3u)$. And worse, the 4 can also be split in two ways: (4)(1) and (2)(2). And even worse, the order in which we arrange the factors of 4 matters, too. The only good news is that all the terms of the trinomial are positive.

Let's start the multiplications, the first three using $(9u)(u)$:

$(9u + 4)(u + 1) = 9u^2 + 9u + 4u + 4 = 9u^2 + 13u + 4$ ☹

$(9u + 1)(u + 4) = 9u^2 + 36u + u + 4 = 9u^2 + 37u + 4$ ☹

$(9u + 2)(u + 2) = 9u^2 + 18u + 2u + 4 = 9u^2 + 20u + 4$ ☹

Now we'll use $(3u)(3u)$:

$(3u + 4)(3u + 1) = 9u^2 + 3u + 12u + 4 = 9u^2 + 15u + 4$ ☹

$(3u + 1)(3u + 4) = 9u^2 + 12u + 3u + 4 = 9u^2 + 15u + 4$ ☹

$(3u + 2)(3u + 2) = 9u^2 + 6u + 6u + 4 = 9u^2 + 12u + 4$ ☺

Eureka! The factorization of $9u^2 + 12u + 4$ is $(3u + 2)(3u + 2)$, which we can write more succinctly as

$$\boxed{(3u+2)^2}$$

EXAMPLE 11: **Factor: $a^2 + 49$**

Solution: This looks simple enough. Let's start with the most obvious choice: $49 = 7 \times 7$

$(a + 7)(a + 7) = a^2 + 7a + 7a + 49 = a^2 + 14a + 49$ ☹

Hey, that didn't work. Let's try 49×1:

$(a + 49)(a + 1) = a^2 + a + 49a + 49 = a^2 + 50a + 49$ ☹

How can we arrange the factors of 49 so that there's no middle term when the factors are double distributed. How about one of each sign?

$(a + 7)(a - 7) = a^2 - 7a + 7a - 49 = a^2 - 49$

The middle term's gone, but the 49 is the wrong sign.

None of our attempts panned out. As we saw before, not every expression can be factored. So we say that $a^2 + 49$ is

$$\boxed{\text{Not factorable}}$$

First Notice: Even though $a^2 + 49$ is not factorable, $a^2 - 49$ is, since $a^2 - 49 = (a + 7)(a - 7)$. The difference between a plus sign and a minus sign makes all the difference in the world -- so be careful!

Second Notice: Some students jump to the conclusion that when two terms are separated by a plus sign, the expression is not factorable. Consider $4x^2 + 16$. It may not factor with two sets of parentheses, but it does have a common factor of 4, which can be factored out to produce $4(x^2 + 4)$. So $4x^2 + 16$ is factorable.

Homework

13. Factor each expression:

a. $2x^2 + 3x + 1$ b. $3n^2 - 7n + 2$ c. $5a^2 + 3a - 2$
d. $3m^2 - 11m - 20$ e. $4x^2 - 3x - 1$ f. $6u^2 + 7u - 10$
g. $4z^2 - 4z - 3$ h. $6y^2 - 5y - 6$ i. $7n^2 - 45n + 18$

14. Factor each expression:

a. $x^2 + 5x + 6$ b. $x^2 - 5x + 6$ c. $x^2 - 5x - 6$
d. $x^2 + 5x - 6$ e. $n^2 + 10n + 9$ f. $z^2 - 4z - 5$
g. $t^2 - 20t + 96$ h. $u^2 - 6u - 16$ i. $Q^2 + 34Q - 72$

15. Factor each expression:

a. $x^2 + 8x + 16$ b. $y^2 - 10y + 25$ c. $a^2 + 18a + 81$
d. $b^2 - 20b + 100$ e. $4z^2 + 4z + 1$ f. $9n^2 - 24n + 16$
g. $25x^2 - 30x + 9$ h. $x^2 + 6x + 36$ i. $2t^2 + 33t + 100$

16. Factor each expression:

a. $p^2 - 1$
b. $c^2 - 4$
c. $R^2 - 16$
d. $z^2 - 36$
e. $x^2 - 25$
f. $y^2 - 81$
g. $n^2 - 10$
h. $w^2 + 16$
i. $a^2 - 144$
j. $e^2 - 72$
k. $m^2 + 100$
l. $W^2 - 1$

17. Factor each expression:

a. $4x^2 - 9$
b. $9y^2 - 49$
c. $u^2 - 2$
d. $v^2 + 1$
e. $16z^2 - 49$
f. $49w^2 - 16$
g. $49a^2 - 144$
h. $121b^2 - 64$
i. $9x^2 + 25$
j. $1 - x^2$
k. $16 - n^2$
l. $25 - 4g^2$
m. $9 + t^2$
n. $144N^2 - 169$
o. $225a^2 - 1$

18. Factor each expression:

a. $3x^2 + 10x - 8$
b. $t^2 - 121$
c. $y^2 + 10y + 25$
d. $16a^2 - 121$
e. $b^2 - 20$
f. $n^2 + 121$
g. $x^2 + 3x + 1$
h. $12q^2 - 23q + 5$
i. $6a^2 - 13a + 6$
j. $x^2 + 14x + 13$
k. $4y^2 - 49$
l. $9Q^2 + 12Q + 4$
m. $25z^2 - 10z + 1$
n. $16x^2 + 34x - 15$
o. $16x^2 +118x - 15$
p. $16x^2 - 77x - 15$
q. $16x^2 - 72x + 45$
r. $16a^2 - 8a + 1$
s. $x^2 + 7x + 5$
t. $8c^2 + 2c - 21$
u. $8c^2 - 13c - 21$

Practice Problems

19. Factor each expression:

a. $4x + 12$
b. $9x - 9$
c. $7y^2 + 13y$
d. $2n^2 + 8n$
e. $10w^2 - 25w$
f. $8x + 11$
g. $-x + 3$
h. $14x^2 + 21x + 28$
i. $10n^2 + 10n$

20. Factor each expression:

a. $3x - 12$
b. $9x + 9$
c. $7y^2 - 14y$
d. $2n^2 - 10n$
e. $10w^2 + 45w$
f. $8x + 13$
g. $-x - 4$
h. $14n^2 - 21n + 35$
i. $20n^2 - 20n$

21. Factor each expression:

a. $x^2 + 17x + 72$
b. $y^2 - 9y + 8$
c. $N^2 + 100$
d. $N^2 - 100$
e. $x^2 - 18x + 81$
f. $a^2 + 10a + 25$
g. $t^2 + 4t - 45$
h. $a^2 - 21a - 22$
i. $a^2 - 9a - 22$
j. $2x^2 - 9x - 5$
k. $6x^2 + x - 40$
l. $6x^2 - 13x - 15$
m. $6x^2 + 11x - 17$
n. $R^2 - 144$
o. $25n^2 - 30n + 9$
p. $T^2 + 144$
q. $49w^2 + 70w + 25$
r. $9a^2 - 9a + 2$
s. $9a^2 - 19a + 2$
t. $x^2 + 16x - 36$
u. $x^2 + 37x + 36$
v. $30c^2 - 11c + 1$
w. $16a^2 - 6a - 1$
x. $36q^2 + 60q + 25$
y. $x^2 + 8x + 18$
z. $32n^2 - 66n + 27$

22. Solve for x: $a(x - 2) = b(c - x)$

23. Factor: $25y^2 - 49$

24. Factor: $4x^2 + 25$

25. Factor: $9n^2 + 6n + 1$

26. Factor: $14x^2 - 43x + 3$

Solutions

1. Here's what I would do. First, the given expression, $20x + 30y$, consists of two terms, and the result, $10(2x + 3y)$, consists of one term. Since factoring is the process of converting two or more terms into a single term, so far so good. Moreover, if I take my answer, $10(2x + 3y)$, and distribute to remove the parentheses, I will get $20x + 30y$, the original problem. I hope your buddy is now convinced.

2. While it may be true that the original expression consists of two terms, and her answer consists of one term, there's still one big problem. Ask her to take her answer, $6(w + 3z)$, and distribute it to remove the parentheses. She will get $6w + 18z$, which is <u>not</u> equal to the original problem. Therefore, her factorization can't possibly be right.

3. a. $x + z$ b. $P - Q$ c. $x - 4$ d. $2y - 3t$
e. $u + 1$ f. $n - 4$ g. $a - b$ h. $c + d$
i. $x + 2y - 4z$ j. $w - u + z$ k. $2x - 3$ l. $2a^2 + 3a - 4$

4. a. $3(P + Q)$ b. $9(n - 3)$ c. $n(c + d)$
d. $x(w - y)$ e. $7(t - 1)$ f. $x(1 + y)$
g. $-2(4L - 5)$ h. $-b(a + c)$ i. $-(u + 5)$
j. $-(z + x - 10)$ k. $2(x + y + z)$ l. $5(a - 2b + 3c)$

5. a. $a + 2b$ b. $3u - 1$ c. $Q - 3R$ d. $3x + 2$
e. $y - 2$ f. $2a + 3b$ g. $t + 4$ h. $3w - 4z$
i. $5a - 4$ j. $y + 1$

6. a. $cn + dn = 3 \;\Rightarrow\; n(c+d) = 3 \;\Rightarrow\; \dfrac{n\cancel{(c+d)}}{\cancel{c+d}} = \dfrac{3}{c+d} \;\Rightarrow\; n = \dfrac{3}{c+d}$

b. $n = \dfrac{d}{a-c}$

c. $n = \dfrac{M}{L+1}$

d. $tn = c - sn \;\Rightarrow\; tn + sn = c \;\Rightarrow\; n(t+s) = c \;\Rightarrow\; n = \dfrac{c}{t+s}$

e. $n = \dfrac{3}{r-t}$

f. $m(n+1) - Qn - R = 0 \;\Rightarrow\; mn + m - Qn - R = 0$
$\Rightarrow\; mn - Qn + m - R = 0 \;\Rightarrow\; mn - Qn = R - m$
$\Rightarrow\; n(m-Q) = R - m \;\Rightarrow\; n = \dfrac{R-m}{m-Q}$

g. $a(n+3) + b(n+c) = R \;\Rightarrow\; an + 3a + bn + bc = R$
$\Rightarrow\; an + bn = R - 3a - bc \;\Rightarrow\; n(a+b) = R - 3a - bc$
$\Rightarrow\; n = \dfrac{R-3a-bc}{a+b}$

h. $n(a+b+c) = d \;\Rightarrow\; n = \dfrac{d}{a+b+c}$

i. $n(a-1) = a \;\Rightarrow\; n = \dfrac{a}{a-1}$

j. $2n + 2 + an = c \;\Rightarrow\; n(2+a) = c - 2 \;\Rightarrow\; n = \dfrac{c-2}{a+2}$

7. a. $x = \dfrac{14}{c+7}$ b. $x = \dfrac{w+v}{r-u}$ c. $x = \dfrac{w}{w-1}$
d. $x = \dfrac{c+b}{a+d}$ e. $x = \dfrac{w+au}{u+1}$ f. $x = \dfrac{b-a}{a+b}$
g. $x = \dfrac{z+u-3w}{m-3}$ h. $x = \dfrac{2q+3p}{p-q}$

i. $x = \dfrac{-b-ac}{c-a}$, or, upon multiplying top and bottom by –1, $\dfrac{b+ac}{a-c}$

j. $x = \dfrac{R-Q-2c-ab}{-a-c}$, or, upon multiplying top and bottom by –1,
$\dfrac{Q+2c+ab-R}{a+c}$

8. a. $2x^2 + 7x - 4$ b. $6n^2 - 21n + 15$ c. $3a^2 + 26a - 9$
d. $12y^2 + 40y - 7$ e. $4m^2 - 49$ f. $16w^2 + 40w + 25$
g. $25x^2 - 1$ h. $14n^2 + n - 30$ i. $49u^2 - 42u + 9$
j. $144a^2 - 169$

9. a. T b. T c. F d. T e. T f. F g. T h. F

10. a. 3 b. 4 c. 6 d. 8 e. 7 f. 1 g. 2 h. 5

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11. a. $x - 2$ b. $n - 2$ c. $a - 8$ d. $q - 7$ e. $c + 3$

12. a. 5, 3 b. 3, 3 c. 1, 1 d. 4, 4 e. 7, 1
f. 7, 7 g. 15, 15 h. 3, 3 i. 1, 8

13. a. $(2x + 1)(x + 1)$ b. $(3n - 1)(n - 2)$ c. $(5a - 2)(a + 1)$
d. $(3m + 4)(m - 5)$ e. $(4x + 1)(x - 1)$ f. $(6u - 5)(u + 2)$
g. $(2z + 1)(2z - 3)$ h. $(2y - 3)(3y + 2)$ i. $(7n - 3)(n - 6)$

14. a. $(x + 3)(x + 2)$ b. $(x - 2)(x - 3)$ c. $(x - 6)(x + 1)$
d. $(x + 6)(x - 1)$ e. $(n + 9)(n + 1)$ f. $(z - 5)(z + 1)$
g. $(t - 12)(t - 8)$ h. $(u - 8)(u + 2)$ i. $(Q + 36)(Q - 2)$

15. a. $(x + 4)^2$ b. $(y - 5)^2$ c. $(a + 9)^2$
d. $(b - 10)^2$ e. $(2z + 1)^2$ f. $(3n - 4)^2$
g. $(5x - 3)^2$ h. Not factorable i. $(2t + 25)(t + 4)$

16. a. $(p + 1)(p - 1)$ b. $(c + 2)(c - 2)$ c. $(R + 4)(R - 4)$
d. $(z + 6)(z - 6)$ e. $(x + 5)(x - 5)$ f. $(y + 9)(y -9)$
g. Not factorable h. Not factorable i. $(a + 12)(a - 12)$
j. Not factorable k. Not factorable l. $(W + 1)(W - 1)$

17. a. $(2x + 3)(2x - 3)$ b. $(3y + 7)(3y - 7)$ c. Not factorable
d. Not factorable e. $(4z + 7)(4z - 7)$ f. $(7w + 4)(7w - 4)$
g. $(7a + 12)(7a - 12)$ h. $(11b + 8)(11b - 8)$ i. Not factorable
j. $(1 + x)(1 - x)$ k. $(4 + n)(4 - n)$ l. $(5 + 2g)(5 - 2g)$
m. Not factorable n. $(12N + 13)(12N - 13)$ o. $(15a + 1)(15a - 1)$

18. a. $(3x - 2)(x + 4)$ b. $(t + 11)(t - 11)$ c. $(y + 5)^2$
d. $(4a + 11)(4a - 11)$ e. Not factorable f. Not factorable
g. Not factorable h. $(3q - 5)(4q - 1)$ i. $(2a - 3)(3a - 2)$
j. $(x + 1)(x + 13)$ k. $(2y + 7)(2y - 7)$ l. $(3Q + 2)^2$
m. $(5z - 1)^2$ n. $(8x - 3)(2x + 5)$ o. $(8x - 1)(2x + 15)$
p. $(16x + 3)(x - 5)$ q. $(4x - 3)(4x - 15)$ r. $(4a - 1)^2$
s. Not factorable t. $(4c + 7)(2c - 3)$ u. $(8c - 21)(c + 1)$

19. a. $4(x + 3)$ b. $9(x - 1)$ c. $y(7y + 13)$
d. $2n(n + 4)$ e. $5w(2w - 5)$ f. Not factorable
g. $-(x - 3)$ h. $7(2x^2 + 3x + 4)$ i. $10n(n + 1)$

20. a. $3(x - 4)$
b. $9(x + 1)$
c. $7y(y - 2)$
d. $2n(n - 5)$
e. $5w(2w + 9)$
f. Not factorable
g. $-(x + 4)$
h. $7(2n^2 - 3n + 5)$
i. $20n(n - 1)$

21. a. $(x + 9)(x + 8)$
b. $(y - 1)(y - 8)$
c. Not factorable
d. $(N + 10)(N - 10)$
e. $(x - 9)^2$
f. $(a + 5)^2$
g. $(t + 9)(t - 5)$
h. $(a - 22)(a + 1)$
i. $(a - 11)(a + 2)$
j. $(2x + 1)(x - 5)$
k. $(3x + 8)(2x - 5)$
l. $(6x + 5)(x - 3)$
m. $(6x + 17)(x - 1)$
n. $(R + 12)(R - 12)$
o. $(5n - 3)^2$
p. Not factorable
q. $(7w + 5)^2$
r. $(3a - 1)(3a - 2)$
s. $(9a - 1)(a - 2)$
t. $(x + 18)(x - 2)$
u. $(x + 36)(x + 1)$
v. $(5c - 1)(6c - 1)$
w. $(8a + 1)(2a - 1)$
x. $(6q + 5)^2$
y. Not factorable
z. $(16n - 9)(2n - 3)$

22. $x = \dfrac{bc + 2a}{a + b}$

23. $(5y + 7)(5y - 7)$

24. Not factorable

25. $(3n + 1)^2$

26. $(14x - 1)(x - 3)$

"Ninety-nine percent of the failures come from people who have the habit of making excuses."

– George Washington

CH 13 – BREAK-EVEN POINT, PART II

❑ *INTRODUCTION*

Back in Chapter 7, we discussed the notions of revenue (R), expenses (E), profit (P), and the break-even point:

Profit: $P = R - E$

Break-even: $R = E$ or $P = 0$

Now that we're getting proficient at factoring, we can solve some more break-even business problems; these problems will result in a **quadratic equation**, which can be defined as an equation where the variable is squared. An example of a quadratic equation is $x^2 + 5x + 6 = 0$.

❑ *CONFIRMING THE SOLUTIONS OF A QUADRATIC EQUATION*

Let's look at the solutions of the quadratic equation

$$x^2 - 10x + 16 = 0$$

First, let's verify that <u>$x = 2$</u> is a solution of this equation (don't worry about where the 2 came from):

$$x^2 - 10x + 16 = 0$$

$$\mathbf{2}^2 - 10(\mathbf{2}) + 16 \overset{?}{=} 0$$

$$\begin{aligned} 4 - 20 + 16 &\overset{?}{=} 0 \\ -16 + 16 &\overset{?}{=} 0 \\ 0 &= 0 \quad ✓ \end{aligned}$$

Fine -- we have a solution. Here comes the (possibly) surprising fact: This equation has another solution, namely $\underline{x = 8}$. Watch this:

$$\begin{aligned} x^2 - 10x + 16 &= 0 \\ \mathbf{8}^2 - 10(\mathbf{8}) + 16 &\overset{?}{=} 0 \\ 64 - 80 + 16 &\overset{?}{=} 0 \\ -16 + 16 &\overset{?}{=} 0 \\ 0 &= 0 \quad ✓ \end{aligned}$$

One equation with <u>two</u> solutions? Yep, that's what we have. This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: we call it a ***quadratic equation***.

Homework

1. For each quadratic equation, verify that the two given solutions are really solutions:

   a. $x^2 + 3x - 10 = 0$ $\qquad x = -5;\ x = 2$

   b. $n^2 - 25 = 0$ $\qquad n = 5;\ n = -5$

   c. $a^2 + 7a = -12$ $\qquad a = -3;\ a = -4$

   d. $w^2 = 7w + 18$ $\qquad w = 9;\ w = -2$

   e. $2y^2 + 8y = 0$ $\qquad y = -4;\ y = 0$

SOLVING QUADRATIC EQUATIONS

We know how to check that a number is indeed a solution to a quadratic equation, and we've learned that a quadratic equation can have two different solutions. It's time to begin the discussion of solving such equations, and we'll begin with a quadratic equation given to us in factored form.

Consider the quadratic equation

$$(x+3)(x-7) \;=\; 0$$

Can you see why this is a quadratic equation?

There are two solutions to this equation -- what are they? Before we present the formal process, here's what you should note: What would happen if we let $x = -3$ in the equation $(x+3)(x-7) \;=\; 0$? We'd get

$$(\mathbf{-3}+3)(\mathbf{-3}-7) \;=\; (0)(-10) \;=\; 0 \quad \checkmark$$

Look at that! We've stumbled upon a solution of the equation $(x+3)(x-7) \;=\; 0$. Let's "stumble" one more time and choose $x = 7$:

$$(\mathbf{7}+3)(\mathbf{7}-7) \;=\; (10)(0) \;=\; 0 \quad \checkmark$$

Can you see how we stumbled across these two solutions, –3 and 7? Each solution was chosen so that one of the two factors would turn into zero. That way, the product of that zero factor with the other factor (no matter what it is) would <u>have</u> to be zero.

If $a \times b = 0$,
then $a = 0$
or $b = 0$

Now let's solve the quadratic equation $(x-9)(x+17) \;=\; 0$ without stumbling upon the solutions. Here's our reasoning: Since we have two factors whose product is 0, we know that either of the two factors could be 0. Setting each factor to 0 gives two possibilities:

$$x - 9 \;=\; 0 \;\;\Rightarrow\;\; \boldsymbol{x \;=\; 9}$$

$x + 17 \;=\; 0 \;\;\Rightarrow\;\; \boldsymbol{x \;=\; -17}$, and we have our two solutions.

A quadratic equation is said to be in **standard form** when the order of the terms in the equation is the squared term first, followed by the linear term, followed by the constant, followed by the equals sign, followed by a zero. For example, our next example, $3x^2 - 7x - 40 = 0$, is already in standard form. If a quadratic equation is not given to us in standard form, a little algebra can always convert it to standard form.

EXAMPLE 1: **Solve for x: $3x^2 - 7x - 40 = 0$**

Solution: This equation is in standard quadratic form, so it's all set to factor:

$3x^2 - 7x - 40 = 0$ (the original equation)

$\Rightarrow \quad (3x + 8)(x - 5) = 0$ (factor the left side)

$\Rightarrow \quad 3x + 8 = 0 \text{ or } x - 5 = 0$ (set each factor to 0)

$\Rightarrow \quad x = \frac{-8}{3} = -\frac{8}{3} \text{ or } x = 5$ (solve each equation)

Thus, the final solutions to the quadratic equation are

$$\boxed{5, -\frac{8}{3}}$$

EXAMPLE 2: **Solve for y: $9y^2 - 16 = 0$**

Solution: It's a good-looking quadratic (even though the middle term is missing), so let's factor and set the factors to zero:

$9y^2 - 16 = 0$ (the original equation)

$\Rightarrow \quad (3y + 4)(3y - 4) = 0$ (factor the left side)

$\Rightarrow \quad 3y + 4 = 0 \text{ or } 3y - 4 = 0$ (set each factor to 0)

$\Rightarrow \quad y = -\frac{4}{3} \text{ or } y = \frac{4}{3}$ (solve each equation)

Therefore, the solutions of the equation are

$$\boxed{\frac{4}{3},\ -\frac{4}{3}} \qquad \text{which can also be written } \pm\frac{4}{3}.$$

<u>EXAMPLE 3:</u> **Solve for u: $9u^2 = 42u - 49$**

<u>Solution:</u> This quadratic equation is <u>not</u> in standard form, so the first steps will be to transform it into standard form:

$$9u^2 = 42u - 49 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad 9u^2 - 42u = -49 \qquad \text{(subtract } 42u\text{)}$$

$$\Rightarrow \quad 9u^2 - 42u + 49 = 0 \qquad \text{(add 49)}$$

$$\Rightarrow \quad (3u - 7)(3u - 7) = 0 \qquad \text{(factor)}$$

$$\Rightarrow \quad 3u - 7 = 0 \ \text{ or } \ 3u - 7 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad u = \frac{7}{3} \ \text{ or } \ u = \frac{7}{3} \qquad \text{(solve each equation)}$$

We obtained two solutions, but they're the same, so we really have just one solution:

$$\boxed{\frac{7}{3}}$$

Homework

2. Solve each quadratic equation:

a. $x^2 + 5x - 14 = 0$

b. $25y^2 = 4$

c. $z^2 + 1 = -2z$

d. $4a^2 = 3 - 4a$

e. $6u^2 = 47u + 8$ f. $0 = 4t^2 + 8t + 3$

g. $-n^2 + n + 56 = 0$ [Hint: multiply each side by -1]

h. $-2x^2 - 7x + 15 = 0$ i. $144n^2 - 49 = 0$

j. $81q^2 = 126q - 49$ k. $-30a^2 + 13a + 3 = 0$

☐ BREAK-EVEN

EXAMPLE 4: **Find the break-even point(s) if the profit formula is given by $P = 2w^2 - 31w + 84$.**

Solution: We find the break-even points by setting the profit formula to zero:

$$
\begin{aligned}
& 2w^2 - 31w + 84 = 0 && \text{(set profit to 0)} \\
\Rightarrow\ & (2w - 7)(w - 12) = 0 && \text{(factor)} \\
\Rightarrow\ & 2w - 7 = 0 \ \text{ or } \ w - 12 = 0 && \text{(set each factor to 0)} \\
\Rightarrow\ & 2w = 7 \ \text{ or } \ w = 12 && \text{(solve each equation)} \\
\Rightarrow\ & w = 3\tfrac{1}{2} \ \text{ or } \ w = 12 &&
\end{aligned}
$$

Thus, the break even points are

$$\boxed{3\tfrac{1}{2} \text{ widgets and 12 widgets}}$$

Of course, $3\frac{1}{2}$ widgets can't really exist, but it's a good enough answer for this problem.

EXAMPLE 5: **Find the break-even point(s) if revenue and expenses are given by the formulas**

$$\boldsymbol{R = 3w^2 - 3w - 8}$$
$$\boldsymbol{E = 2w^2 + 30w - 268}$$

Solution: Recall that one of the two ways to describe the **break-even points** is by equating revenue and expenses. Notice that we put the resulting equation in standard form by bringing all the terms on the right side to the left side so that the right side will be zero.

$R = E$ (to find break-even)

$\Rightarrow \quad 3w^2 - 3w - 8 = 2w^2 + 30w - 268$ (use the given formulas)

$\Rightarrow \quad w^2 - 3w - 8 = 30w - 268$ (subtract $2w^2$)

$\Rightarrow \quad w^2 - 33w - 8 = -268$ (subtract $30w$)

$\Rightarrow \quad w^2 - 33w + 260 = 0$ (add 268 $\Rightarrow$ standard form)

$\Rightarrow \quad (w - 20)(w - 13) = 0$ (factor)

$\Rightarrow \quad w - 20 = 0 \quad \text{or} \quad w - 13 = 0$ (set each factor to 0)

$\Rightarrow \quad w = 20 \quad \text{or} \quad w = 13$ (solve each equation)

And so the two break-even points are

$$\boxed{\text{20 widgets and 13 widgets}}$$

Homework

3. Find the **break-even points** for the given profit formula:

a. $P = w^2 - 12w + 35$

b. $P = 2w^2 - 13w + 15$

c. $P = w^2 - 25w + 150$

d. $P = 6w^2 - 31w + 40$

4. Find the **break-even points** given the revenue and expense formulas:

a. $R = 2w^2 + 8w - 20$ $E = w^2 + 24w - 75$

b. $R = 2w^2 - 3w + 1$ $E = -w^2 + 16w - 29$

c. $R = 5w^2 - 26w + 80$ $E = 3w^2 + w + 10$

☐ *COMPLETE FACTORING AND MORE QUADRATICS*

Just as factoring 12 as 3 × 4 isn't complete using just one step (the complete factorization is 12 = 2 × 2 × 3), factoring an algebraic expression may require more than one step.

EXAMPLE 6: **Factor completely: $10x^2 + 50x + 60$**

Solution: Look at the 10. Its factor pairs are 1 and 10, or 2 and 5. Now take a gander at the 60. It's downright scary to consider all the pairs of factors of that number. But watch what happens if we deal with the greatest common factor first, and then worry about the rest later.

The variable x is not common to all three terms, so we'll ignore it. But each of the three terms does contain a factor of 10. Thus,

$$\begin{aligned} & 10x^2 + 50x + 60 && \text{(the given expression)} \\ = \; & 10(x^2 + 5x + 6) && \text{(pull out the GCF of 10)} \\ = \; & 10(x + 3)(x + 2) && \text{(factor the quadratic)} \end{aligned}$$

Not so difficult, after all. Therefore, the complete factorization of $10x^2 + 50x + 60$ is

$$\boxed{10(x + 3)(x + 2)}$$

**** The key to complete factoring is to FIRST pull out the GCF! ****

Homework

5. Factor each expression completely:

a. $4a^2 + 8b^2$
b. $6x^2 - 9x$
c. $15y^2 - 5y$
d. $30z^2 + 20z$
e. $7x - 10y$
f. $9x^2 + 10x$

6. Factor each expression completely:

a. $7x^2 - 35x + 42$
b. $10n^2 - 10$
c. $5a^2 - 30a + 45$
d. $50u^2 - 25u - 25$
e. $7w^2 - 700$
f. $9n^2 + 9$
g. $5y^2 - 125$
h. $3x^2 + 15x + 12$
i. $14x^2 - 7x - 7$
j. $13t^2 + 117$
k. $48z^2 - 28z + 4$
l. $24a^2 - 120a + 150$

Additional Quadratic Equations

Now we'll combine the GCF method of factoring with the methods of this chapter to solve more quadratic equations. The following example should convince you that factoring out a simple number first makes the rest of the factoring, and thus the solving of the equation, vastly easier.

EXAMPLE 7: **Solve for k: $16k^2 = 40k + 24$**

Solution: Solving a quadratic equation requires that we make one side of the equation zero. To this end, we will first bring the

40k and the 24 to the left side, factor in two steps, divide each side by the greatest common factor, set each factor to 0, and then solve each resulting equation.

$$16k^2 = 40k + 24 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad 16k^2 - 40k - 24 = 0 \qquad \text{(subtract } 40k \text{ and 24)}$$

$$\Rightarrow \quad 8(2k^2 - 5k - 3) = 0 \qquad \text{(factor out 8, the GCF)}$$

$$\Rightarrow \quad \frac{\cancel{8}(2k^2 - 5k - 3)}{\cancel{8}} = \frac{0}{8} \qquad \text{(divide by 8, the GCF)}$$

$$\Rightarrow \quad 2k^2 - 5k - 3 = 0 \qquad \text{(simplify)}$$

$$\Rightarrow \quad (2k + 1)(k - 3) = 0 \qquad \text{(factor)}$$

$$\Rightarrow \quad 2k + 1 = 0 \quad \text{or} \quad k - 3 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad \boxed{k = -\frac{1}{2} \quad \text{or} \quad k = 3} \qquad \text{(solve each equation)}$$

Warning!!

Do you see the step in the preceding example where we divided both sides of the equation by 8? This was legal because we did the same thing to both sides of the equation, and we did not divide by zero. Do not ever fall into the trap of dividing each side of an equation by an expression with the variable in it; that expression might be equal to zero. The upshot is that you may lose a solution to the equation.

For example, the correct way to solve the quadratic equation $x^2 + x = 0$ is as follows:

$$x^2 + x = 0$$

$$\Rightarrow \quad x(x + 1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x + 1 = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -1$$

That is, we have <u>two</u> solutions: **0** and **–1**.

<u>Check:</u> $\mathbf{0}^2 + \mathbf{0} = 0 + 0 = 0$ ✓

$(-\mathbf{1})^2 + (-\mathbf{1}) = 1 + (-1) = 1 - 1 = 0$ ✓

Now let's do it the <u>wrong</u> way:

$$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow \frac{\cancel{x}(x+1)}{\cancel{x}} = \frac{0}{x}$$

$$\Rightarrow x + 1 = 0 \Rightarrow x = -1,$$

which is merely one of the two solutions. That is, we lost a solution when we divided by the variable. Since the purpose of algebra is to obtain solutions -- not throw them away -- we see that dividing by the variable was a really bad idea.

Homework

7. Solve each quadratic equation:

a. $7x^2 - 35x + 42 = 0$
b. $10n^2 = 10$
c. $5a^2 + 45 = 30a$
d. $50u^2 = 25u + 25$
e. $7w^2 - 700 = 0$
f. $180z^2 - 30z - 60 = 0$
g. $4x^2 + 4x - 24 = 0$
h. $16x^2 = 6 - 4x$
i. $10x^2 - 490 = 0$
j. $75w^2 + 48 = 120w$

Practice Problems

8. Solve for x: $40x^2 + x - 6 = 0$

9. Solve for y: $100y^2 = 49$

10. Solve for n: $25n^2 + 9 = -30n$

11. If the profit formula is given by $P = w^2 - 57w + 350$, find the two break-even points.

12. If the revenue and expenses are given by the formulas $R = 5w^2 - 30w + 100$ and $E = 4w^2 + 4w - 180$, find the break-even points.

13. Solve by factoring: $30q^2 + 68q = -30$

14. Solve by factoring: $15x^2 = 95x + 70$

15. Solve by factoring: $16x^2 + 80x + 100 = 0$

Solutions

1. For each quadratic equation, substitute each "solution" (separately) into the original equation. Then work the arithmetic on each side of the equation separately. [Do NOT swap things back and forth across the equals sign.] In all cases, the two sides should balance at the end of the calculations.

2. a. $x = -7, 2$ b. $y = \pm\frac{2}{5}$ c. $z = -1$
d. $x = \frac{1}{2}, -\frac{3}{2}$ e. $u = -\frac{1}{6}, 8$ f. $t = -\frac{1}{2}, -\frac{3}{2}$
g. $n = 8, -7$ h. $x = -5, \frac{3}{2}$ i. $n = \pm\frac{7}{12}$
j. $q = \frac{7}{9}$ k. $a = \frac{3}{5}, -\frac{1}{6}$

3. a. $w = 5, 7$ b. $w = \frac{3}{2}, 5$ c. $w = 10, 15$ d. $w = \frac{8}{3}, \frac{5}{2}$

4. a. $w = 5, 11$ b. $w = \frac{10}{3}, 3$ c. $w = \frac{7}{2}, 10$

5. a. $4(a^2 + 2b^2)$ b. $3x(2x - 3)$ c. $5y(3y - 1)$
d. $10z(3z + 2)$ e. Not factorable f. $x(9x + 10)$

6. a. $7(x - 3)(x - 2)$ b. $10(n + 1)(n - 1)$ c. $5(a - 3)^2$
d. $25(2u + 1)(u - 1)$ e. $7(w + 10)(w - 10)$ f. $9(n^2 + 1)$
g. $5(y + 5)(y - 5)$ h. $3(x + 1)(x + 4)$ i. $7(2x + 1)(x - 1)$
j. $13(t^2 + 9)$ k. $4(4z - 1)(3z - 1)$ l. $6(2a - 5)^2$

7. a. $2, 3$ b. ± 1 c. 3 d. $1, -\frac{1}{2}$
e. ± 10 f. $\frac{2}{3}, -\frac{1}{2}$ g. $2, -3$ h. $\frac{1}{2}, -\frac{3}{4}$
i. ± 7 j. $\frac{4}{5}$

8. $x = \frac{3}{8}, -\frac{2}{5}$ **9**. $y = \pm\frac{7}{10}$ **10**. $n = -\frac{3}{5}$

11. $w = 7$ and $w = 50$ **12**. $w = 14$ and $w = 20$

13. $-\frac{3}{5}, -\frac{5}{3}$ **14**. $7, -\frac{2}{3}$ **15**. $x = -\frac{5}{2}$

“There is one purpose to life and one only: to bear witness to and understand as much as possible of the complexity of the world – its beauty, its mysteries, its riddles. The more you understand, the more you look – the greater is your enjoyment of life and your sense of peace. That's all there is to it. If an activity is not grounded in ‘*to love*’ or ‘*to learn*,’ it does not have value.”

– **Anne Rice**

CH 14 – INEQUALITIES AND ABSOLUTE VALUE EQUATIONS

❑ *INTRODUCTION*

You must score *between* 80% and 89% to get a B in your math class. You must be *at least* 18 years of age to vote. You can be *no taller* than 48 inches to play in the park. These are all examples of quantities being greater than something or less than something. Since they are not equalities, they are called ***inequalities***.

❑ *INEQUALITY SYMBOLS*

We know that 5 is bigger than 3, which we can write as "5 > 3." The symbol ">" can also be read as "is larger than" or "is greater than."

But, of course, the fact that 5 is larger than 3 is the same as the fact that 3 is less than 5. This is written "3 < 5."

> means "is greater than"

< means "is less than"

≥ means "is greater than or equal to"

≤ means "is less than or equal to"

The symbol "≥" can be read "is greater than or equal to." For example, $9 \ge 7$ because 9 is indeed greater than or equal to 7. (Actually, it's greater than 7, but that doesn't change the fact that it's greater than or equal to 7.) And believe it or not, $12 \ge 12$ is a true statement -- after all, since 12 = 12, it's certainly the case that 12 is greater than or equal to 12.

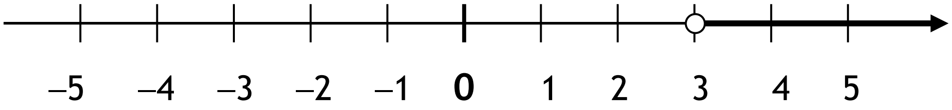
The symbol "≤" is read "less than or equal to." A couple of examples are $6 \le 10$ and $8 \le 8$.

☐ INTERVALS ON THE LINE

First Example: Consider all the real numbers greater than 3. One simple way to express this set of numbers is the following:

$$x > 3$$

We can also graph this set of numbers on a number line:

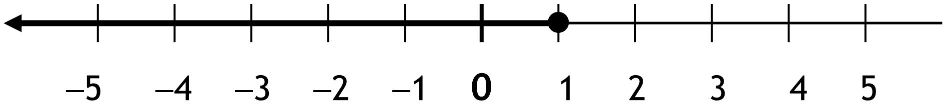


Notice that we put an "open dot" at $x = 3$ to indicate the 3 is not part of the set of numbers. But the arrow goes infinitely to the right because $x > 3$ is the set of numbers greater than 3. Whether written as an inequality or a graph on a number line, note that the numbers 3.01, π, 17, and 200 are part of the set; but the numbers –5, 0, 2.5 and 3 are not part of the set.

Second Example: Now we consider all the real numbers less than or equal to 1. This set of numbers can be written as an inequality like this:

$$x \le 1$$

As a graph on a number line, we write:



The “solid dot” is used to indicate that $x = 1$ is part of the set of the numbers. And since x must be less than or equal to 1, the arrow goes infinitely to the left. Either as an inequality or a graph, you should see that the numbers -3, -1.1, $\frac{7}{8}$, and 1 are part of the set, while the numbers 1.001 and $\sqrt{2}$ are not part of the set.

Homework

1. T/F:

   a. $7 > 3$
   b. $-2 < 1$
   c. $13 \geq 13$
   d. $-9 \leq -9$
   e. $12 \geq 9$
   f. $-18 \leq -20$
   g. $\pi > 0$
   h. $-\sqrt{2} \leq 0$

2. Express each statement as an inequality:

   a. Your age, a, must be at least 18 years.
   b. Your height, h, can be no taller than 48 inches.
   c. Your years of experience, y, must exceed 10 years.
   d. The number of driving tickets, t, must be fewer than 5.
   e. the mean, μ (Greek letter mu), must be at least 75.
   f. the standard deviation, σ (Greek letter sigma), must be no more than 10.
   g. the energy, E, must be greater than 100.
   h. the mass, m, must be less than 3.7.

☐ LINEAR INEQUALITIES

To solve an inequality such as $\frac{-2x+7}{3} \ge -4$, we have to perform the operations necessary to isolate the *x*. We will accomplish this goal by using the usual *do the same thing to each side* rule, but since we're talking about an inequality -- not an equation -- we have to be very careful. The following experiment should illustrate the potential problems, and how we can resolve them.

Let's perform six experiments. Consider the true statement

$$4 < 6$$

i.	Add 10 to each side:	$4 + \mathbf{10} < 6 + \mathbf{10} \Rightarrow 14 < 16$ ✓
ii.	Subtract 3 from each side:	$4 - \mathbf{3} < 6 - \mathbf{3} \Rightarrow 1 < 3$ ✓
iii.	Multiply each side by 5:	$4(\mathbf{5}) < 6(\mathbf{5}) \Rightarrow 20 < 30$ ✓
iv.	Divide each side by 2:	$\frac{4}{2} < \frac{6}{2} \Rightarrow 2 < 3$ ✓
v.	Multiply each side by –3:	$4(-3) < 6(-3) \Rightarrow -12 < -18$ ☹
vi.	Divide each side by –2:	$\frac{4}{-2} < \frac{6}{-2} \Rightarrow -2 < -3$ ☹

What can we deduce from these six calculations? The first two show that adding or subtracting the same number in an inequality is legal -- they lead to a true inequality, with no issues to worry about.

The next two calculations indicate that multiplying or dividing by a <u>positive</u> number always leads to an inequality without problems.

The last two cases, however -- multiplying or dividing by a <u>negative</u> number -- have led to false statements. So in these two scenarios, we must <u>reverse</u> the order of the inequality sign in order to maintain a true statement.

These numerical experiments are by no means a complete proof of the principle we are about to state, but they're convincing enough.

The Basic Principle of Inequalities

If you multiply or divide each side of an inequality by a negative number, you must reverse the inequality symbol.

EXAMPLE 1: **Solve each inequality:**

A. $\boldsymbol{x + 3 > 4}$

Subtracting 3 from each side gives $\boldsymbol{x > 1}$.

Note: The inequality symbol was not reversed.

B. $\boldsymbol{-2n - 9 \le 13}$

Add 9 to each side to get $-2n \le 22$.

(The inequality symbol was not reversed.)

Divide each side by -2: $\boldsymbol{n \ge -11}$.

This time the inequality symbol was reversed.

C. $\boldsymbol{\frac{u}{5} + 3 < -4}$

Subtracting 3 gives $\frac{u}{5} < -7$.

Multiplying by 5 gives $\boldsymbol{u < -35}$.

Neither operation required reversing the inequality symbol

D. $\mathbf{\dfrac{-2x-5}{-3} \ge 11}$

Multiplying each side of the inequality by –3, and reversing the inequality sign gives:

$$-2x - 5 \le -33$$

Adding 5 to each side of the inequality does not require reversing the inequality sign:

$$-2x \le -28$$

Dividing each side of the inequality by –2, and remembering to reverse the inequality sign, we get the solution:

$$\mathbf{x \ge 14}$$

Note: Although the final inequality symbol (≥) is the same as the one in the given problem, notice that we reversed it twice to get that answer. If you get the right answer on an exam <u>without</u> reversing it twice, it will probably be marked wrong.

Homework

3. Solve each inequality:

   a. $x + 7 > -10$ b. $x - 3 \le 5$ c. $2x \ge 14$

   d. $-3x < -42$ e. $\frac{x}{8} > -3$ f. $\frac{x}{-5} \ge 10$

4. a. Solve for y: $6(y - 5) - (y + 1) \le 12y + 21$

   b. Solve for u: $-2(7 + 3u) - (1 - u) > 2u - 10$

5. a. Solve for x: $\frac{-3x+5}{-2}+9 > 16$ Note: The process is more important than the answer. Do you know what this means?

   b. Solve for a: $\frac{7a-5}{-3}-12 < 12$

   c. Solve for y: $\frac{9-y}{-2} \ge -2$

   d. Solve for n: $\frac{-14-2n}{12} \le -1$

6. Marty was trying to solve the inequality $ax + b > c$ for x, and wrote

$$ax + b > c$$
$$\Rightarrow \quad ax > c - b$$
$$\Rightarrow \quad x > \frac{c-b}{a}$$

   Explain the fallacy in Marty's reasoning.

☐ *THE MEANING OF ABSOLUTE VALUE*

The Prologue gave us our first encounter with the notion of absolute value, while Chapter 2 taught us about the graphs of absolute values. We recall that we use a pair of vertical bars to represent absolute value; the absolute value of x is written $|x|$.

If a number is greater than or equal to 0, then its absolute value is that same number. If a number is less than 0 (which means it's a negative number), then its absolute value is the <u>opposite</u> of that number (which will be a positive number). We can write this in the following way:

If $x \ge 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

This definition ensures that the absolute value of a quantity is never negative.

Here are some examples:

$$|9| = 9 \qquad |0| = 0 \qquad |-13| = 13$$

An Alternative Definition of Absolute Value

Consider a number line and ask “What is the distance between a given number and 0? The answer to that question is the *absolute value* of that number.

What is the distance between 9 and 0? It’s 9, so $|9| = 9$.

What is the distance between 0 and 0? The distance is 0, so $|0| = 0$.

And what is the distance between –13 and 0? Since it’s 13, $|-13| = 13$.

Note:

The absolute value of a quantity is either positive or zero.

If x is any real number, then $|x| \geq 0$.

In other words, the absolute value of a quantity is never negative. To illustrate this point, although you may not be able to calculate

$$\left|\sin^2(\pi/6) + \ln(e-1)\right|$$

until pre-calculus, you should still be able to understand that the answer to this problem must be greater than or equal to zero. In other words, this problem, no matter what all that gibberish means, is definitely not negative.

EXAMPLE 2: **Evaluate each expression:**

A. $|7-2| = |5| = \mathbf{5}$

B. $|3^2-15| = |9-15| = |-6| = \mathbf{6}$

C. $-|2-7| = -|-5| = \mathbf{-5}$

D. $|-3-4|-|10-2| = |-7|-|8| = 7-8 = \mathbf{-1}$

Homework

7. The absolute value of any real number is ______.

8. True/False:

   a. Every real number has an absolute value.
   b. There is a number whose absolute value is 0.
   c. There is a number whose absolute value is negative.
   d. There are two different numbers whose absolute value is 9.

9. Simplify each expression:

   a. $|4-14|$ b. $|2(-3)-7|$ c. $-|-9|$ d. $|2^0-\pi^0|$

☐ *ABSOLUTE VALUE EQUATIONS*

EXAMPLE 3: **Solve for x:** $|x| = 12$

Solution: What can you take the absolute value of and get a result of 12? Well, the absolute value of 12 is 12, so x could be 12. But -12 also has an absolute value of 12, so x could be -12, too.

In other words, since $|12| = 12$ and $|-12| = 12$, it appears that this equation has two solutions, 12 and –12.

$$\boxed{x = \pm 12}$$

EXAMPLE 4: **Solve for n: $|n| = -5$**

Solution: This equation is a statement that the absolute value of some number is –5. But the absolute value of any number is greater than or equal to zero; that is, the absolute value of any number can never be negative. Thus, there is no number n that will work in this equation. Our conclusion:

$$\boxed{\text{No solution}}$$

EXAMPLE 5: **Solve for w: $|w| = 0$**

Solution: What number has an absolute value of 0? There's only one such number, and it's 0: $|0| = 0$. Therefore,

$$\boxed{w = 0}$$

EXAMPLE 6: **Solve for x: $|8-2x| = 20$**

Solution: Here's what we ask ourselves: "What has an absolute value of 20?" There are two numbers that have an absolute value of 20, and they are 20 and –20. This means that the quantity inside the absolute value sign can be either 20 or –20. In other words, the quantity 8 – $2x$ can be either 20 or –20. This gives us two equations to solve:

$$\begin{array}{l|l} 8 - 2x = \mathbf{20} & 8 - 2x = \mathbf{-20} \\ -2x = 12 & -2x = -28 \\ x = -6 & x = 14 \end{array}$$

Our absolute value equation has two solutions:

$$\boxed{x = -6,\ 14}$$

Check:

$\underline{x = -6:}$ $|8-2x| = |8-2(-6)| = |8+12| = |20| = 20$ ✓

$\underline{x = 14:}$ $|8-2x| = |8-2(14)| = |8-28| = |-20| = 20$ ✓

Homework

10. Solve each absolute value equation:

a. $|t| = 4$

b. $|n| = 0$

c. $|R| = -1$

d. $|x+1| = 9$

e. $|x-3| = 5$

f. $|2x+8| = 0$

g. $|2-5x| = -3$

h. $|2x+8| = 10$

i. $|3y-6| = 9$

j. $|2a+1| = 19$

k. $|7-5y| = 3$

l. $|7x+\sqrt{7}| = -\frac{\pi}{2}$

m. $|t|-7 = 4$

n. $|n|-\pi = -\pi$

o. $|R|+7 = 3$

p. $|x-4| = 9$

q. $|x+20| = 5$

r. $|2x-17| = 0$

s. $|2-5x|-2 = -3$

t. $|9-2x| = 10$

u. $|3y-20| = 9$

v. $|2a+3| = 21$

w. $|5-7y| = 5$

x. $|\sqrt{7}\,x+\sqrt{7}| = -\sqrt{7}$

Practice Problems

11. a. What is the distance between –200 and 0 on the number line?
    b. What is the distance between 7π and 0 on the number line?
    c. What is the distance between 0 and 0 on the number line?
    d. What is the distance between $-7\sqrt{11}$ and 0 on the number line?

12. Evaluate: $|7| - |-4| + |\pi - \pi|$

13. a. If $x \ge 0$, then $|x| =$ ______
    b. If $x < 0$, then $|x| =$ ______

14. Which is the best statement regarding $|x|$, where x is any real number?

    a. $|x| < 0$ b. $|x| > 0$ c. $|x| \le 0$ d. $|x| \ge 0$

Solve each inequality or equation:

15. $-2n - 9 \ge -12$

16. $\dfrac{8+3x}{-5} < -10$

17. $-2(3y - 9) + 1 \le 10 - (3y - 4) - y$

18. $|x| = 44$

19. $|y| + 17 = 15$

20. $|z - 1| = 0$

21. $|2x + 9| = 10$

22. $|7n - 20| = 13$

23. $2a - 9 < 9a + 7$

24. $-5x + 5 \ge -9x - 10$

25. $\dfrac{10-2x}{-4} < -14$

26. $\dfrac{-8t+2t-10}{-3} \le -9$

Solutions

1. a. T b. T c. T d. T
e. T f. F g. T h. T

2. a. $a \geq 18$ b. $h \leq 48$ c. $y > 10$ d. $t < 5$
e. $\mu \geq 75$ f. $\sigma \leq 10$ g. $E > 100$ h. $m < 3.7$

3. a. $x > -17$ b. $x \leq 8$ c. $x \geq 7$
d. $x > 14$ e. $x > -24$ f. $x \leq -50$

4. a. $y \geq -\frac{52}{7}$ b. $u < -\frac{5}{7}$

5. a. $x > \frac{19}{3}$ b. $a < -\frac{67}{7}$ c. $y \geq 5$ d. $n \geq -1$

6. Marty divided each side of the inequality by a, which could be positive or negative. Do you see the problem now?

7. ≥ 0

8. a. T b. T c. F d. T

9. a. 10 b. 13 c. –9 d. 0

10. a. $t = \pm 4$ b. $n = 0$ c. No solution d. $x = 8, -10$
e. $x = 8, -2$ f. $x = -4$ g. No solution h. $x = 1, -9$
i. $y = 5, -1$ j. $a = 9, -10$ k. $y = \frac{4}{5}, 2$ l. No solution
m. $t = \pm 11$ n. $n = 0$ o. No solution p. $x = 13, -5$
q. $x = -15, -25$ r. $x = \frac{17}{2}$ s. No solution t. $x = \frac{-1}{2}, \frac{19}{2}$
u. $y = \frac{29}{3}, \frac{11}{3}$ v. $a = 9, -12$ w. $y = 0, \frac{10}{7}$ x. No solution

11. a. 200 b. 7π c. 0 d. $7\sqrt{11}$

12. $7 - 4 + 0 = 3$

13. a. x b. $-x$

14. d.

15. $n \le \frac{3}{2}$

16. $x > 14$

17. $y \ge \frac{5}{2}$

18. $x = \pm 44$

19. No solution

20. $z = 1$

21. $z = \frac{1}{2}, -\frac{19}{2}$

22. $n = \frac{33}{7}, 1$

23. $a > -\frac{16}{7}$

24. $x \ge -\frac{15}{4}$

25. $x < -23$

26. $t \le -\frac{37}{6}$

> "*Effort* only fully releases its reward after a person refuses to quit."
>
> – Napoleon Hill

CH 15 – VARIATION

☐ *INTRODUCTION*

Let's start right off with an example.

Let D = number of hard Drives sold
C = the Capacity of the drive, in gigabytes (GB)
P = selling Price

Now let's make up a formula that will illustrate the ideas of this chapter:

$$D = \frac{5C}{P}$$

① What happens to drives sold, *D,* when the capacity, *C,* is increased? Let's assume that last month the capacity was 200 GB and that the price was \$100. The number of drives sold was

$$D = \frac{5C}{P} = \frac{5 \cdot 200}{100} = \underline{\text{10 drives}}$$

Now increase the capacity to 600 GB; the number of drives sold will be

$$D = \frac{5C}{P} = \frac{5 \cdot 600}{100} = \underline{\text{30 drives}}$$

Make sense? Increasing the capacity (making the drives better) should increase sales.

② Now let's see what happens if we increase the price. Assume that last month the price P was \$150 when the capacity was 450 GB. We calculate the number of drives sold:

$$D = \frac{5C}{P} = \frac{5 \cdot 450}{150} = \underline{\text{15 drives}}$$

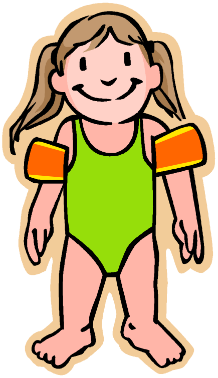
Let's predict future sales if we increase the price to \$225:

$$D = \frac{5C}{P} = \frac{5 \cdot 450}{225} = \underline{10 \text{ drives}}$$

Make sense? Increasing the price produced a decrease in sales.

Without going into the details, we could also demonstrate that if the capacity, *C,* goes down, so will the number of drives sold. And if the price, *P,* goes down, *D* will go up. Be sure these two statements seem logical to you.

☐ DIRECT AND INVERSE VARIATION



Direct Variation What happens to the sale of bathing suits, *B*, when the temperature, *T*, goes up? Probably, as the temperature rises, so do the sales of bathing suits. We say that *B* is ***directly proportional*** to *T*, or that *B* ***varies directly*** as *T*. A formula of this type might be

$$B = 3T$$

For example, if the temperature is 80°, then 240 suits will be sold. But if the temperature rises to 100°, then 300 units will be sold. When it gets cold again, *B* will decrease. Whatever *T* does, *B* does the **same**.

Inverse Variation What happens to the air pressure, *P*, as you increase your elevation, *E*? As your elevation goes up, the air pressure goes down (less air the higher you go). We say that *P* is ***inversely proportional*** to *E*, or that *P* ***varies inversely*** as *E*. An example might be the formula

$$P = \frac{2000}{E}$$

For instance, if $E = 200$, then $P = 10$. But if E is increased to 500, then the pressure P is reduced to 4. When you return to lower altitudes, the pressure goes back up. Whatever E does, P does the **reverse**.

See the numbers "3" and "2000" in our two formulas above? These numbers are called the ***constants of proportionality*** (or ***constants of variation***).

In the following definitions, the letter ***k*** is the positive constant of proportionality (or constant of variation).

$y = kx$ is read "*y* is ***directly proportional to*** *x*," or "*y* ***varies directly as*** *x*," and means: If *x* increases, then *y* increases; and if *x* decreases, then *y* decreases. In other words, *y* does whatever *x* does.

$y = \frac{k}{x}$ is read "*y* is ***inversely proportional to*** *x*," or "*y* ***varies inversely as*** *x*," and means: If *x* increases, then *y* decreases; and if *x* decreases, then *y* increases. In other words, *y* does the reverse of what *x* does.

Homework

1. Consider the **direct variation** $D = 12Q$. We can pretend that Q is the quality of a car and D is the demand for that car.

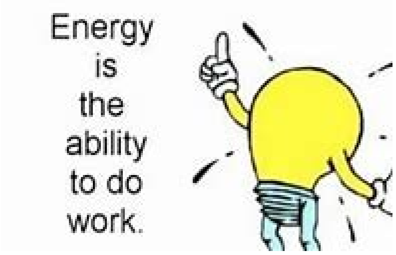
   a. Find the value of D if Q is 40.
   b. Double the Q to 80 and recalculate D.
   c. When the quality increased, what happened to the demand?
   d. Now reduce Q to 3 and recalculate D.
   e. As the quality decreased, what happened to the demand?

2. Consider the **inverse variation** $t = \frac{200}{r}$. Let's assume that r is rate (speed) and t is time.

   a. Find the value of t if $r = 10$.
   b. Quadruple the r to 40 and recalculate t.
   c. When the rate increased, what happened to the time?
   d. Now reduce r to 5 and recalculate t.
   e. As the rate decreased, what happened to the time?

☐ *EXTENSIONS OF THE BASIC VARIATION FORMULAS*

To extend the usefulness of problems in variation, we can add exponents and square roots to our direct and inverse variation formulas; for example, a fact of physics is that the kinetic energy of an object (energy of motion) varies directly as the square of its velocity, which can be written $E = kv^2$.



We can also combine direct and inverse variation into a single formula. For example, a chemistry principle states that "the volume of a gas is directly proportional to its temperature and inversely proportional to its pressure." This is summarized by the formula $V = \frac{kT}{P}$.

EXAMPLE 1: **Translate each variation statement into an algebraic formula, using *k* as the constant of variation:**

A. z varies directly as the cube of T: $z = kT^3$

B. R is inversely proportional to the square root of V: $R = \frac{k}{\sqrt{V}}$

C. P varies directly as the square of Q, and inversely as the square root of R: $P = \dfrac{kQ^2}{\sqrt{R}}$

D. B varies directly as the product of the square root of A and the cube of C: $B = k\sqrt{A}\,C^3$

E. y varies directly as the product of x and z, and inversely as the fourth power of w: $y = \dfrac{kxz}{w^4}$

Homework

3. Translate each variation statement into an algebraic formula, using ***k*** as the constant of proportionality:

   a. b varies inversely as the 6th power of t.

   b. n varies directly as the square root of b.

   c. p is inversely proportional to the square of s.

   d. v is directly proportional to c.

   e. p varies inversely as the cube of x.

   f. u is directly proportional to the square of t.

   g. t varies directly as the square of h.

   h. L is inversely proportional to the cube of s.

   i. w varies directly as the 9th power of u.

   j. R varies directly as the product of w and y.

   k. g is directly proportional to the cube of c.

   l. h varies directly as c, and inversely as y.

   m. x varies directly as the cube of t and inversely as the cube of h.

   n. u is directly proportional to the product of c and the cube of r, and inversely proportional to m.

o. y varies directly as the square of v and inversely as the cube of d.

p. n varies directly as the product of t and the square of d, and inversely as b.

q. m is directly proportional to w and inversely proportional to z.

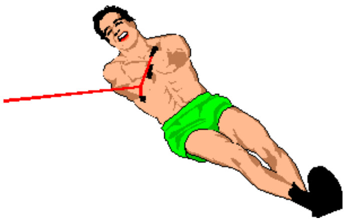
r. A varies directly as the product of u and the square of x, and inversely as p.

s. n is directly proportional to r and inversely proportional to v.

t. c varies directly as the cube of u and inversely as the cube of z.

❑ APPLICATIONS

<u>EXAMPLE 2:</u> **The number of bathing suits sold is directly proportional to the outside temperature, and inversely proportional to the selling price. The number of suits sold was 1,600 when the temperature was 80° and the selling price was $50. Find the number of suits sold when the temperature rises to 95° and the price is reduced to $40.**



<u>Solution:</u> We'll start by letting

B = bathing suits sold

T = temperature

P = price

The first sentence of the problem, "The number of bathing suits sold is directly proportional to the outside temperature, and inversely proportional to the selling price," gives us our variation formula:

$$B = \frac{kT}{P}$$

The second sentence "The number of suits sold was 1,600 when the temperature was 80° and the selling price is $50," allows us to find the value of k:

$$1{,}600 = \frac{k(80)}{50} \qquad \text{(substitute the given values)}$$

$$\Rightarrow \quad 80k = 80{,}000 \qquad \text{(multiply each side by 50)}$$

$$\Rightarrow \quad \underline{k = 1{,}000} \qquad \text{(divide each side by 80)}$$

The third sentence, "Find the number of suits sold when the temperature rises to 95° and the price is reduced to $40," (with the value $k = 1{,}000$ we just calculated) gives us all the parts needed to compute the number of bathing suits sold under the new set of conditions:

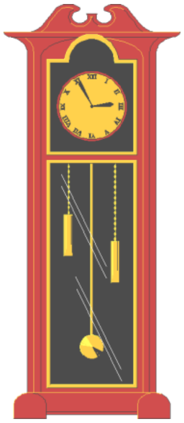
$$B = \frac{kT}{P} \qquad \text{(our variation formula)}$$

$$\Rightarrow \quad B = \frac{1{,}000T}{P} \qquad \text{(the constant } k \text{ is 1,000)}$$

$$\Rightarrow \quad B = \frac{1{,}000(95)}{40} \qquad \text{(use the new values of } T \text{ and } P\text{)}$$

$$\Rightarrow \quad B = \frac{95{,}000}{40}$$

$$\Rightarrow \quad \boxed{B = 2{,}375 \text{ bathing suits}}$$



EXAMPLE 3: **The *period* of the pendulum (the amount of time for one full swing) in a grandfather clock varies directly as the square root of its length. If the period is 50π when the length is 25, find the period when the length is 49.**

Solution: The first sentence gives us our variation formula:

$$P = k\sqrt{L}$$

Substituting a period of 50π and a length of 25 gives:

$$50\pi = k\sqrt{25}$$

$$\Rightarrow \quad 50\pi = 5k \qquad \text{(the positive square root of 25 is 5)}$$

$$\Rightarrow \quad \underline{k = 10\pi} \qquad \text{(divide each side by 5)}$$

Now we rewrite our variation formula using the k we just found:

$$P = k\sqrt{L} = 10\pi\sqrt{L}$$

Finally, find the period when the length is 49:

$$P = 10\pi\sqrt{49}$$

$$\Rightarrow \quad P = 10\pi(7)$$

$$\Rightarrow \quad \boxed{P = 70\pi}$$

Homework

4. The area of a rectangle is directly proportional to its length. If the area is 247 when the length is 19, find the area when the length is 11.

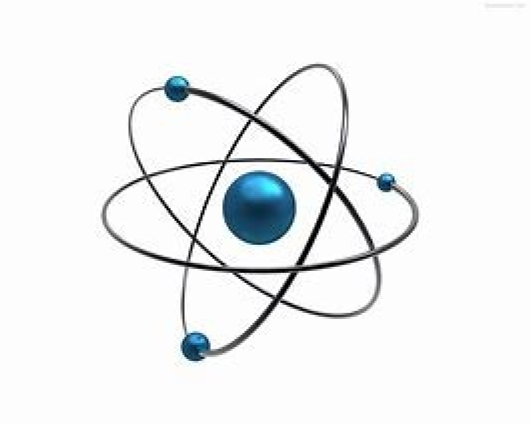
5. The circumference of a circle is directly proportional to its radius. If the circumference is 40π when the radius is 20, then what is the circumference when the radius is 8?

6. The current in a circuit varies directly as the voltage. If the current is 336 when the voltage is 21, find the current when the voltage is 35.

In a variation problem, DON'T forget "***k***", the *constant of variation*.

7. The area of a circle varies directly as the square of its radius. If the area is 9π when the radius is 3, then what is the area when the radius is 6?

8. The number of electrons revolving around the nucleus of an atom varies directly as the square of the energy level. If the number of electrons is 32 when the energy level is 4, then how many electrons are there when the energy level is 3?



9. The energy density is directly proportional to the fourth power of the temperature. If the energy is 512 when the temperature is 4, then what is the energy when the temperature is 5?

10. The density of an object varies inversely as the object's volume. If the density is 22 when the volume is 18, then what is the density when the volume is 6?

11. The acceleration of an object is inversely proportional to the object's mass. If the acceleration is 9 when the mass is 19, then what is the acceleration when the mass is 1?

12. The force of gravity varies inversely as the square of the distance between the objects. If the force is 4 when the distance is 8, then what is the force when the distance is 1?

13. The period of a pendulum varies directly as the square root of its length. If the period is 22π when the length is 121, then what is the period when the length is 49?

14. The velocity of an object is directly proportional to the square root of its kinetic energy. If the velocity is 524 when the kinetic energy is 4, then what is the velocity when the kinetic energy is 25?

15. The potential energy of an object varies directly as the product of its mass and its height. If the potential energy is 80 when the mass is 4 and the height is 2, find the potential energy if the mass is 14 and the height is 13.

16. The fluid force on an object is directly proportional to the product of its area and its depth. If the fluid force is 1920 when the area is 10 and the depth is 24, find the fluid force if the area is 23 and the depth is 5.

17. The volume of a gas varies directly as its temperature, and inversely as its pressure. If the volume is 12 when the temperature is 9 and the pressure is 12, find the volume when the temperature is 5 and the pressure is 16.

18. The electric field is directly proportional to the charge, and inversely proportional to the area. If the electric field is 13 when the charge is 13 and the area is 9, find the electric field when the charge is 15 and the area is 3.

Review Problems

19. Translate the variation statement into an algebraic formula, using k as the constant of variation: "E varies directly as the product of x and the cube of y, and inversely as the square root of z."

20. The volume of a gas varies directly as its temperature, and inversely as its pressure. If the volume is 80 when the temperature is 40 and the pressure is 5, find the volume when the temperature is 30 and the pressure is 6.

21. The current in a circuit varies directly as the voltage. If the current is 336 when the voltage is 21, find the current when the voltage is 35.

22. The acceleration of an object is inversely proportional to the object's mass. If the acceleration is 9 when the mass is 19, then what is the acceleration when the mass is 1?

23. The potential energy of an object varies directly as the product of its mass and its height. If the potential energy is 462 when the mass is 6 and the height is 7, find the potential energy if the mass is 24 and the height is 17.

24. The electric field is directly proportional to the charge, and inversely proportional to the area. If the electric field is 10 when the charge is 15 and the area is 3, find the electric field when the charge is 12 and the area is 6.

25. The kinetic energy of a particle varies directly as the product of its mass and the square of its velocity. Assume that a particle of mass 10 and traveling at a velocity of 8 has a kinetic energy of 320. Find the kinetic energy of a particle with mass 7 and velocity 10.

Solutions

1. a. 480 b. 960 c. It increased d. 36 e. It decreased

2. a. 20 b. 5 c. It decreased d. 40 e. It increased

3. a. $b = \frac{k}{t^6}$ b. $n = k\sqrt{b}$ c. $p = \frac{k}{s^2}$ d. $v = kc$

e. $p = \frac{k}{x^3}$ f. $u = kt^2$ g. $t = kh^2$ h. $L = \frac{k}{s^3}$

i. $w = ku^9$ j. $R = kwy$ k. $g = kc^3$ l. $h = \frac{kc}{y}$

m. $x = \frac{kt^3}{h^3}$ n. $u = \frac{kcr^3}{m}$ o. $y = \frac{kv^2}{d^3}$ p. $n = \frac{ktd^2}{b}$

q. $m = \frac{kw}{z}$ r. $A = \frac{kux^2}{p}$ s. $n = \frac{kr}{v}$ t. $c = \frac{ku^3}{z^3}$

4. 143 **5**. 16π **6**. 560 **7**. 36π

8. 18 **9**. 1,250 **10**. 66 **11**. 171

12. 256 **13**. 14π **14**. 1,310 **15**. 1,820

16. 920 **17**. 5 **18**. 45 **19**. $E = \frac{kxy^3}{\sqrt{z}}$

20. 50

21. In electronics, current is denoted by the letter *i*.

$i = kV \Rightarrow 336 = k \cdot 21 \Rightarrow k = 16 \Rightarrow i = 16V$,
so when $V = 35$, $i = 16 \cdot 35 = 560$.

22. $a = 171$

23. $E = kmh \Rightarrow 462 = k \cdot 6 \cdot 7 \Rightarrow k = 11 \Rightarrow E = 11mh$
$\Rightarrow E = 11(24)(17) = 4488$

24. $F = \frac{kC}{A}$; $k = 2$; $F = 4$

25. $E = kmv^2 \qquad E = 350$

"It is never too late to be what you might have been."

- George Eliot

CH 16 – MOTION PROBLEMS

❑ INTRODUCTION

Whether it's the CHP pursuing a bank robber, or a physicist determining the velocity of a proton in a cyclotron, the concepts of time, distance, and speed are at the heart of all science and technology.

If you travel from L.A. to San Francisco, 400 miles away, and you travel for 8 hours at an average speed of 50 miles per hour, then

you traveled a **DISTANCE** of 400 miles
at a **RATE** (speed) of 50 mph
during a **TIME** interval of 8 hours.

Notice that in this example, if you multiply the rate by the time (50 mph × 8 hrs), you get the distance (400 mi). This idea always holds:

Rate × Time = Distance

Homework

1. a. Moe traveled at a rate of 120 km/hr for 12 hours. Find Moe's distance.

   b. Larry flew a distance of 3000 miles in 6 hours. What was Larry's rate?

   c. Curly jogged 12 miles at a rate of 3 mph. How long was Curly jogging?

2. Which is the proper formula for distance?

   a. $d = rt$ b. $d = \frac{r}{t}$ c. $d = \frac{t}{r}$

3. Two skaters leave the skate park and skate in opposite directions, one at 10 mph and the other at 8 mph. After some time, they are 18 miles apart. If d_1 is the distance traveled by the first skater, and if d_2 is the distance traveled by the second skater, write an appropriate equation.

4. A jet ski leaves the beach. Nine hours later a motorboat begins to pursue the jet ski and finally catches up with it. If d_1 is the distance the jet ski travels, and if d_2 is the distance the motorboat travels, write an appropriate equation.

5. A woodpecker traveled from the maple tree to the oak tree at 13 mph, and then made a return trip at 19 mph. If d_1 is the distance he traveled to the oak tree, and if d_2 is the distance from the oak back to the maple, write an appropriate equation.

6. A 1096-mi trip took a total of 16 hours. The speed was 71 mph for the first part of the trip, and then decreased to 67 mph for the rest of the trip. If d_1 is the distance traveled on the first part of the trip, and if d_2 is the distance traveled on the second part of the trip, write an appropriate equation.

7. Mutt and Jeff leave the mall at the same time and head in the same direction. Jeff's speed is 9 mph more than 6 times Mutt's speed. Four hours later Jeff is 1036 miles ahead of Mutt. If d_1 is the distance Mutt traveled, and if d_2 is the distance Jeff traveled, write an appropriate equation.

☐ SOLVING A SYSTEM OF EQUATIONS BY SUBSTITUTION

A system of two equations in two variables is a pair of equations like

$$\begin{aligned} a+b &= 10 \\ a-b &= 4 \end{aligned}$$

A solution of this system of equations is a pair of numbers a and b which make both equations true.

For example, if $a = 5$ and $b = 5$, then the first equation is true but the second is false, so this is not a solution of the system of equations.

Similarly, $a = 12$ and $b = 8$ is a solution to the second equation, but not the first; this pair is also not a solution of the system.

But what if $a = 7$ and $b = 3$? Then both equations are satisfied, and we conclude by saying that $a = 7$ and $b = 3$ is a solution of the system of equations.

There are many ways to solve a system of two equations in two unknowns, including the Addition (Elimination) method. For the problems in this chapter, we'll use a form of the Substitution Method, but you should use any method you and your instructor like.

EXAMPLE 1: **Solve the system** $\begin{aligned} \boldsymbol{x+y} &= \mathbf{10} \\ \boldsymbol{2x-3y} &= \mathbf{15} \end{aligned}$ **by substitution.**

Solution: To apply the substitution method, we select one of the two equations, then select one of the two variables in that equation. We then solve for that variable and then *substitute* that result into the other equation. It's a lot easier to show than to explain.

First step: Select an equation. Let's choose the first equation because it looks a lot simpler than the second equation:

$$x+y = 10$$

Second step: Select a variable. For no particular reason, we'll choose the y.

Third step: Solve for that variable; that is, isolate it:

$$x + y = 10$$

$$\Rightarrow \quad y = 10 - x \qquad \text{[EQU 1]} \qquad \text{(subtract } x \text{ from each side)}$$

Fourth step: Substitute $10 - x$ for y in the other equation. Here's the other equation:

$$2x - 3y = 15$$

Now replace the y in this equation with its value of $10 - x$:

$$2x - 3(\mathbf{10 - x}) = 15 \qquad \text{(since } 10 - x \text{ and } y \text{ are the same)}$$

How is this equation any improvement over the original pair of equations? I'll tell you -- it has only one variable in it! That's a good thing; we can solve for x rather easily:

$$2x - 3(10 - x) = 15$$

$$\Rightarrow \quad 2x - 30 + 3x = 15 \qquad \text{(distribute)}$$

$$\Rightarrow \quad 5x - 30 = 15 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 5x = 45 \qquad \text{(add 30 to each side)}$$

$$\Rightarrow \quad \underline{x = 9} \qquad \text{(divide each side by 5)}$$

Now we know that $x = 9$. But what about y? To find the value of y, we could use either of the two original equations, but the easiest way to find y is to use EQU 1 -- after all, it's already solved for y. So we write EQU 1, place the number we got for x, and figure out y:

$$y = 10 - x \qquad \text{(EQU 1)}$$

$$\Rightarrow \quad y = 10 - \mathbf{9} \qquad \text{(substitute 9 for } x\text{)}$$

$$\Rightarrow \quad y = 1$$

We're done. Now don't panic at how long it took to solve this problem. With a little practice, you'll be doing them quickly. We'll write our final answer as

$$\boxed{x = 9 \ \ \& \ \ y = 1}$$

Homework

8. Solve each system of equations by Substitution:

a. $\begin{aligned} x + y &= 20 \\ 4x - 3y &= -25 \end{aligned}$

b. $\begin{aligned} y &= 2x + 4 \\ 3x + 5y &= 7 \end{aligned}$

c. $\begin{aligned} x - 2y &= -8 \\ 4x + 7y &= 28 \end{aligned}$

☐ OPPOSITE DIRECTIONS

EXAMPLE 2: **Mike and Sarah start from the burger stand and skate in opposite directions. Mike's speed is 5 less than 3 times Sarah's speed. In 10 hours they are 70 miles apart. Find the speed of both skaters.**

Solution: Let's organize all the information in a table using the basic $rt = d$ formula we're learning in this chapter. Down the first column are the names of our two skaters. Across the first row are the three components of motion, the rate (speed), the time, and the distance. We've written the formula to help us remember the basic relationship among these three concepts.

	Rate ×	**Time** =	**Distance**
Mike			
Sarah			

Since each skater's speed is being asked for (they're the unknowns), we'll let M stand for Mike's speed and S stand for Sarah's speed, and so these variables go into the Rate column.

As for the Time column, the problem states that each skater skated for exactly 10 hours, so each of their travel times is 10.

Since Distance = Rate × Time, the Distance column is simply the product of the Rate and Time columns for both Mike and Sarah.

	Rate ×	**Time** =	**Distance**
Mike	M	10	$10M$
Sarah	S	10	$10S$

Since there are two unknowns in this problem, it's likely we'll need two equations. Let's look at the rates first: From the phrase in the problem "Mike's speed is 5 less than 3 times Sarah's speed" we create the equation

$$M = 3S - 5 \qquad \text{[Equation 1]}$$

To determine the second equation, we have to picture where the skaters are going. They start in the same place and then proceed to skate in opposite directions and end up 70 miles from each other. Therefore, the <u>sum</u> of their individual distances must be 70. Well, Mike skated a distance of $10M$ miles while Sarah went $10S$ miles. So 70 must be the sum of $10M$ and $10S$:

$$10M + 10S = 70 \qquad \text{[Equation 2]}$$

Now substitute the first equation into the second equation:

$$10(\mathbf{3\boldsymbol{S} - 5}) + 10S = 70 \qquad \text{(replaced } M \text{ with } 3S - 5\text{)}$$

$$\Rightarrow \quad 30S - 50 + 10S = 70 \qquad \text{(distribute)}$$

$$\Rightarrow \quad 40S - 50 = 70 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad 40S = 120 \qquad \text{(add 50 to each side)}$$

$$\Rightarrow \quad \underline{S = 3} \qquad \text{(divide each side by 40)}$$

Recall that S stood for Sarah's speed, so we know for sure that Sarah skated 3 mph. To find Mike's speed we use Equation 1 and Sarah's speed:

$$\begin{aligned} M &= 3S - 5 \\ &= 3(\mathbf{3}) - 5 \\ &= 9 - 5 \\ &= 4 \end{aligned}$$

This shows that Mike skated at a rate of 4 mph. We now have the complete answer to the question:

Mike's speed was 4 mph and Sarah's speed was 3 mph.

Homework

9. Two pedestrians leave the same place and walk in opposite directions. The speed of one of the pedestrians is 5 mph less than 7 times the other. In 6 hours they are 354 miles apart. Find the speed of each pedestrian.

10. Two skaters leave the same place and skate in opposite directions. The speed of one of the skaters is 8 mph less than 10 times the other. In 9 hours they are 819 miles apart. Find the speed of each skater.

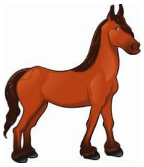
11. Two joggers leave the same place and jog in opposite directions. The speed of one of the joggers is 9 mph more than 5 times the other. In 7 hours they are 357 miles apart. Find the speed of each jogger.

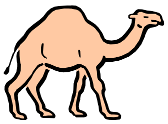
12. Two pedestrians leave the same place and walk in opposite directions. The speed of one of the pedestrians is 7 mph less than 9 times the other. In 10 hours they are 930 miles apart. Find the speed of each pedestrian.

13. Two skaters leave the same place and skate in opposite directions. The speed of one of the skaters is 1 mph more than 7 times the other. In 9 hours they are 513 miles apart. Find the speed of each skater.

❑ PURSUIT

EXAMPLE 3: **A camel leaves the oasis traveling 5 mph. Eight hours later a horse begins to pursue the camel at a speed of 45 mph. How many hours after the horse leaves the oasis will it catch up with the camel?**





Solution: This problem gives us the rates of both animals, so those aren't an issue. In fact, the question is asking for the travel time of the horse. In addition, we don't know the travel time of the camel, either. So how about we let

c = the travel *time* of the camel, and

h = the travel *time* of the horse.

These variables will go into the Time column of our table. And since the rates (speeds) of the animals are given, we'll simply place them in the Rate column. As in the previous example, the

Distance is found by multiplying the Rate by the Time. Here's the table with all the known and unknown information in it:

	Rate	**× Time**	**= Distance**
camel	5	c	$5c$
horse	45	h	$45h$

Again, two variables will require two equations. We'll start with the Time column. Notice that the horse left <u>after</u> the camel (by 8 hours). This implies that the horse's travel time was 8 hours <u>less</u> than the camel's. This observation (which is not very obvious) leads to the first equation:

$$h = c - 8 \qquad \text{[Equation 1]}$$

To determine the second equation, we have to visualize where the animals are going. They start in the same place, leave one after the other, and then go in the same direction and end up in the same place. Thus, each of them went the <u>same distance</u> even though the camel left before the horse. This fact means that we can set the two distances in the table equal to each other:

$$5c = 45h \qquad \text{[Equation 2]}$$

Substituting Equation 1 into Equation 2:

$$5c = 45(\boldsymbol{c - 8}) \qquad \text{(since } h = c - 8\text{)}$$

$$\Rightarrow \quad 5c = 45c - 360 \qquad \text{(distribute)}$$

$$\Rightarrow \quad -40c = -360 \qquad \text{(subtract } 45c \text{ from each side)}$$

$$\Rightarrow \quad c = 9 \qquad \text{(divide each side by } -40\text{)}$$

This tells us that the camel traveled for 9 hours. But the question asked for the horse's travel time, so we use Equation 1 to find h:

$$h = c - 8 = 9 - 8 = 1$$

We conclude that

> It takes the horse 1 hour to catch up with the camel.

Homework

14. A camel leaves the oasis traveling 10 mph. Five hours later a dune buggy begins to pursue the camel at a speed of 15 mph. How many hours after the dune buggy leaves the oasis will it catch up with the camel?

15. A sailboat leaves the island traveling 14 mph. Five hours later a hydrofoil begins to pursue the sailboat at a speed of 24 mph. How many hours after the hydrofoil leaves the island will it catch up with the sailboat?

16. A robber leaves the bank traveling 15 mph. Four hours later a sheriff begins to pursue the robber at a speed of 35 mph. How many hours after the sheriff leaves the bank will he catch up with the robber?

17. A rowboat leaves the harbor traveling 26 mph. Seven hours later a speedboat begins to pursue the rowboat at a speed of 39 mph. How many hours after the speedboat leaves the harbor will it catch up with the rowboat?

18. A robber leaves the bank traveling 26 mph. Three hours later a sheriff begins to pursue the robber at a speed of 39 mph. How many hours after the sheriff leaves the bank will she catch up with the robber?

❑ ROUND TRIP

EXAMPLE 4: **It takes a helicopter a total of 13 hours to travel from the mountain to the valley at a speed of 30 mph and return at a speed of 35 mph. How long does it take to get from the mountain to the valley?**

Solution: We'll let x represent the time it takes to go from the mountain to the valley (since this is what's being asked for). Let's also choose y to stand for the time it takes to return from the valley to the mountain. The two rates (speeds) are given, and we are getting pretty good at knowing that each distance is the product of the rate and the time. So here's the table:

	Rate ×	**Time =**	**Distance**
to valley	30	x	$30x$
to mtn	35	y	$35y$

Since the <u>total</u> travel time is 13 hours, we get our first equation:

$$x + y = 13 \qquad \text{[Equation 1]}$$

Now what about the distances, $30x$ and $35y$? Wouldn't you agree that the distance from the mountain to the valley is the same as the distance from the valley to the mountain? That is, $30x$ and $35y$ must be equal:

$$30x = 35y \qquad \text{[Equation 2]}$$

To solve this system of two equations in two unknowns, let's take Equation 1 and solve it for y:

$$x + y = 13 \qquad \text{(Equation 1)}$$

$$\Rightarrow \quad y = 13 - x \qquad \text{(subtract } x \text{ from each side)}$$

We now replace the variable y in Equation 2 with the result just obtained:

$$30x = 35(\mathbf{13 - \boldsymbol{x}})$$

$$\Rightarrow \quad 30x = 455 - 35x \qquad \text{(distribute)}$$

$$\Rightarrow \quad 65x = 455 \qquad \text{(add } 35x \text{ to each side)}$$

$$\Rightarrow \quad \underline{x = 7} \qquad \text{(divide each side by 65)}$$

Now, what did x represent? Go back to the table and see that x represented the travel time from the mountain to the valley. But this is exactly what was being asked for in the problem, so we're done.

It takes 7 hours to travel from the mountain to the valley.

Homework

19. A helicopter traveled from the hospital to the battlefield at a speed of 22 mph and returned at a speed of 44 mph. If the entire trip took 18 hours, find the travel times to and from the battlefield.

20. A hang glider traveled from the oceanside to the mountaintop at a speed of 18 mph and returned at a speed of 21 mph. If the entire trip took 13 hours, find the travel times to and from the mountaintop.

21. A helicopter traveled from the hospital to the battlefield at a speed of 36 mph and returned at a speed of 24 mph. If the entire trip took 20 hours, find the travel times to and from the battlefield.

22. A tractor traveled from the wheat field to the chicken coop at a speed of 27 mph and returned at a speed of 36 mph. If the entire trip took 21 hours, find the travel times to and from the chicken coop.

23. A hang glider traveled from the oceanside to the mountaintop at a speed of 34 mph and returned at a speed of 51 mph. If the entire trip took 15 hours, find the travel times to and from the mountaintop.

☐ TWO-PART JOURNEY

EXAMPLE 5: **A limousine traveled at 29 mph for the first part of a 540-mile trip, and then increased its speed to 53 mph for the rest of the trip. How many hours were traveled at each rate if the total trip took 12 hours?**

Solution: The rates for each part of the trip are given, so just put them in the table in the right places. Let x be the travel time for the first part of the trip and let y be the travel time for the second part of the trip. Finally, the Distance column is the product of the Rate and Time columns.

	Rate	× **Time**	= **Distance**
1st part	29	x	$29x$
2nd part	53	y	$53y$

The total travel is given to be 12 hours. Therefore,

$$x + y = 12$$

Since the total distance traveled was 540 miles, adding the distance of the 1st part of the trip plus the distance of the 2nd part of the trip should give a total of 540:

$$29x + 53y = 540$$

Solving the first equation for y gives $y = 12 - x$. Substituting $12 - x$ for y in the second equation gives:

$$29x + 53(\mathbf{12 - x}) = 540$$

$$\Rightarrow \quad 29x + 636 - 53x = 540 \qquad \text{(distribute)}$$

$$\Rightarrow \quad -24x + 636 = 540 \qquad \text{(combine like terms)}$$

$$\Rightarrow \quad -24x = -96 \qquad \text{(subtract 636)}$$

$$\Rightarrow \quad \underline{x = 4}$$

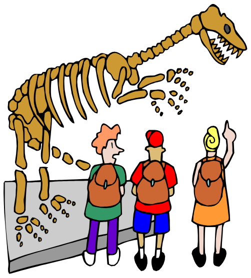
This means that the first part of the limo trip took 4 hours. Using the equation $y = 12 - x$, we calculate the time for the rest of the trip as $y = 12 - x = 12 - 4 = 8$. In short,

$$\boxed{\text{4 hours at 29 mph and 8 hours at 53 mph}}$$

Homework

24. A 1096-mi trip took a total of 16 hours. The speed was 71 mph for the first part of the trip, and then decreased to 67 mph for the rest of the trip. How many hours were traveled at each speed?
25. A 730-mi trip took a total of 11 hours. The speed was 68 mph for the first part of the trip, and then decreased to 59 mph for the rest of the trip. How many hours were traveled at each speed?
26. A 664-mi trip took a total of 12 hours. The speed was 30 mph for the first part of the trip, and then increased to 68 mph for the rest of the trip. How many hours were traveled at each speed?
27. A 489-mi trip took a total of 9 hours. The speed was 45 mph for the first part of the trip, and then increased to 59 mph for the rest of the trip. How many hours were traveled at each speed?
28. A 556-mi trip took a total of 14 hours. The speed was 38 mph for the first part of the trip, and then increased to 42 mph for the rest of the trip. How many hours were traveled at each speed?

❑ SAME DIRECTION



<u>EXAMPLE 6:</u> **Six hours after Mary and Moe leave the museum at the same time and head in the same direction, Moe is 252 miles ahead of Mary. If Moe's speed is 8 mph less than 3 times Mary's speed, find the speeds of Mary and Moe.**

<u>Solution:</u> Mary and Moe left the museum at the same time and each traveled for 6 hours, so both times in the table must be 6. Since the rates are unknown, we let x represent Mary's speed and let y represent Moe's speed.

	Rate ×	**Time** =	**Distance**
Mary	x	6	$6x$
Moe	y	6	$6y$

First we need an equation relating x and y. The phrase "Moe's speed is 8 mph less than 3 times Mary's speed" translates to

$$y = 3x - 8.$$

Now, according to the table, Mary traveled $6x$ miles, while Moe traveled $6y$ miles. The problem says that at the end of the 6 hours, Moe is 252 miles ahead of Mary. This means that Moe's distance ($6y$) is 252 miles more than Mary's distance ($6x$), which translates into our second equation:

$$6y = 6x + 252$$

Substituting the first equation into the second equation gives us

$$6(3x - 8) = 6x + 252$$

$$\Rightarrow \quad 18x - 48 = 6x + 252 \qquad \text{(distribute)}$$

$\Rightarrow \quad 12x = 300$ (subtract $6x$ and add 48)

$\Rightarrow \quad \underline{x = 25}$ (divide by 12)

which implies that Moe's rate is $y = 3x - 8 = 3(\mathbf{25}) - 8 = 67$.

Mary's speed was 25 mph and
Moe's speed was 67 mph.

Homework

29. Moe and Curly leave the airport at the same time and head in the same direction. Curly's speed is 8 mph less than 3 times Moe's speed. Five hours later Curly is 430 miles ahead of Moe. Find the speeds of Moe and Curly.

30. Sally and Maria leave the mall at the same time and head in the same direction. Maria's speed is 6 mph less than 2 times Sally's speed. Ten hours later Maria is 410 miles ahead of Sally. Find the speeds of Sally and Maria.

31. Lucy and Ethyl leave the mall at the same time and head in the same direction. Ethyl's speed is 4 mph more than 4 times Lucy's speed. Four hours later Ethyl is 604 miles ahead of Lucy. Find the speeds of Lucy and Ethyl.

32. Sally and Maria leave the stadium at the same time and head in the same direction. Maria's speed is 3 mph less than 4 times Sally's speed. Eight hours later Maria is 1080 miles ahead of Sally. Find the speeds of Sally and Maria.

33. Mutt and Jeff leave the stadium at the same time and head in the same direction. Jeff's speed is 8 mph less than 3 times Mutt's speed. Ten hours later Jeff is 660 miles ahead of Mutt. Find the speeds of Mutt and Jeff.

Practice Problems

34. Two skaters leave from the same place and skate in opposite directions. The speed of one of the skaters is 5 mph more than 9 times the other. In 10 hours they are 850 miles apart. Find the speed of each skater.
35. A camel leaves the oasis traveling 18 mph. Four hours later a dune buggy begins to pursue the camel at a speed of 26 mph. How many hours after the dune buggy leaves the oasis will it catch up with the camel?
36. A pickup truck traveled from the house to the ballpark at a speed of 16 mph and returned at a speed of 20 mph. If the entire trip took 18 hours, find the travel times to and from the ballpark.
37. A 450-mi trip took a total of 9 hours. The speed was 46 mph for the first part of the trip, and then increased to 64 mph for the rest of the trip. How many hours were traveled at each speed?
38. Sally and Maria leave the museum at the same time and head in the same direction. Maria's speed is 1 mph less than 10 times Sally's speed. Six hours later Maria is 1938 miles ahead of Sally. Find the speeds of Sally and Maria.
39. Mutt and Jeff leave the mall at the same time and head in the same direction. Jeff's speed is 9 mph more than 6 times Mutt's speed. Four hours later Jeff is 1036 miles ahead of Mutt. Find the speeds of Mutt and Jeff.
40. A 430-mi trip took a total of 11 hours. The speed was 35 mph for the first part of the trip, and then increased to 50 mph for the rest of the trip. How many hours were traveled at each speed?
41. A hang glider traveled from the oceanside to the mountaintop at a speed of 34 mph and returned at a speed of 17 mph. If the entire trip took 18 hours, find the travel times to and from the mountaintop.

42. A robber leaves the bank traveling 26 mph. Three hours later a sheriff begins to pursue the robber at a speed of 39 mph. How many hours after the sheriff leaves the bank will she catch up with the robber?

43. Two skaters leave from the same place and skate in opposite directions. The speed of one of the skaters is 4 mph less than 6 times the other. In 4 hours they are 180 miles apart. Find the speed of each skater.

44. A robber leaves the bank traveling 15 mph. Seven hours later a sheriff begins to pursue the robber at a speed of 30 mph. How many hours after the sheriff leaves the bank will he catch up with the robber?

45. An 1127-mi trip took a total of 17 hours. The speed was 55 mph for the first part of the trip, and then increased to 79 mph for the rest of the trip. How many hours were traveled at each speed?

46. A helicopter traveled from the hospital to the battlefield at a speed of 45 mph and returned at a speed of 35 mph. If the entire trip took 16 hours, find the travel times to and from the battlefield.

47. Sally and Maria leave the museum at the same time and head in the same direction. Maria's speed is 4 mph less than 4 times Sally's speed. Three hours later Maria is 186 miles ahead of Sally. Find the speeds of Sally and Maria.

48. Two spaceships leave from the same place and fly in opposite directions. The speed of one of the spaceships is 6 mph more than 4 times the other. In 8 hours they are 248 miles apart. Find the speed of each spaceship.

Solutions

1. a. 1440 km b. 500 mi/hr c. 4 hrs

2. a. $d = rt$ or $rt = d$

3. $d_1 + d_2 = 18$ **4**. $d_1 = d_2$ **5**. $d_1 = d_2$ **6**. $d_1 + d_2 = 1096$

7. $d_1 + 1036 = d_2$ OR $d_2 - d_1 = 1036$ OR $d_2 - 1036 = d_1$

8. $x = 5$ & $y = 15$ b. $x = -1$ & $y = 2$ c. $x = 0$ & $y = 4$

9. 8 mph & 51 mph **10**. 9 mph & 82 mph **11**. 7 mph & 44 mph

12. 10 mph & 83 mph **13**. 7 mph & 50 mph **14**. 10 hours

15. 7 hours **16**. 3 hours **17**. 14 hours

18. 6 hours **19**. 12 hrs & 6 hrs **20**. 7 hrs & 6 hrs

21. 8 hrs & 12 hrs **22**. 12 hrs & 9 hrs **23**. 9 hrs & 6 hrs

24. 6 hrs & 10 hrs **25**. 9 hrs & 2 hrs **26**. 4 hrs & 8 hrs

27. 3 hrs & 6 hrs **28**. 8 hrs & 6 hrs **29**. 47 mph & 133 mph

30. 47 mph & 88 mph **31**. 49 mph & 200 mph **32**. 46 mph & 181 mph

33. 37 mph & 103 mph **34**. 8 mph & 77 mph **35**. 9 hours

36. 10 hrs & 8 hrs **37**. 7 hrs & 2 hrs **38**. 36 mph & 359 mph

39. 50 mph & 309 mph **40**. 8 hrs & 3 hrs **41**. 6 hrs & 12 hrs

42. 6 hours **43**. 7 mph & 38 mph **44**. 7 hours

45. 9 hrs & 8 hrs **46**. 7 hrs & 9 hrs **47**. 22 mph & 84 mph

48. 5 mph & 26 mph

"Formal education is but an incident in the lifetime of an individual. Most of us who have given the subject any study have come to realize that education is a continuous process ending only when ambition comes to a halt."

– R.I. Rees

CH 17 – PERCENT MIXTURE PROBLEMS

❑ INTRODUCTION

NaOCl
sodium hypochlorite

Suppose you have 12 quarts of a bleach solution which is at a 25% concentration. This means that 25% of the solution is bleach, while the other 75% is some neutral substance (usually water). How much of the 12 quarts is actually bleach? Since the solution is 25% bleach, we take 25% of 12, which is 12 × 0.25 = 3 quarts of bleach.

The 12 quarts of solution is called the total **quantity**.

The 25% is called the **concentration**.

The 3 quarts of bleach is called the actual **amount**.

The calculation above shows us the formula which ties all these ideas together, and is the basis for this chapter and the next one:

Quantity × % Concentration = Amount

Two Important Notes:

1. To convert 63% to a decimal, move the decimal point (it's after the 3) two places to the left and drop the percent sign, giving 0.63. To convert the decimal 0.08 to a percent, move the decimal point two places to the right and add a percent sign to get 8%.

2. Pure bleach would have a 100% bleach concentration, while pure water would have a 0% bleach concentration. Be sure this makes sense to you.

Homework

1. A 40-quart solution of sulfuric acid is at a 30% concentration. How many quarts of the solution are sulfuric acid? How many quarts are water?
2. A 25-kg solution of salt water is 20% NaCl (salt). How many kg of the solution consist of NaCl? How many kg are water?
3. What is the percent concentration of sodium in pure sodium?
4. What is the percent concentration of nitric acid in pure water?
5. Consider a gallon of pure antifreeze. What is the percent concentration of antifreeze? What is the percent concentration of water?
6. Consider a liter of pure water. What is the percent concentration of hydrochloric acid? What is the percent concentration of water?

☐ *SOLVING A SYSTEM OF EQUATIONS BY ADDITION*

To solve the applications in this chapter and future courses, we need to be able to solve two equations in two variables. A method that works very well in many cases is called the ***Addition Method***. We multiply one or both equations by appropriate numbers (whatever that means), **add** the resulting equations to eliminate a variable, and then solve for the variable that survived. The *elimination method* is another term used to describe this procedure.

<u>EXAMPLE 1:</u> **Solve the system:** $\mathbf{5x - 2y = 20}$

$\mathbf{3x + 7y = -29}$

<u>Solution:</u> In the Addition Method we may eliminate either variable. But there's a certain "orderliness" that comes in handy in future math courses if we always eliminate the first variable, so in this case we will eliminate the x. As mentioned above, we multiply one or both equations by some numbers, add the resulting equations to kill off one of the variables, and then solve for the variable that still lives. How do we find these numbers? Rather than some mystifying explanation, just watch -- you'll catch on.

$$\begin{aligned} 5x - 2y &= 20 \xrightarrow{\text{times 3}} & 15x - 6y &= 60 \\ 3x + 7y &= -29 \xrightarrow{\text{times -5}} & -15x - 35y &= 145 \end{aligned}$$

<u>Add</u> the equations: $0x - 41y = 205$

The x's are gone!

Divide by –41: $\dfrac{-41y}{-41} = \dfrac{205}{-41}$

$\underline{y = -5}$

Now that we have the value of y, we can substitute its value of –5 into either of the two original equations to find the value of x. Using the first equation:

$$\begin{aligned} & 5x - 2(\mathbf{-5}) = 20 \\ \Rightarrow\ & 5x + 10 = 20 \\ \Rightarrow\ & 5x = 10 \\ \Rightarrow\ & \underline{x = 2} \end{aligned}$$

Therefore, the final solution to the system of equations is

$$\boxed{x = 2 \ \& \ y = -5}$$

Homework

7. Solve each system using the Addition Method, and be sure you practice checking your solution (your pair of numbers) in both of the original equations:

a. $$\begin{aligned} 2x + y &= 5 \\ -2x + 7y &= 19 \end{aligned}$$

b. $$\begin{aligned} 5a - 3b &= 5 \\ 10a + 4b &= -40 \end{aligned}$$

c. $$\begin{aligned} -2u - 3v &= -16 \\ -7u + 8v &= -56 \end{aligned}$$

d. $$\begin{aligned} 7x + 12y &= -24 \\ 6x - 7y &= 14 \end{aligned}$$

☐ *MIXING CHEMICALS*

EXAMPLE 2: **How many quarts each of a 62% poison solution and a 6% poison solution must a foreign spy need to get 14 quarts of a mixture that is 30% poison?**

Solution: Let x represent the quarts of the 62% poison. Let y represent the quarts of the 6% poison.

	Quantity x	Concentration =	Amount
62% poison	x	62%	$0.62x$
6% poison	y	6%	$0.06y$
final mixture	14	30%	14×0.30

Looking at the Quantity column, we're mixing x quarts with y quarts to get a total of 14 quarts in the final mixture. It makes sense to say that the sum of x and y must be 14:

$$x + y = 14 \qquad \textbf{[Equation \#1]}$$

Consider the Concentration column. Does adding the concentrations together make any sense? Of course not: $62\% + 6\% \neq 30\%$. In fact, our intuition tells us that the concentration of the final solution ought to be somewhere <u>between</u> the concentrations of the ingredients being mixed. But we need another equation -- since we have two variables x and y, we'll need two equations in order to find x and y.

Now look at the Amount column. Each ingredient being mixed together contains poison and water. Does it make sense that the actual amount of poison in the final solution must be the sum of the actual amounts of poison in the ingredients? This leads to the second equation:

$$0.62x + 0.06y = 14 \times 0.30$$

$$\text{or,} \quad 0.62x + 0.06y = 4.2 \qquad \textbf{[Equation \#2]}$$

Equations 1 and 2 constitute a system of two equations in two variables, which we will solve using the Addition Method.

$$\begin{array}{lcrcr}
x + y = 14 & \text{(times } -0.62\text{)} & \Rightarrow & -0.62x - 0.62y = & -8.68 \\
0.62x + 0.06y = 4.2 & \text{(leave alone)} & \Rightarrow & 0.62x + 0.06y = & 4.2 \\
\hline
 & \text{Adding} & \Rightarrow & 0 - 0.56y = & -4.48 \\
 & & \Rightarrow & y = & 8
\end{array}$$

Since y stood for the number of quarts of the 6% solution, we see that the detective needs 8 quarts of the 6% solution. Moreover, since $x + y = 14$, we see that $x + 8 = 14 \Rightarrow x = 6$, which means that the detective also needs 6 quarts of the 62% solution.

6 quarts of the 62% poison, &
8 quarts of the 6% poison

EXAMPLE 3: **A druggist wants to create 4 liters of a 94% anti-malaria medicine. How many liters each of pure anti-malaria medicine and a 92% anti-malaria medicine must she mix together?**

Solution:

Let x represent the liters of pure anti-malaria medicine.
Let y represent the liters of the 92% anti-malaria medicine.

Remembering that pure anti-malaria medicine has an anti-malaria concentration of 100%, we put all our information in the chart:

	Quantity x	Concentration =	Amount
pure medicine	x	100%	$1.00x$
92% medicine	y	92%	$0.92y$
final mixture	4	94%	4×0.94

One of our equations comes from the fact that the quantities x and y must add up to 4:

$$x + y = 4$$

The second equation comes from the fact that the amount of anti-malaria medicine in the 100% medicine plus the amount of anti-malaria medicine in the 92% medicine must equal the amount of anti-malaria medicine in the final mixture:

$$1.00x + 0.92y = 0.94 \times 4,$$

$$\text{or,} \quad x + 0.92y = 3.76$$

Let's solve this system of equations via the Addition Method.

$$\begin{array}{lllr}
x + y = 4 & \text{(leave alone)} & \Rightarrow & x + y = 4 \\
x + 0.92y = 3.76 & \text{(times } -1\text{)} & \Rightarrow & -x - 0.92y = -3.76 \\
\cline{4-4}
 & \text{Adding} & \Rightarrow & 0 + 0.08y = 0.24 \\
 & & \Rightarrow & y = 3
\end{array}$$

Since $y = 3$, and $x + y = 4$, it follows that $x = 1$. We now know how many liters of each medicine she must mix together.

3 liters of the 92% medicine, &
1 liter of the pure medicine

<u>EXAMPLE 4:</u> **How many fluid ounces each of pure water and a 10% albuterol inhalant must an allergist mix to get 5 fluid ounces of an inhalant that is 6% albuterol?**

<u>Solution:</u> This is the same scenario as the two previous examples, so we can get right to it; note, however, that pure water contains 0% albuterol (that is, there is <u>no</u> albuterol in pure water).

	Quantity x Concentration = Amount		
pure water	x	0%	$0x$
10% albuterol	y	10%	$0.10y$
final mixture	5	6%	5×0.06

The two equations are $x + y = 5$ and $0x + 0.10y = 5 \times 0.06$, or $0.10y = 0.3$. Setting up the system of equations neatly gives

$$\begin{aligned} x + y &= 5 \\ 0.10y &= 0.3 \end{aligned}$$

The second equation can be easily solved for y, so we don't need the Addition method. Dividing each side of the second equation by 0.10, we get $y = 3$. Using $x + y = 5$, we find that $x = 2$. Thus,

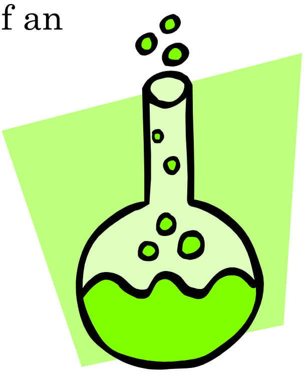
2 ounces of water, &
3 ounces of the 10% inhalant

Homework

8. How many pounds each of a 97% poison solution and a 67% poison solution must a spy mix to get 15 pounds of a solution that is 91% poison?

9. A druggist wants to create 24 ounces of a 92% alcohol medicine. How many ounces each of pure alcohol medicine and a 76% alcohol medicine must he mix together?

10. How many ounces each of pure water and an 80% albuterol inhalant must an allergist mix to get 20 ounces of an inhalant that is 64% albuterol?



11. A druggist wants to mix some 59% alcohol medicine with some 21% alcohol medicine. How many mL of each substance must she use to get a 19-mL mixture that is 57% alcohol?

12. A detective wants to mix some pure poison with some 20% poison solution. How many pounds of each substance must she use to get a 25-pound mixture that is 52% poison?

13. How many liters each of a 99% solution and an 88% solution must be mixed together to get 22 liters of a mixture whose concentration is 97%?

14. How many liters each of a pure solution and a 12% solution must be mixed together to get 88 liters of a mixture whose concentration is 42%?

15. How many liters each of a 24% solution and a 68% solution must be mixed together to get 22 liters of a mixture whose concentration is 44%?

16. How many liters each of pure water and a 57% solution must be mixed together to get 38 liters of a mixture whose concentration is 42%?

17. How many liters each of a 79% solution and a 70% solution must be mixed together to get 36 liters of a mixture whose concentration is 76%?

18. How many liters each of pure water and a 48% solution must be mixed together to get 72 liters of a mixture whose concentration is 14%?

19. How many liters each of a 9% solution and a 94% solution must be mixed together to get 68 liters of a mixture whose concentration is 69%?

20. How many liters each of pure water and a 20% solution must be mixed together to get 100 liters of a mixture whose concentration is 9%?

21. How many liters each of a 73% solution and a 97% solution must be mixed together to get 12 liters of a mixture whose concentration is 93%?

22. How many liters each of a pure solution and a 58% solution must be mixed together to get 48 liters of a mixture whose concentration is 93%?

Solutions

1. 12 qts acid; 28 qts H_2O

2. 5 kg NaCl; 20 kg H_2O

3. 100%

4. 0%

5. 100% antifreeze; 0% water

6. 0% hydrochloric acid; 100% water

7. a. $x = 1, y = 3$

Complete Check:

$2x + y = 5$	$-2x + 7y = 19$
$2(\mathbf{1}) + \mathbf{3} = 5$	$-2(\mathbf{1}) + 7(\mathbf{3}) = 19$
$2 + 3 = 5$	$-2 + 21 = 19$
$5 = 5$ ✓	$19 = 19$ ✓

b. $a = -2, b = -5$

c. $u = 8, v = 0$

d. $x = 0, y = -2$

8. 12 pounds of the 97% solution and 3 pounds of the 67% solution

9. 16 ounces of pure alcohol and 8 ounces of the 76% alcohol medicine

10. 4 ounces of water and 16 ounces of the 80% albuterol inhalant

11. 18 mL of the 59% alcohol medicine and 1 mL of the 21% alcohol medicine

12. 10 pounds of pure poison and 15 pounds of the 20% poison solution

13. 18 L of the 99% solution and 4 L of the 88% solution

14. 30 L of the pure solution and 58 L of the 12% solution

15. 12 L of the 24% solution and 10 L of the 68% solution

16. 10 L of pure water and 28 L of the 57% solution

17. 24 L of the 79% solution and 12 L of the 70% solution

18. 51 L of pure water and 21 L of the 48% solution

19. 20 L of the 9% solution and 48 L of the 94% solution

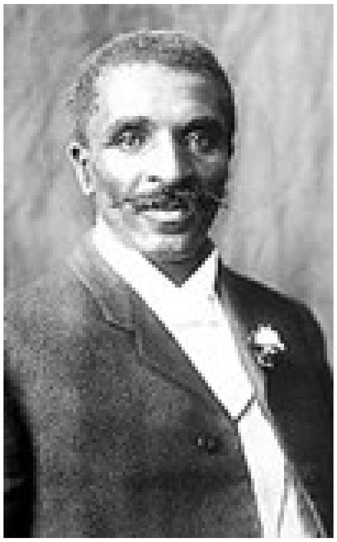
20. 55 L of pure water and 45 L of the 20% solution

21. 2 L of the 73% solution and 10 L of the 97% solution

22. 40 L of the pure solution and 8 L of the 58% solution

“When you do the
common things in life in
an uncommon way,
you will command the
attention of the world.”

- George Washington Carver (1864-1943)



CH 18 – EXPONENTS

❑ INTRODUCTION

We already know that an exponent usually means repeated multiplication, and sometimes we find expressions with two or more exponents involved. We need to know how to simplify such expressions so that they're easier to understand. This chapter also introduces us to the idea of using zero as an exponent.

❑ THE MEANING OF EXPONENTS

$$x^5 = \boldsymbol{x \cdot x \cdot x \cdot x \cdot x}$$

5 factors of *x*

$$3^4 = 3 \times 3 \times 3 \times 3 = \mathbf{81}$$

4 factors of 3

$$2^5 = (2)(2)(2)(2)(2) = \mathbf{32}$$

5 factors of 2

$$1^{10} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \mathbf{1}$$

10 factors of 1

$$0^8 = 0 \times 0 \times 0 \times 0 \times 0 \times 0 \times 0 \times 0 = \mathbf{0}$$

8 factors of 0

$$(-4)^6 = \underbrace{(-4)(-4)(-4)(-4)(-4)(-4)}_{\text{6 factors of }-4} = \mathbf{4{,}096}$$

$$(-2)^7 = \underbrace{(-2)(-2)(-2)(-2)(-2)(-2)(-2)}_{\text{7 factors of }-2} = \mathbf{-128}$$

Homework

1. Rewrite each expression using exponents:

a. $nnnn$ b. $x \cdot x \cdot x$ c. $a \times a \times a \times a \times a$

d. $(yyy)(zzz)$ e. $ababa$ f. $10qqq$

2. Evaluate each expression:

a. 3^3 b. 2^{10} c. 1^{321}

d. 0^{4231} e. 4^4 f. 5^3

g. $(-2)^3$ h. $(-3)^4$ i. $(-3)^2$

j. $(-1)^{123}$ k. $(-1)^{234}$ l. $(-2)^8$

m. -5^2 n. -10^3 o. -2^4

☐ *EXPONENTS WITH NUMBERS*

This section will introduce the Five Laws of Exponents using numbers only. For each calculation we use the Order of Operations, since these rules are the only ones we know for sure. Also remember that *product* means multiply, *quotient* means divide, and *power* means exponent.

I. Let's calculate a **product of powers**. For example,

$$2^3 \times 2^5 \;=\; 8 \times 32 \;=\; 256$$

On the other hand, 256 can also be written as 2^8. So it should be clear that we can legally write

$$2^3 \times 2^5 \;=\; 2^8$$

Is it possible that we can *multiply powers of the same base* simply by adding the exponents?

II. Now let's raise a **power to a power**. For instance,

$$(5^2)^3 \;=\; 25^3 \;=\; 15{,}625 \qquad \text{(Order of Operations)}$$

But you can also calculate that $5^6 = 15{,}625$. Hence,

$$(5^2)^3 \;=\; 5^6$$

Can calculating *a power of a power* be as simple as multiplying the exponents?

III. It's time for a **power of a product**. We can try this:

$$(2 \times 3)^4 \;=\; 6^4 \;=\; 1{,}296 \qquad \text{(Order of Operations)}$$

But here's another way to get the same result:

$$2^4 \times 3^4 \;=\; 16 \times 81 \;=\; 1{,}296$$

Do we raise *a product to a power* merely by raising each factor to the power?

IV. Let's try a **quotient of powers**. Here's an example:

$$\frac{10^6}{10^2} \;=\; \frac{1{,}000{,}000}{100} \;=\; 10{,}000 \qquad \text{(Order of Operations)}$$

Now watch this: If we subtract the exponents and keep the base of 10, we get 10^4, which is also equal to 10,000.

Let's do a second example where we put the bigger exponent on the bottom:

$$\frac{2^3}{2^7} = \frac{8}{128} = \frac{1}{16}$$

But $\frac{1}{16}$ can also be written as $\frac{1}{2^4}$, where the 4 results from subtracting the exponents. There's something going on here.

V. Our fifth example in this section will look at a **power of a quotient**.

$$\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{16}{81} \qquad \text{(the meaning of exponent)}$$

But if we simply raise 2 to the 4th power, and then raise 3 to the 4th power, and write one over the other, we get the same result:

$$\frac{2^4}{3^4} = \frac{16}{81}$$

Homework

3. Use the Order of Operations to evaluate each expression:

a. $2^2 \times 2^3$

b. $3^1 \times 3^4$

c. $4^2 \cdot 4^2$

d. $(3^2)^3$

e. $(2^3)^3$

f. $(5^2)^1$

g. $(2 \times 3)^3$

h. $(3 \times 4)^2$

i. $(1 \cdot 10)^6$

j. $\frac{2^7}{2^5}$

k. $\frac{10^6}{10^3}$

l. $\frac{3^3}{3^5}$

m. $\left(\frac{2}{5}\right)^3$

n. $\left(\frac{1}{8}\right)^2$

o. $\left(\frac{3}{4}\right)^5$

☐ *EXPONENTS WITH VARIABLES*

It's now time for a change in tactics, in order to give us a deeper understanding of the laws of exponents. For each of the following five examples, we will "stretch and squish," and then we'll generalize what we observe to an official law of exponents.

I. We start by finding the product of x^3 and x^4:

$$x^3x^4 = (x\,x\,x)(x\,x\,x\,x) = x\,x\,x\,x\,x\,x\,x = x^7$$

$$\boxed{x^3x^4 = x^7}$$

Notice that the bases (the x's) are the same, and it's a multiplication problem. As long as the bases are the same, and it's a multiplication problem, it appears that we merely need to write down the base, and then <u>add</u> the exponents together to get the exponent of the answer. That is, $\boldsymbol{x^a x^b = x^{a+b}}$.

II. For our second example, let's raise a power to a power:

$$(x^4)^2 = (x\,x\,x\,x)^2 = (x\,x\,x\,x)(x\,x\,x\,x) = x\,x\,x\,x\,x\,x\,x\,x = x^8$$

$$\boxed{(x^4)^2 = x^8}$$

We appear to have a shortcut at hand. Simply <u>multiply</u> the two exponents together and we're done. So, to raise a power to a power, we can write a general rule: $\boldsymbol{(x^a)^b = x^{ab}}$.

III. Now we're to try raising a product to a power; for instance,

$$(ab)^5 = (ab)(ab)(ab)(ab)(ab) = (a\,a\,a\,a\,a)(b\,b\,b\,b\,b) = a^5b^5$$

$$\boxed{(ab)^5 = a^5b^5}$$

In general, when raising a product to a power, <u>raise each factor</u> to the power: $\boldsymbol{(xy)^n = x^n y^n}$.

Note that the quantity in the parentheses is a <u>single</u> term -- there's no adding or subtracting in the parentheses. In fact, if there are two or more terms in the parentheses, this law of exponents does <u>not</u> apply.

IV. Next we divide powers of the same base. We'll need two examples for this concept.

A. $$\frac{x^6}{x^2} = \frac{xxxxxx}{xx} = \frac{\cancel{x}\,\cancel{x}\,xxxx}{\cancel{x}\,\cancel{x}} = x^4$$

B. $$\frac{y^3}{y^6} = \frac{yyy}{yyyyyy} = \frac{\cancel{yyy}}{\cancel{yyy}\,yyy} = \frac{1}{y^3}$$

$$\boxed{\begin{array}{l} \dfrac{x^6}{x^2} = x^4 \\[2ex] \dfrac{y^3}{y^6} = \dfrac{1}{y^3} \end{array}}$$

In general, when dividing powers of the same base, <u>subtract</u> the exponents, leaving the remaining factors on the top if the top exponent is bigger, and on the bottom if the bottom exponent is bigger: $\boldsymbol{\dfrac{x^a}{x^b} = x^{a-b}}$

V. Our last example in this section is the process of raising a quotient to a power. As usual, we stretch and squish; then we generalize to a law of exponents.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aaaa}{bbbb} = \frac{a^4}{b^4}$$

$$\boxed{\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}}$$

In general, we can raise a quotient to a power by raising both the top and bottom to the power: $\boldsymbol{\left(\dfrac{a}{b}\right)^n = \dfrac{a^b}{b^n}}$.

Homework

4. Use the stretch-and-squish technique to simplify each expression:

a. x^2x^4 b. yy^5 c. $a^4a^3a^2$ d. $z^{10}z^{10}$

e. $(a^2)^3$ f. $(b^3)^2$ g. $(y^3)^3$ h. $(w^4)^2$

i. $(xy)^3$ j. $(ab)^2$ k. $(cd)^4$ l. $(wz)^5$

m. $\dfrac{a^6}{a^2}$ n. $\dfrac{x^{10}}{x^9}$ o. $\dfrac{b^5}{b^8}$ p. $\dfrac{y}{y^7}$

q. $\left(\dfrac{x}{y}\right)^2$ r. $\left(\dfrac{a}{b}\right)^3$ s. $\left(\dfrac{w}{z}\right)^5$ t. $\left(\dfrac{g}{h}\right)^6$

❑ *SUMMARY OF THE FIVE LAWS OF EXPONENTS*

Exponent Law	Example
$x^a x^b = x^{a+b}$	$x^2 x^6 = x^8$
$(x^a)^b = x^{ab}$	$(a^4)^3 = a^{12}$
$(xy)^n = x^n y^n$	$(wz)^7 = w^7 z^7$
For $a > b$, $\dfrac{x^a}{x^b} = x^{a-b}$ For $b > a$, $\dfrac{x^a}{x^b} = \dfrac{1}{x^{b-a}}$	$\dfrac{x^{10}}{x^2} = x^8$ $\dfrac{a^3}{a^7} = \dfrac{1}{a^4}$
$\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$	$\left(\dfrac{a}{b}\right)^5 = \dfrac{a^5}{b^5}$

☐ *SINGLE-STEP EXAMPLES*

EXAMPLE 1:

A. $A^7A^5 = A^{7+5} = A^{12}$

The bases are the same, and it's a multiplication problem. So we can simply write the base and add the exponents.

B. $x^2x^3x^4 = x^{2+3+4} = x^9$

All the bases are the same, and it's a multiplication problem, and so we simply add the exponents.

C. $(x+y)^4(x+y)^9 = (x+y)^{13}$

It doesn't matter what the base is, as long as we're multiplying powers of the same base.

EXAMPLE 2:

A. $(c^{10})^2 = c^{20}$

Raising a base to a power, and then raising that result to a further power requires simply that we multiply the exponents.

B. $\left((x^2)^3\right)^4 = x^{24}$

Power to a power to a power? Just multiply all three exponents.

EXAMPLE 3:

A. $(ax)^5 = a^5x^5$

It's a power of a product (a single term). So just raise each factor to the 5th power.

B. $(abc)^7 = a^7b^7c^7$

Even a term with three factors can be raised to the 7th power by raising each factor to the 7th power.

EXAMPLE 4:

A. $\dfrac{x^7}{x^5} = x^2$

Since 7 > 5, we divide powers of the same base by subtracting the exponents.

B. $\dfrac{w^{15}}{w^{25}} = \dfrac{1}{w^{10}}$

Since 25 > 15, we divide the powers of w by subtracting the exponents, leaving the result on the bottom.

EXAMPLE 5:

A. $\left[\dfrac{x}{z}\right]^7 = \dfrac{x^7}{z^7}$

To raise a quotient to a power, just raise both the top and bottom to the 7th power.

B. $\left(\dfrac{a+b}{u-w}\right)^{23} = \dfrac{(a+b)^{23}}{(u-w)^{23}}$

Just raise top and bottom to the 23rd power.

Homework

5. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^4
b. $x^5x^6x^2$
c. y^3y^3
d. $z^{12}z$
e. $(x^3)^4$
f. $(z^8)^2$
g. $(n^{10})^{10}$
h. $(a^1)^7$
i. $(ab)^3$
j. $(xyz)^5$
k. $(RT)^1$
l. $(math)^5$
m. $\dfrac{a^8}{a^2}$
n. $\dfrac{b^3}{b^9}$
o. $\dfrac{w^5}{w^5}$
p. $\dfrac{Q^{100}}{Q^{50}}$
q. $\left(\dfrac{k}{w}\right)^4$
r. $\left(\dfrac{a}{b}\right)^{99}$
s. $\left(\dfrac{1}{m}\right)^{20}$
t. $a(bc)^2$

6. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^5
b. $u^5u^7u^2$
c. $y^{30}y^{30}$
d. $z^{14}z$
e. $(x^4)^5$
f. $(z^9)^2$
g. $(n^{100})^{10}$
h. $(a^1)^9$
i. $(xy)^4$
j. $(abc)^{17}$
k. $(pn)^1$
l. $(love)^4$
m. $\dfrac{a^{10}}{a^2}$
n. $\dfrac{b^3}{b^{12}}$
o. $\dfrac{w^9}{w^9}$
p. $\dfrac{Q^{100}}{Q^{20}}$
q. $\left(\dfrac{x}{w}\right)^3$
r. $\left(\dfrac{a}{b}\right)^{999}$
s. $\left(\dfrac{1}{z}\right)^{22}$
t. $w(xy)^3$

☐ *WHEN __NOT__ TO USE THE FIVE LAWS OF EXPONENTS*

a^5b^6 cannot be simplified. Although the first law of exponents demands that the expressions be multiplied -- and they are -- it also requires that the bases be the same -- and they aren't.

$x^3 + x^4$ cannot be simplified. Even though the bases are the same, the first law of exponents requires that the two powers of x be multiplied.

$w^3 + w^3$ can be simplified, but not by the first law of exponents, since the powers of w are not being multiplied. But the two terms are like terms, which means we simply add them together to get $2w^3$.

$(a + b)^{23}$ does not equal $a^{23} + b^{23}$. The third law of exponents $(xy)^n = x^n y^n$ does not apply because xy is a single term, whereas $a + b$ consists of two terms. However, the last chapter of the book will show us a pretty cool way to consider expanding $(a + b)^{23}$, which actually expands out to 24 terms.

Homework

7. Simplify each expression:

a. y^4y^4
b. a^3b^4
c. $x^4x^3x^2$
d. $p^3t^2p^2$
e. $a^3 + a^5$
f. a^3a^5
g. $n^4 + n^4$
h. $x^3 - x^3$
i. $(x + y)^{55}$
j. $Q^2 + Q^2$
k. u^5w^6
l. $h^6 - h^2$
m. $(a - b)^2$
n. $(ab)^2$
o. $(x^3)^3$
p. $x^4 + x^5$
q. $x^{14} + x^{14}$
r. $y^{12} - y^{12}$
s. $a^8 + a^9$
t. $a^{10} + a^{10}$
u. $(xy)^2$
v. $(x + y)^2$
w. a^3b^4
x. $a^3 + b^4$
y. $a(a^2)(b^2)b$
z. $n^6 + n^6$

❑ *MULTI-STEP EXAMPLES*

EXAMPLE 6:

A. $(-3x^2y^3)(-5xy^7) = (-3)(-5)\,(x^2x)(y^3y^7) = 15x^3y^{10}$

B. $-2x^2y(xy - 4x^3y^4) = -2x^3y^2 + 8x^5y^5$

C. $\left(2a^2b^3\right)^4 = 2^4(a^2)^4(b^3)^4 = 16a^8b^{12}$

D. $7\left(xy^{10}\right)^5 = 7x^5\,(y^{10})^5 = 7x^5y^{50}$

E. $\left(\dfrac{a^2}{b^3}\right)^7 = \dfrac{(a^2)^7}{(b^3)^7} = \dfrac{a^{14}}{b^{21}}$

F. $\left(\dfrac{x^3y^9}{xy^{12}}\right)^5 = \left(\dfrac{x^2}{y^3}\right)^5 = \dfrac{(x^2)^5}{(y^3)^5} = \dfrac{x^{10}}{y^{15}}$

Homework

8. Simplify each expression:

a. $(-5a^3b^4)(-2a^2b)$

b. $(7xy)(-7x^2y^5)$

c. $(-2uw)(2uw)$

d. $x^3(2x^2 - x - 1)$

e. $3y^2(3y^2 - y + 3)$

f. $\left(a^2b^3\right)^4$

g. $\left(-5m^3n^{10}\right)^3$

h. $\left[-3p^3q^3\right]^4$

i. $4\left(xy^7\right)^{10}$

j. $-10(-2c^3y^4)^3$

k. $\left(\dfrac{a^3}{c^2}\right)^{10}$

l. $\left[\dfrac{2x^3}{3xy^4}\right]^3$

m. $\left(\dfrac{a^2b^3}{a^4b}\right)^5$

n. $2(3x^2y^3)^4$

☐ *ZERO AS AN EXPONENT*

Approach #1: Now for the interesting exponent of zero -- what in the world could 2^0, for instance, possibly mean? If your first instinct is 0, then I might be inclined to agree with you -- but we'd both be wrong! Let's make a list of known ***powers of* 2**, determine the pattern which exists, and then extend that pattern to figure out what 2^0 is.

$$\begin{aligned} 2^4 &= 16 \\ 2^3 &= 8 \\ 2^2 &= 4 \\ 2^1 &= 2 \end{aligned}$$

The exponents in the first column are clearly decreasing by 1 at each step, and each number in the right column is one-half of the number above it.

Look at the pattern in the powers of 2 in the first column: The exponents are decreasing one at a time -- the next power of 2 in the sequence ought to be 2^0. Now look at the sequence of numbers 16, 8, 4, and 2 in the second column. Each number is one-half the preceding number; that is, we're dividing by 2 at each step. Therefore, the next number in the sequence should be 1 (which is half of 2). So, continuing the two patterns extends the list above to the following:

$$\begin{array}{rcl} 2^4 &=& 16 \\ 2^3 &=& 8 \\ 2^2 &=& 4 \\ 2^1 &=& 2 \\ \hline \mathbf{2^0} &\mathbf{=}& \mathbf{1} \end{array}$$

This leads to the conclusion that $2^0 = 1$ -- a very strange result, indeed.

Approach #2: Here's a more powerful way to deduce the meaning of the zero exponent: Consider the expression

$$x^3x^0, \text{ where we assume } x \neq 0.$$

To figure out the meaning of x^0, we can use the first law of exponents to calculate

$$x^3x^0 = x^{3+0} = x^3$$

That is,

$$x^3x^0 = x^3$$

Now "isolate" the x^0, since that's what we're trying to find the value of. We do this by dividing each side of the equation by x^3:

$$\frac{x^3x^0}{x^3} = \frac{x^3}{x^3}$$

It's legal to divide by x^3, since we've stipulated that $x \neq 0$.

which implies that

$$x^0 = 1$$

and we're done:

Any number (except 0) raised to the zero power is 1.

EXAMPLE 7:

A. $(x - 3y + z)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

B. $(abc)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

C. $a + b^0 = \boldsymbol{a + 1}$ (the exponent is on the b only)

D. $uw^0 = u(1) = \boldsymbol{u}$ (the exponent is on the w only)

E. $(-187)^0 = \mathbf{1}$ (the exponent is on the -187)

F. $-14^0 = \mathbf{-1}$ (the exponent is on the 14, not on the minus sign)

Homework

9. Evaluate each expression:

a. $2^0 + 2^0$ b. $2^0 \cdot 2^0$ c. $2^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 1)^0$ e. $2^5 - 2^3 + 2^1 - 2^0$ f. $(8 - 6)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{6}\right)^0 + \left(\frac{100}{50}\right)^0 - (20 - 15 - 3)^0 + \left(3^2 - 7\right)^1$

10. Simplify each expression:

a. x^0 b. xy^0 c. $x + y^0$ d. $(x + y)^0$

e. $(ab)^0$ f. $\left(\frac{a^2}{b^3}\right)^0$ g. $\frac{(x^2)^0}{y^3}$ h. m^0m

i. $x^0 + x^0$ j. Q^0Q^0 k. $a^0 - a^0$ l. $\left(\frac{2x^2y^0}{-3ab^{10}}\right)^0$

Practice Problems

11. Simplify: $\left(-3x^3y^4x^7\right)^3$

12. Simplify: $-3\left(x^5x^4x^8\right)^3$

13. Simplify: $\frac{a^2b^3c^9}{ab^4c^3}$

14. Simplify: $-2y^3(3y^4 - 2y^3 + 1)$

15. Simplify: $x^{12} + x^{14}$

16. Simplify: $u^{22} + u^{22}$

17. Simplify: $abcd^0e^0$

18. Simplify: $\left[\dfrac{10^0a^0b^{14}}{a^3b^7}\right]^5$

Solutions

1. a. n^4 b. x^3 c. a^5 d. y^3z^3 e. a^3b^2 f. $10q^3$

2. a. 27 b. 1024 c. 1 d. 0 e. 256 f. 125

g. –8 h. 81 i. 9 j. –1 k. 1 l. 256

m. This is <u>not</u> the square of –5 (there are no parentheses). It's the opposite of 5^2; so the answer is –25.

n. –1000 o. –16

3. a. $2^2 \times 2^3 = 4 \times 8 = 32$ b. 243 c. 256

d. $(3^2)^3 = 9^3 = 729$ e. 512 f. 25

g. $(2 \times 3)^3 = 6^3 = 216$ h. 144 i. 1,000,000

j. $\dfrac{2^7}{2^5} = \dfrac{128}{32} = 4$ k. 1000 l. $\dfrac{1}{9}$

m. $\left(\dfrac{2}{5}\right)^3 = \dfrac{2}{5} \times \dfrac{2}{5} \times \dfrac{2}{5} = \dfrac{8}{125}$ n. $\dfrac{1}{64}$ o. $\dfrac{243}{1024}$

4. a. $x^2x^4 = (xx)(xxxx) = xxxxxx = x^6$

b. y^6 c. a^9 d. z^{20}

e. $(a^2)^3 = (aa)^3 = (aa)(aa)(aa) = aaaaaa = a^6$

f. b^6 g. y^9 h. w^8

i. $(xy)^3 = (xy)(xy)(xy) = xxxyyy = x^3y^3$

j. a^2b^2 k. c^4d^4 l. w^5z^5

m. $\frac{a^6}{a^2} = \frac{aaaa\cancel{aa}}{\cancel{aa}} = aaaa = a^4$ n. x

o. $\frac{b^5}{b^8} = \frac{\cancel{bbbbb}}{\cancel{bbbbb}bbb} = \frac{1}{bbb} = \frac{1}{b^3}$ p. $\frac{1}{y^6}$

q. $\left(\frac{x}{y}\right)^2 = \left(\frac{x}{y}\right)\left(\frac{x}{y}\right) = \frac{x^2}{y^2}$ r. $\frac{a^3}{b^3}$ s. $\frac{w^5}{z^5}$ t. $\frac{g^6}{h^6}$

5. a. a^7 b. x^{13} c. y^6 d. z^{13}
e. x^{12} f. z^{16} g. n^{100} h. a^7
i. a^3b^3 j. $x^5y^5z^5$ k. RT l. $m^5a^5t^5h^5$
m. a^6 n. $\frac{1}{b^6}$ o. 1 p. Q^{50}
q. $\frac{k^4}{w^4}$ r. $\frac{a^{99}}{b^{99}}$ s. $\frac{1}{m^{20}}$ t. ab^2c^2

6. a. a^8 b. u^{14} c. y^{60} d. z^{15}
e. x^{20} f. z^{18} g. n^{1000} h. a^9
i. x^4y^4 j. $a^{17}b^{17}c^{17}$ k. pn l. $l^4o^4v^4e^4$
m. a^8 n. $\frac{1}{b^9}$ o. 1 p. Q^{80}
q. $\frac{x^3}{w^3}$ r. $\frac{a^{999}}{b^{999}}$ s. $\frac{1}{z^{22}}$ t. wx^3y^3

7. a. y^8 b. As is c. x^9 d. p^5t^2
e. As is f. a^8 g. $2n^4$ h. 0
i. As is (for now) j. $2Q^2$ k. As is l. As is
m. $a^2 - 2ab + b^2$ n. a^2b^2 o. x^9 p. As is
q. $2x^{14}$ r. 0 s. As is t. $2a^{10}$
u. x^2y^2 v. $x^2 + 2xy + y^2$ w. As is
x. As is y. a^3b^3 z. $2n^6$

8. a. $10a^5b^5$ b. $-49x^3y^6$ c. $-4u^2w^2$ d. $2x^5 - x^4 - x^3$
e. $9y^4 - 3y^3 + 9y^2$ f. a^8b^{12} g. $-125m^9n^{30}$

h. $81p^{12}q^{12}$ i. $4x^{10}y^{70}$ j. $80c^{9}y^{12}$ k. $\frac{a^{30}}{c^{20}}$

l. $\frac{8x^{6}}{27y^{12}}$ m. $\frac{b^{10}}{a^{10}}$ n. $162x^{8}y^{12}$

9. a. 2 b. 1 c. 31 d. 1

e. 25 f. 3 g. 1024 h. 3

10. a. 1 b. x c. $x + 1$ d. 1 e. 1 f. 1

g. $\frac{1}{y^{3}}$ h. m i. 2 j. 1 k. 0 l. 1

11. $-27x^{30}y^{12}$ **12**. $-3x^{51}$ **13**. $\frac{ac^{6}}{b}$

14. $-6y^{7} + 4y^{6} - 2y^{3}$ **15**. As is **16**. $2u^{22}$

17. abc **18**. $\frac{b^{35}}{a^{15}}$

"What sculpture is to a block of marble, education is to the human soul."

Joseph Addison

CH 19 – FACTORING, PART II

❑ *INTRODUCTION*

We can now factor lots of quadratic binomials and trinomials. Sorry to have to tell you this, but we're not done with factoring just yet. In this chapter, we learn how to factor expressions with the exponent 4 in them, expressions containing four terms, expressions containing GCFs you might never have seen before, and expressions that are the sum or difference of cubes.

❑ *FACTORING QUARTICS*

EXAMPLE 1: **Factor each quartic (4th degree) polynomial:**

A. $\boldsymbol{c^4 - 256}$

$= (c^2 + 16)(c^2 - 16)$ (difference of squares)

$= \boxed{(c^2 + 16)(c + 4)(c - 4)}$ (difference of squares again)

Note: $c^2 + 16$ cannot be factored any further.

B. $\boldsymbol{9a^4 - 37a^2 + 4}$

$= (9a^2 - 1)(a^2 - 4)$ (factor trinomial)

Now we notice that each factor is quadratic and is the difference of two squares. Therefore, each factor can be factored further to get a final answer of

$$\boxed{(3a + 1)(3a - 1)(a + 2)(a - 2)}$$

Homework

1. Factor each quartic polynomial:

a. $x^4 - 1$	b. $x^4 - x^2 - 6$
c. $n^4 - 10n^2 + 9$	d. $a^4 - 81$
e. $36w^4 - 25w^2 + 4$	f. $9x^4 - 34x^2 + 25$
g. $c^4 - 16$	h. $x^4 - 8x^2 - 9$
i. $x^4 - 3x^2 - 10$	j. $g^4 - 256$
k. $36u^4 - 85u^2 + 9$	l. $y^4 + 81$

❑ *THE GCF REVISITED*

EXAMPLE 2: **Factor: $(a + b)^2 + 4(a + b)$**

Solution: There are two terms in this expression: $(a + b)^2$ and $4(a + b)$. Notice that each of these two terms contains the same factor, namely $a + b$. In other words, the GCF of the two terms is $a + b$. Factoring out this GCF gives us the final factored form, a single term:

$$\boxed{(a + b)(a + b + 4)}$$

The thing not to do in this kind of problem is to distribute the original expression; if you do, you'll be going in the wrong direction. Check it out:

$(a + b)^2 + 4(a + b) = a^2 + 2ab + b^2 + 4a + 4b$
Do you really want to try to factor that last expression?

So, when you see an expression, like $a + b$ in this problem, occurring multiple times in an expression, it's usually best to leave it intact. Also notice that we have converted a 2-termed expression into 1 term -- we have factored.

Alternate Method: Let's try a substitution method. We might be able to better see the essence of the problem if we replace $a + b$ with a simpler symbol -- for example, x will represent $a + b$. Then the original expression

$$(a + b)^2 + 4(a + b)$$

is transformed into

$$x^2 + 4x$$

The GCF in this form is clearly x, so we pull it out in front:

$$x(x + 4)$$

Now substitute in the reverse direction, to get $a + b$ back in the problem:

$$(a + b)(a + b + 4) \qquad \text{(the same answer as before)}$$

<u>EXAMPLE 3:</u> **Factor: $x^2(u - w) - 100(u - w)$**

<u>Solution:</u> The two given terms have a GCF of $u - w$. Factoring this GCF out gives

$$(u - w)(x^2 - 100)$$

But we're not done yet. The second factor is a difference of squares. Factoring that part gives us our final factorization:

$$\boxed{(u - w)(x + 10)(x - 10)}$$

EXAMPLE 4: **Factor:** $w^2(x + z) - 4w(x + z) + 3(x + z)$

Solution: Let's use substitution to make this expression appear a little less intimidating; we'll convert every occurrence of $x + z$ to the symbol A:

$$w^2A - 4wA + 3A$$

Pulling out the GCF of A, we get

$$A(w^2 - 4w + 3)$$

Factor the trinomial in the usual way:

$$A(w - 3)(w - 1)$$

Last, replace the A with its original definition of $x + z$:

$$\boxed{(x + z)(w - 3)(w - 1)}$$

Homework

2. Factor each expression:

a. $(x + y)^2 + 7(x + y)$

b. $(a - b)^2 - c(a - b)$

c. $x^2(c + d) + 5(c + d)$

d. $n^2(a - b) - 9(a - b)$

e. $x^2(a + 4) + 5x(a + 4) + 6(a + 4)$

f. $y^2(m + n) + 7y(m + n)$

g. $2x^2(a + b) + 3x(a + b) - 5(a + b)$

h. $4x^2(w + z) - 9(w + z)$

i. $(u - w)^2 - 9(u - w)$

j. $n^2(a + b) - 9n(a + b)$

k. $(t + r)y^2 - 100(t + r)$

l. $3ax^2 - 20ax - 7a$

☐ GROUPING WITH FOUR TERMS

EXAMPLE 5: **Factor: $a^2 + ac + ab + bc$**

Solution: Group the first two terms and the last two terms:

$$(a^2 + ac) + (ab + bc)$$

Now factor each pair of grouped terms separately (using the GCF) :

$$a(a + c) + b(a + c)$$

Even though we've grouped and factored, we can't be done because there are still two terms, and we need one term in the final answer to a factoring question. So we continue -- using our knowledge of the previous section -- and factor out the GCF, which is $a + c$:

$$\boxed{(a + c)(a + b)}$$

By the commutative property of multiplication, the final answer could also be written $(a + b)(a + c)$. Also, to check our answer, just double distribute the answer and you should get the original expression.

EXAMPLE 6: **Factor: $x^3 - 7x^2 - 9x + 63$**

Solution: Group the first two terms and the last two terms:

$$(x^3 - 7x^2) + (-9x + 63)$$

Now factor the GCF in each pair of grouped terms. The first GCF is obvious: x^2. Choosing the GCF in the second grouping is a little trickier -- should we choose 9 or –9? Ultimately, it's a trial-and-error process. Watch what happens if we choose –9 for the GCF:

$x^2(x-7)-9(x-7)$ (check the signs carefully)

We now see two terms whose GCF is $x-7$:

$(x-7)(x^2-9)$

All this, and we're still not done. The second factor is the difference of two squares -- now we're done:

$\boxed{(x-7)(x+3)(x-3)}$

EXAMPLE 7: **Factor:** $\boldsymbol{ab+cd+ad+bc}$

Solution: Group the first two terms and the last two terms (after all, this technique worked quite well in the previous two examples):

$(ab+cd)+(ad+bc)$

We're stuck; there's no way to factor either pair of terms (the GCF = 1 in each case), so let's swap the two middle terms of the original problem and again group in pairs:

$(ab+ad)+(cd+bc)$

Pull out the GCF from each set of parentheses:

$a(b+d)+c(d+b)$

Do we have a common factor in these two terms? Well, does $b+d = d+b$? Since addition is commutative, of course they are equal. So the GCF is $b+d$, and when we pull it out in front, we're done:

$\boxed{(b+d)(a+c)}$

EXAMPLE 8: **Factor: $2ax - bx - 2ay + by$**

Solution: Group in pairs, as usual:

$$(2ax - bx) + (-2ay + by)$$

Pull out the GCF in each grouping:

$$x(2a - b) + y(-2a + b)$$

Problem: There's no common factor; however, the factors $2a - b$ and $-2a + b$ are opposites of each other, and that gives us a clue. Let's go back to our first step and factor out $-y$ rather than y:

$$x(2a - b) - y(2a - b) \qquad \text{(distribute to make sure we're right)}$$

Now we see a good GCF, so we pull it out in front, and we're done:

$$\boxed{(2a - b)(x - y)}$$

Homework

3. Factor each expression:

a. $xw + xz + wy + yz$
b. $a^2 + ac + ab + bc$
c. $x^3 - 4x^2 + 3x - 12$
d. $n^3 - n^2 - 5n + 5$
e. $x^3 + x^2 - 9x - 9$
f. $ac - bd + bc - ad$
g. $xw + yz - xz - wy$
h. $2ac - 2ad + bc - bd$
i. $6xw - yz + 3xz - 2wy$
j. $hj - j^2 - hk + jk$
k. $ax + ay - bx - by$
l. $x^3 - 2x^2 - 25x + 50$
m. $xw + 2wy - xz - 2yz$
n. $a^3 - a^2 - 5a + 5$
o. $4tw - 2tx + 2w^2 - wx$
p. $6x^3 + 2x^2 - 9x - 3$
q. Not factorable
r. $6a^3 - 15a^2 + 10a - 25$

❑ MORE GROUPING AND SUBSTITUTION PROBLEMS

EXAMPLE 9: **Factor: $(w + z)^2 - a^2$**

Solution: After some practice, you might not need a substitution for this kind of problem, but we'll use one for this problem. Let $n = w + z$. The starting problem then becomes

$$n^2 - a^2$$

This is just a standard difference of squares:

$$(n + a)(n - a)$$

Now substitute in the other direction:

$$\boxed{(w + z + a)(w + z - a)}$$

EXAMPLE 10: **Factor: $x^2 + 6x + 9 - y^2$**

Solution: Grouping in pairs has worked quite well so far, so let's try it again:

$$(x^2 + 6x) + (9 - y^2)$$

We see that the first pair of terms has a nice GCF of x, and the second is the difference of squares:

$$x(x + 6) + (3 + y)(3 - y)$$

Good try, but there's no common factor in these two terms. In fact, no grouping into pairs will result in a common factor -- a dead end. Let's go back to the original problem and regroup so that the first three terms are together:

$$(x^2 + 6x + 9) - y^2$$

The first set of three terms is a perfect square trinomial, and factors into the square of a binomial:

$$(x + 3)^2 - y^2$$

leaving us with another difference of squares (just like the previous example), which factors to

$$\boxed{(x + 3 + y)(x + 3 - y)}$$

Homework

4. Factor each expression:

a. $(x + y)^2 - z^2$

b. $(a - b)^2 - c^2$

c. $x^2 + 4x + 4 - y^2$

d. $n^2 - 6n + 9 - Q^2$

e. $(u + w)^2 - T^2$

f. $y^2 + 10y + 25 - x^2$

g. $a^2 + 2ab + b^2 - c^2$

h. $w^2 - 2wy + y^2 - 49$

i. $4x^2 + 4x + 1 - t^2$

j. $9x^2 - 12x + 4 - y^2$

☐ *FACTORING CUBICS USING THE GCF*

EXAMPLE 11: **Factor each cubic (3rd degree) polynomial:**

A. $\mathbf{5q^3 + 10q^2 + 5q}$

This is not as bad as it looks, if we remember to start with the GCF:

$5q^3 + 10q^2 + 5q$ (the polynomial to factor)

$= 5q(q^2 + 2q + 1)$ (factor out $5q$, the GCF)

$= \quad 5q(q + 1)(q + 1)$ (factor the trinomial)

$= \quad \boxed{5q(q + 1)^2}$ (write it more simply)

B. $\mathbf{4x^3 - x}$

$= \quad x(4x^2 - 1)$ (factor out x, the GCF)

$= \quad \boxed{x(2x + 1)(2x - 1)}$ (difference of squares)

Homework

5. Factor each cubic polynomial:

a. $x^3 - x$

b. $2n^3 + 6n^2 + 4n$

c. $10a^3 - 5a^2 - 5a$

d. $7y^3 + 70y^2 + 175y$

e. $36w^3 - 9w$

f. $24z^3 - 20z^2 - 24z$

FACTORING THE SUM AND DIFFERENCE OF CUBES

We've learned that we can factor the *difference of squares* $x^2 - y^2$ into $(x + y)(x - y)$. We've also determined that the *sum of squares* $x^2 + y^2$ cannot be factored. Now we're about to show that the ***difference of cubes*** $x^3 - y^3$ can also be factored -- and perhaps surprisingly -- even the ***sum of cubes*** $x^3 + y^3$ can be factored. We begin with a discussion of division, remainders, and factors.

Is 3 a factor of 161? No -- divide 161 by 3 and you'll get 53 remainder 2. Since the remainder is not zero, 3 is not a factor of 161. In other words, 3 does not go into 161 "evenly."

Is 7 a factor of 161? Yes -- divide 161 by 7 and you'll get 23, remainder 0. Thus, 7 divides into 161 exactly 23 times. And therefore, 161 = 7 × 23. We have factored 161 into 7 × 23 by showing that the factor 7 divides into 161 without remainder. These observations are the key to factoring the sum and difference of cubes.

Perfect Cubes

We know that $2^3 = 8$. Since the cube of 2 is 8, we say that 8 is a ***perfect cube***. Here are some more examples of perfect cubes:

125 is a perfect cube because it's the cube of 5.

1 is a perfect cube because it's the cube of 1.

x^3 is a perfect cube because it's the cube of x.

$27y^3$ is a perfect cube because it's the cube of $3y$.

$8n^6$ is a perfect cube because it's the cube of $2n^2$.

$64z^{12}$ is a perfect cube because it's the cube of $4z^4$.

Homework

6. a. $64m^3$ is a perfect cube because it's the cube of _____.

b. $216n^3$ is a perfect cube because it's the cube of _____.

c. $27A^6$ is a perfect cube because it's the cube of _____.

d. _____ is a perfect cube because it's the cube of $7z^2$.

e. _____ is a perfect cube because it's the cube of $-3a^3$.

Factoring a *Sum* of Cubes

We're now ready to try to factor a sum of cubes; for example, what is the factorization of $x^3 + 8$? To answer this question, we should try to divide $x^3 + 8$ by something that goes into it evenly; that is, divide $x^3 + 8$ by something that will leave a remainder of zero. But what should we divide by? Since both terms of $x^3 + 8$ are perfect cubes, let's divide it by the binomial $x + 2$, since these two terms are the cube roots of x^3 and 8. Maybe this will work and maybe it won't, but we've got to try something.

$$\begin{array}{r l}
 & \quad\; x^2 \;-\; 2x + 4 \\
x+2 & \overline{)\,x^3 + 0x^2 + 0x + 8} \\
 & \underline{x^3 + 2x^2} \\
 & \quad -2x^2 + 0x \\
 & \quad \underline{-2x^2 - 4x} \\
 & \qquad\qquad 4x + 8 \\
 & \qquad\qquad \underline{4x + 8} \\
 & \qquad\qquad\qquad 0
\end{array}$$

Here's the division of $x^3 + 8$ by $x + 2$. Note that the dividend has two zeros placed in it to account for the missing terms.

Also note that the remainder is 0. This means that x + 2 is a factor of $x^3 + 8$ and therefore, that $x^2 - 2x + 4$ is the other factor.

Now we write the results of our long division in the form of a multiplication problem, giving us the factorization of x3 + 8:

$$\boxed{x^3 + 8 \;=\; (x+2)(x^2 - 2x + 4)}$$

Factoring a *Difference* of Cubes

For our difference of cubes, let's try to factor $n^3 - 27$. What do you think one of the factors will be? Consider the binomial consisting of the individual cube roots of n^3 and -27, namely $n - 3$. This time it's your turn to carry out the long division. Here's what you should end up with:

$$\begin{array}{r} n^2 + 3n + 9 \\ n-3\,\overline{\big)\, n^3 + 0n^2 + 0n - 27} \end{array}$$

We now have our factorization:

$$\boxed{n^3 - 27 = (n-3)(n^2+3n+9)}$$

EXAMPLE 12:

A. Factor: $N^3 - 1$. Divide $N^3 - 1$ by $N - 1$ and you should get the factorization $\boldsymbol{N^3 - 1 = (N - 1)(N^2 + N + 1)}$.

B. Factor: $8p^3 + 27$. Divide $8p^3 + 27$ by $2p + 3$. It should divide evenly, thus giving $\boldsymbol{8p^3 + 27 = (2p + 3)(4p^2 - 6p + 9)}$.

C. Factor: $(a + b)^3 - 125$. This is tricky, and it will be much easier to perform the long division if we make a substitution first. If we let $x = a + b$, then the expression to factor becomes $x^3 - 125$. The appropriate quantity to divide this by would be $x - 5$. When the long division is finished, the quotient is $x^2 + 5x + 25$ with remainder 0. We therefore get the factorization

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25)$$

But the original problem didn't have any x's in it. So we need to substitute back the other way -- converting each x back into $a + b$, we get the factorization

$$(a + b)^3 - 125 = \big((a + b) - 5\big)\big((a + b)^2 + 5(a + b) + 25\big),$$

which can be simplified to the final answer of

$$\boldsymbol{(a + b)^3 - 125 = (a + b - 5)(a^2 + 2ab + b^2 + 5a + 5b + 25)}.$$

Homework

7. In the discussion above, we arrived at the following factorizations:

   a. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   b. $n^3 - 27 = (n - 3)(n^2 + 3n + 9)$

   Verify each result by simplifying the right side of the statement so that it becomes the left side.

8. Factor each expression:

   a. $x^3 - 8$　　b. $n^3 + 27$　　c. $z^3 + 1$

   d. $8x^3 - 27$　　e. $27y^3 + 125$　　f. $64a^3 - 1$

9. Factor each expression:

   a. $(x + y)^3 + 8$　　b. $(a - b)^3 - 27$　　c. $(p + q)^3 + 1$

10. Factor $x^5 + 1$. Hint: Divide by $x + 1$.

11. Factor $A^5 - 32$. Hint: Divide by $A - 2$.

12. Factor each expression:

    a. $w^5 + 1$　　b. $c^5 - 1$　　c. $y^5 - 32$

    d. $z^5 + 32$　　e. $n^5 + 243$　　f. $m^5 - 243$

13. Factor each expression:

    a. $x^7 - 1$　　b. $y^7 + 1$　　c. $u^7 - 128$　　d. $z^7 + 128$

Practice Problems

14. Factor each expression:

a. $10ax^4 - 160a$

b. $Z^2(P - Q) - 144(P - Q)$

c. $50x^3 - 75x^2 - 2x + 3$

d. $12ac - 10bd + 8bc - 15ad$

e. $a^2 - 2ab + b^2 - c^2$

f. $x^2 + 2xy + y^2 - 144$

g. $x^4 - 34x^2 + 225$

h. $x^4 - 8x^2 - 9$

i. $x^3 - 7x^2 + 9x - 63$

j. $n^3 + 3n^2 - 16n - 48$

k. $(a + b)^2 - 5(a + b) + 6$

l. $(x - y)^2 + 7(x - y) + 6$

m. $(a - b)^2 + 6(a - b) - 16$

n. $hm - hn + km - kn$

15. Factor each expression:

a. $n^3 + 64$

b. $a^3 - 125$

c. $8T^3 - 27$

d. $27x^3 + 1$

Solutions

1. a. $(x^2 + 1)(x + 1)(x - 1)$

b. $(x^2 + 2)(x^2 - 3)$

c. $(n + 1)(n - 1)(n + 3)(n - 3)$

d. $(a^2 + 9)(a + 3)(a - 3)$

e. $(2w + 1)(2w - 1)(3w + 2)(3w - 2)$

f. $(x + 1)(x - 1)(3x + 5)(3x - 5)$

g. $(c^2 + 4)(c + 2)(c - 2)$

h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 2)(x^2 - 5)$

j. $(g^2 + 16)(g +4)(g - 4)$

k. $(2u + 3)(2u - 3)(3u + 1)(3u - 1)$

l. Not factorable

2. a. $(x + y)(x + y + 7)$
b. $(a - b)(a - b - c)$
c. $(c + d)(x^2 + 5)$
d. $(a - b)(n + 3)(n - 3)$
e. $(a + 4)(x + 3)(x + 2)$
f. $y(m + n)(y + 7)$
g. $(a + b)(2x + 5)(x - 1)$
h. $(w + z)(2x + 3)(2x - 3)$
i. $(u - w)(u - w - 9)$
j. $n(a + b)(n - 9)$
k. $(t + r)(y + 10)(y - 10)$
l. $a(3x + 1)(x - 7)$

3. a. $(x + y)(w + z)$
b. $(a + b)(a + c)$
c. $(x^2 + 3)(x - 4)$
d. $(n^2 - 5)(n - 1)$
e. $(x + 1)(x + 3)(x - 3)$
f. $(a + b)(c - d)$
g. $(x - y)(w - z)$
h. $(2a + b)(c - d)$
i. $(3x - y)(2w + z)$
j. $(h - j)(j - k)$
k. $(a - b)(x + y)$
l. $(x - 2)(x + 5)(x - 5)$
m. $(x + 2y)(w - z)$
n. $(a^2 - 5)(a - 1)$
o. $(2t + w)(2w - x)$
p. $(2x^2 - 3)(3x + 1)$
q. Not factorable
r. $(3a^2 + 5)(2a - 5)$

4. a. $(x + y + z)(x + y - z)$
b. $(a - b + c)(a - b - c)$
c. $(x + 2 + y)(x + 2 - y)$
d. $(n - 3 + Q)(n - 3 - Q)$
e. $(u + w + T)(u + w - T)$
f. $(y + 5 + x)(y + 5 - x)$
g. $(a + b + c)(a + b - c)$
h. $(w - y + 7)(w - y - 7)$
i. $(2x + 1 + t)\ (2x + 1 - t)$
j. $(3x - 2 + y)(3x - 2 - y)$

5. a. $x(x + 1)(x - 1)$
b. $2n(n + 1)(n + 2)$
c. $5a(2a + 1)(a - 1)$
d. $7y(y + 5)^2$
e. $9w(2w + 1)(2w - 1)$
f. $4z(3z + 2)(2z - 3)$

6. a. $4m$ b. $6n$ c. $3A^2$ d. $343z^6$ e. $-27a^9$

7. a. $(x + 2)(x^2 - 2x + 4) = x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 = x^3 + 8$ ✓
b. You try it.

8. a. $(x-2)(x^2+2x+4)$

b. $(n+3)(n^2-3n+9)$

c. $(z+1)(z^2-z+1)$

d. $(2x-3)(4x^2+6x+9)$

e. $(3y+5)(9y^2-15y+25)$

f. $(4a-1)(16a^2+4a+1)$

9. a. $(x+y+2)(x^2+2xy+y^2-2x-2y+4)$

b. $(a-b-3)(a^2-2ab+b^2+3a-3b+9)$

c. $(p+q+1)(p^2+2pq+q^2-p-q+1)$

10. $(x+1)(x^4-x^3+x^2-x+1)$

11. $(A-2)(A^4+2A^3+4A^2+8A+16)$

12. a. $(w+1)(w^4-w^3+w^2-w+1)$

b. $(c-1)(c^4+c^3+c^2+c+1)$

c. $(y-2)(y^4+2y^3+4y^2+8y+16)$

d. $(z+2)(z^4-2z^3+4z^2-8z+16)$

e. $(n+3)(n^4-3n^3+9n^2-27n+81)$

f. $(m-3)(m^4+3m^3+9m^2+27m+81)$

13. a. $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

b. $(y+1)(y^6-y^5+y^4-y^3+y^2-y+1)$

c. $(u-2)(u^6+2u^5+4u^4+8u^3+16u^2+32u+64)$

d. $(z+2)(z^6-2z^5+4z^4-8z^3+16z^2-32z+64)$

14. a. $10a(x^2 + 4)(x + 2)(x - 2)$
b. $(P - Q)(Z + 12)(Z - 12)$
c. $(2x - 3)(5x + 1)(5x - 1)$
d. $(3a + 2b)(4c - 5d)$
e. $(a - b + c)(a - b - c)$
f. $(x + y + 12)(x + y - 12)$
g. $(x + 5)(x - 5)(x + 3)(x - 3)$
h. $(x^2 + 1)(x + 3)(x - 3)$
i. $(x^2 + 9)(x - 7)$
j. $(n + 4)(n - 4)(n + 3)$
k. $(a + b - 3)(a + b - 2)$
l. $(x - y + 6)(x - y + 1)$
m. $(a - b + 8)(a - b - 2)$
n. $(m - n)(h + k)$

15. a. $(n + 4)(n^2 - 4n + 16)$
b. $(a - 5)(a^2 + 5a + 25)$
c. $(2T - 3)(4T^2 + 6T + 9)$
d. $(3x + 1)(9x^2 - 3x + 1)$

> "A college degree is not a sign that one is a finished product, but an indication a person is prepared for life."

Reverend Edward A. Malloy, *Monk's Reflections*

CH 20 – FRACTIONS, PART I

❑ *INTRODUCTION*

No matter what a fraction is used for, or no matter how complicated it looks, a fraction ultimately represents division. For example, $\frac{6}{3}$ is a fraction, but it is also the division problem 6 ÷ 3, which is why we write

$$\frac{6}{3} = 2$$

The fraction $\frac{1}{4}$ is a division problem (it equals 0.25) even if we never actually carry out the division.

The top of a fraction is called the ***numerator***, while the bottom is called the ***denominator***.

The ***reciprocal*** of a fraction is obtained by swapping the numerator and denominator; that is, by "inverting" the fraction. As examples, the reciprocal of $\frac{2}{7}$ is $\frac{7}{2}$, the reciprocal of $-\frac{9}{5}$ is $-\frac{5}{9}$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, the reciprocal of x is $\frac{1}{x}$, and the reciprocal of $\frac{1}{x}$ is x.

The only number which does not possess a reciprocal is 0 -- the reciprocal of 0 would have to be $\frac{1}{0}$, but we've learned that this fraction is undefined.

Last, an interesting property of reciprocals is that the product of a number and its reciprocal is always 1. For example, $\frac{2}{3}\times\frac{3}{2} = \frac{6}{6} = 1$.

Homework

1. True/False: Every real number has a reciprocal.

2. Find the **reciprocal** of each number:

a. $\frac{3}{2}$ b. $-\frac{4}{5}$ c. $-\frac{8}{7}$ d. 7 e. -14

f. $-\frac{1}{7}$ g. a h. $\frac{1}{x}$ i. $\frac{x}{y}$ j. $-\frac{y}{x}$

k. $-T$ l. $-\frac{1}{w}$ m. 1 n. -1 o. 0

3. a. What is the reciprocal of $-\frac{7}{19}$?

b. What is the reciprocal of $-\frac{19}{7}$?

c. Prove that $-\frac{17}{9}$ and $-\frac{9}{17}$ are reciprocals of each other by calculating their product (multiply them).

❑ REDUCING NUMBER FRACTIONS

Most of us learned to reduce an arithmetic fraction by dividing the top and the bottom of the fraction by the same (non-zero) number. For example,

$$\frac{30}{75} = \frac{30 \div 15}{75 \div 15} = \frac{2}{5}$$

When it comes to algebra, though, we need to look at reducing a fraction a little differently, since it's kind of hard to divide letters by letters. So now we reduce the same fraction using a different method, a method more appropriate to algebraic fractions. Watch this:

$$\frac{30}{75} = \frac{2 \cdot 3 \cdot 5}{3 \cdot 5 \cdot 5} = \frac{2 \cdot \cancel{3} \cdot \cancel{5}}{\cancel{3} \cdot \cancel{5} \cdot 5} = \frac{2}{5},$$

which is, of course, the same answer as before. Here's what we did. First we factored the top and the bottom into prime factors. Then we "divided out" any factor that was common to both the top <u>and</u> the bottom (since any number divided by itself is 1). Whatever factors remain constitute the final answer. Let's do another example.

$$\frac{18}{54} = \frac{2\cdot3\cdot3}{2\cdot3\cdot3\cdot3} = \frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{3}\cdot3} = \frac{1}{3}$$

Notice that we don't leave a zero in the numerator just because every factor in the numerator canceled out. Remember that canceling is really dividing, so each time we cancel a pair of factors, we're really dividing some number by itself, which is always 1.

Homework

4. Reduce each fraction to lowest terms by factoring into primes:

a. $\frac{20}{22}$ b. $\frac{17}{51}$ c. $\frac{84}{174}$ d. $\frac{22}{33}$ e. $\frac{20}{41}$

f. $\frac{30}{50}$ g. $\frac{34}{51}$ h. $\frac{13}{39}$ i. $\frac{26}{65}$ j. $\frac{32}{128}$

☐ REDUCING FRACTIONS BY FACTORING OUT THE GCF

If we can reduce an arithmetic fraction by factoring and dividing out common factors, we should be able to reduce an algebraic fraction the same way. Here are some examples.

<u>EXAMPLE 1:</u> **Reduce each fraction to lowest terms:**

A. $\frac{2x+8}{2a-10} = \frac{2(x+4)}{2(a-5)} = \frac{\cancel{2}(x+4)}{\cancel{2}(a-5)} = \frac{\boldsymbol{x+4}}{\boldsymbol{a-5}}$

B. $\dfrac{ax+bx}{xy-xz} = \dfrac{x(a+b)}{x(y-z)} = \dfrac{\cancel{x}(a+b)}{\cancel{x}(y-z)} = \dfrac{\boldsymbol{a+b}}{\boldsymbol{y-z}}$

C. $\dfrac{bn+an}{aq+bq} = \dfrac{n(b+a)}{q(a+b)} = \dfrac{n(a+b)}{q(a+b)} = \dfrac{n\cancel{(a+b)}}{q\cancel{(a+b)}} = \dfrac{\boldsymbol{n}}{\boldsymbol{q}}$

D. $\dfrac{ux-uw}{ax+aw} = \dfrac{u(x-w)}{a(x+w)}$

There's no common factor to cancel. So the original fraction is **not reducible**.

<u>EXAMPLE 2:</u> **Reduce each fraction to lowest terms:**

A. $\dfrac{x^2+x}{x} = \dfrac{x(x+1)}{x} = \dfrac{\cancel{x}(x+1)}{\cancel{x}} = \boldsymbol{x+1}$

B. $\dfrac{n^2-n}{n-1} = \dfrac{n(n-1)}{n-1} = \dfrac{n\cancel{(n-1)}}{\cancel{n-1}} = \boldsymbol{n}$

C. $\dfrac{a}{a^2-3a} = \dfrac{a}{a(a-3)} = \dfrac{\cancel{a}}{\cancel{a}(a-3)} = \dfrac{\mathbf{1}}{\boldsymbol{a-3}}$

D. $\dfrac{u-4}{u^2-4u} = \dfrac{u-4}{u(u-4)} = \dfrac{\cancel{u-4}}{u\cancel{(u-4)}} = \dfrac{\mathbf{1}}{\boldsymbol{u}}$

Here's a little trick that might help you understand better; we factor a 1 from the numerator:

$$\dfrac{u-4}{u^2-4u} = \dfrac{1(u-4)}{u(u-4)} = \dfrac{1\cancel{(u-4)}}{u\cancel{(u-4)}} = \dfrac{\mathbf{1}}{\boldsymbol{u}}$$

The Two Steps to Reduce a Fraction to Lowest Terms

1) <u>Factor</u> the top and the bottom of the fraction.

2) <u>Divide</u> out (cancel) any factor common to the top and bottom.

We now have a technique for reducing fractions to lowest terms, but it may still be a little hard to believe, for instance, that $\frac{n^2-n}{n-1} = n$, as in part B of the previous example. Perhaps you'll feel a little more confident in this answer if we substitute a number for *n* and verify the equality ourselves. So let's choose $n = 10$; then

$$\frac{n^2-n}{n-1} = \frac{\mathbf{10}^2-\mathbf{10}}{\mathbf{10}-1} = \frac{100-10}{10-1} = \frac{90}{9} = 10, \text{ which does equals } n.$$

EXAMPLE 3: **Reduce each fraction to lowest terms:**

A. $\frac{w-u}{w-u}$

Since anything (except zero) divided by itself is 1, the fraction reduces to **1**. For example,

$$\frac{10-3}{10-3} = \frac{7}{7} = 1$$

B. $\frac{x-y}{y-x}$

The top and bottom are not the same, but if you look carefully, you'll see that they're *opposites* of each other. Our trick here will be to factor a −1 out of the bottom:

$$\frac{x-y}{y-x} = \frac{1(x-y)}{-1(-y+x)} = \frac{1(x-y)}{-1(x-y)} = \frac{1\cancel{(x-y)}}{-1\cancel{(x-y)}} = \mathbf{-1}$$

For instance,

$$\frac{7-2}{2-7} = \frac{5}{-5} = -1$$

C. $\dfrac{a+b}{a-b}$

Unlike part A, the top and bottom are not the same, so the final answer is not 1. Unlike part B, the top and bottom are not opposites of each other, so the answer is not –1. What relationship do the top and bottom have? Basically, none at all; there's nothing we can do here. Thus, the fraction is **not reducible**.

Here's an example: $\dfrac{10+1}{10-1} = \dfrac{11}{9}$, and that's the end of it.

Homework

5. Reduce each fraction to lowest terms:

a. $\dfrac{3x-12}{3n+21}$ b. $\dfrac{ax+bx}{ay+by}$ c. $\dfrac{tx+tz}{ty-tz}$

d. $\dfrac{xy+yz}{ay-by}$ e. $\dfrac{aR+aT}{bR+bT}$ f. $\dfrac{mx+my}{ax-ay}$

g. $\dfrac{ax+bx}{ax-cx}$ h. $\dfrac{ax-bx}{ay-by}$ i. $\dfrac{a}{am+an}$

j. $\dfrac{PT-QT}{T}$ k. $\dfrac{gn+hn}{gn-hn}$ l. $\dfrac{ab-ac}{cx-xy}$

6. Reduce each fraction to lowest terms:

a. $\dfrac{x^2+3x}{x}$ b. $\dfrac{n^2-n}{n-1}$ c. $\dfrac{z}{z+z^2}$ d. $\dfrac{Q-3}{QR-3R}$

e. $\dfrac{y^2-y}{y}$ f. $\dfrac{a^2+a}{a+1}$ g. $\dfrac{t-1}{t^2-t}$ h. $\dfrac{n}{n^2+9n}$

i. $\dfrac{a-3}{a^2-3a}$ j. $\dfrac{c+2}{c^2+c}$ k. $\dfrac{z^2-3z}{z-3}$ l. $\dfrac{Q^2-10Q}{Q}$

m. $\dfrac{c}{ac-c^2}$ n. $\dfrac{rx+x^2}{r+1}$ o. $\dfrac{ax+bx^2}{xy-xz^2}$ p. $\dfrac{an+b}{n}$

7. Reduce each fraction to lowest terms:

a. $\dfrac{x+z}{z+x}$ b. $\dfrac{Q-R}{R-Q}$ c. $\dfrac{b-a}{-a+b}$ d. $\dfrac{m+n}{m-n}$

e. $\dfrac{d+e}{-e-d}$ f. $\dfrac{w-z}{z+w}$ g. $\dfrac{f+g}{g-f}$ h. $\dfrac{c+d}{d-c}$

i. $\dfrac{t-n}{n-t}$ j. $\dfrac{t-n}{n+t}$ k. $\dfrac{c-x}{-x+c}$ l. $\dfrac{y-w}{w-y}$

❑ *REDUCING FRACTIONS BY STANDARD FACTORING*

EXAMPLE 4: **Reduce each fraction to lowest terms:**

A. $$\frac{x^2+5x+6}{x^2-4} = \frac{(x+3)(x+2)}{(x+2)(x-2)} = \frac{(x+3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \boldsymbol{\frac{x+3}{x-2}}$$

B. $$\frac{6n^2+5n-21}{3n^2+22n+35} = \frac{(2n-3)(3n+7)}{(n+5)(3n+7)} = \frac{(2n-3)\cancel{(3n+7)}}{(n+5)\cancel{(3n+7)}} = \boldsymbol{\frac{2n-3}{n+5}}$$

C. $$\frac{x+5}{2x^2+7x-15} = \frac{x+5}{(2x-3)(x+5)} = \frac{\cancel{x+5}}{(2x-3)\cancel{(x+5)}} = \boldsymbol{\frac{1}{2x-3}}$$

D. $$\frac{3w^2-2w-5}{3w-5} = \frac{(3w-5)(w+1)}{3w-5} = \frac{\cancel{(3w-5)}(w+1)}{\cancel{3w-5}} = \boldsymbol{w+1}$$

E. $\dfrac{a^2-25}{a^2+5a+6} = \dfrac{(a+5)(a-5)}{(a+3)(a+2)}$ which contains no common factors.

Therefore, this fraction is **not reducible**.

EXAMPLE 5: **Reduce to lowest terms:** $\frac{5a^2-45}{a^2+6a+9}$

Solution: We need to factor the numerator and the denominator. If we then see any common factors, we can divide them out.

$$\frac{5a^2-45}{a^2+6a+9} = \frac{5(a^2-9)}{a^2+6a+9} = \frac{5(a+3)(a-3)}{(a+3)(a+3)} = \frac{5\cancel{(a+3)}(a-3)}{\cancel{(a+3)}(a+3)}$$

Notice that factoring the numerator required two steps: pulling out the greatest common factor of 5, followed by factoring the $a^2 - 9$. If we hadn't factored out the 5 first, we would never have been able to divide out anything, and we would have reached the erroneous conclusion that the fraction is not reducible.

Thus, the final answer (after distributing the 5 to the $a - 3$) is

$$\boxed{\frac{5a-15}{a+3}}$$

EXAMPLE 6: **Reduce to lowest terms:** $\frac{x^3-5x^2-2x+10}{x^2-15x+50}$

Solution: Let's factor the top first, by grouping the first two terms and the last two terms:

$$x^3 - 5x^2 - 2x + 10$$

$$= x^2(x - 5) - 2(x - 5)$$

$$= (x - 5)(x^2 - 2)$$

This result, together with factoring the bottom gives the fraction

$\frac{(x-5)(x^2-2)}{(x-5)(x-10)}$, which then clearly reduces to

$$\boxed{\frac{x^2-2}{x-10}}$$

Homework

8. Reduce each fraction to lowest terms:

a. $\dfrac{n^2-9}{n^2+6n+9}$ b. $\dfrac{4a^2+4a+1}{2a^2+5a+2}$ c. $\dfrac{c+7}{c^2-49}$

d. $\dfrac{4w^2-9}{2w-3}$ e. $\dfrac{6x^2+11x-7}{2x^2+17x-9}$ f. $\dfrac{h^2+3h+2}{h^2-3h+2}$

g. $\dfrac{3k^2-17k+1}{3k^2-17k+1}$ h. $\dfrac{100-w^2}{w^2+10w}$ i. $\dfrac{10x^2+11x-6}{5x^2-12x+4}$

j. $\dfrac{x^2+7x+10}{x^2+3x+2}$ k. $\dfrac{n^2-9}{n^2+4n+3}$ l. $\dfrac{y-7}{y^2-14y+49}$

m. $\dfrac{w^2-81}{w+9}$ n. $\dfrac{m^2+10m+25}{m^2-25}$ o. $\dfrac{x^2+2x+1}{x^2-4}$

p. $\dfrac{6a^2+13a-5}{9a^2+12a-5}$ q. $\dfrac{k^2-6k+7}{7-6k+k^2}$ r. $\dfrac{16u^2+34u-15}{2u^2+3u-5}$

9. Reduce each fraction to lowest terms:

a. $\dfrac{n^2-4}{n^2-4n+4}$ b. $\dfrac{2x^2+8x+6}{6x^2+18x+12}$

c. $\dfrac{x^2-4x+1}{x^2-4x+1}$ d. $\dfrac{10y^2-30y+20}{5y^2-15y+10}$

e. $\dfrac{2x^2-2}{4x-4}$ f. $\dfrac{3n^2-3n-90}{3n^2+30n+75}$

g. $\dfrac{14x+98}{21x^2-63x-1470}$ h. $\dfrac{5a^2-30a-135}{10a^2-60a-270}$

10. Reduce each fraction to lowest terms:

a. $\dfrac{x^3+4x^2+x+4}{x^2+x-12}$ b. $\dfrac{n^2+2n+1}{n^3+n^2+3n+3}$

c. $\dfrac{a^3-a^2-5a+5}{a^2+3a-4}$ d. $\dfrac{w^2-25}{w^3-5w^2+3w-15}$

☐ *ADDING AND SUBTRACTING FRACTIONS WITH THE SAME DENOMINATOR*

$\dfrac{2}{9}+\dfrac{5}{9} = \dfrac{2+5}{9} = \dfrac{7}{9}$ $\Rightarrow$ $\dfrac{a}{b}+\dfrac{c}{b} = \dfrac{a+c}{b}$

To **add** fractions with the same (common) denominator

- i) the numerator of the answer is the sum of the two numerators
- ii) the denominator of the answer is the common denominator

---

$\dfrac{5}{11}-\dfrac{3}{11} = \dfrac{5-3}{11} = \dfrac{2}{11}$

$\dfrac{2}{13}-\dfrac{8}{13} = \dfrac{-6}{13} = -\dfrac{6}{13}$ $\Rightarrow$ $\dfrac{x}{y}-\dfrac{z}{y} = \dfrac{x-z}{y}$

To **subtract** fractions with the same (common) denominator

- i) the numerator of the answer is the difference of the two numerators
- ii) the denominator of the answer is the common denominator

Homework

11. Perform the indicated operation:

a. $\frac{7}{10}+\frac{2}{10}$ b. $\frac{43}{101}+\frac{44}{101}$ c. $\frac{7}{8}-\frac{3}{8}$ d. $\frac{1}{5}-\frac{3}{5}$

e. $\frac{x}{y}+\frac{z}{y}$ f. $\frac{m}{u}-\frac{q}{u}$ g. $\frac{c}{a}+\frac{d}{a}$ h. $\frac{t}{w}-\frac{z}{w}$

i. $\frac{b}{c}+\frac{7}{c}$ j. $\frac{9}{Q}-\frac{7}{Q}$ k. $\frac{5}{R}-\frac{10}{R}$ l. $\frac{1}{3a}+\frac{7}{3a}$

m. $\frac{a}{bc}-\frac{d}{bc}$ n. $\frac{a}{x^2}+\frac{b}{x^2}$ o. $\frac{3}{10x}+\frac{4}{10x}$ p. $\frac{u}{a^3}-\frac{b}{a^3}$

☐ ADDING AND SUBTRACTING FRACTIONS WITH DIFFERENT DENOMINATORS

$$\begin{aligned} & \frac{2}{3}+\frac{5}{7} \\ = \quad & \frac{2}{3}\left[\frac{7}{7}\right]+\frac{5}{7}\left[\frac{3}{3}\right] \\ = \quad & \frac{14}{21}+\frac{15}{21} \\ = \quad & \frac{29}{21} \end{aligned}$$

⇨

$$\begin{aligned} & \frac{a}{b}+\frac{c}{d} \\ = \quad & \frac{a}{b}\left[\frac{d}{d}\right]+\frac{c}{d}\left[\frac{b}{b}\right] \\ = \quad & \frac{ad}{bd}+\frac{bc}{bd} \\ = \quad & \frac{ad+bc}{bd} \end{aligned}$$

$$\begin{aligned} & 3+\frac{4}{5} \\ = \quad & \frac{3}{1}+\frac{4}{5} \\ = \quad & \frac{3}{1}\left[\frac{5}{5}\right]+\frac{4}{5} \\ = \quad & \frac{15}{5}+\frac{4}{5} \\ = \quad & \frac{19}{5} \end{aligned}$$

⇨

$$\begin{aligned} & x+\frac{y}{z} \\ = \quad & \frac{x}{1}+\frac{y}{z} \\ = \quad & \frac{x}{1}\left[\frac{z}{z}\right]+\frac{y}{z} \\ = \quad & \frac{xz}{z}+\frac{y}{z} \\ = \quad & \frac{xz+y}{z} \end{aligned}$$

To **add** or **subtract** fractions with different denominators, each fraction must be written with the same denominator. This is accomplished by multiplying one or both fractions by a fraction equal to 1. A fraction is equal to 1 when the top and bottom are the same.

In the first example above, we see that we need to change each denominator into 21. We do this by multiplying the top and bottom of the first fraction by 7, and then multiplying the top and bottom of the second fraction by 3.

Similarly, in the second example, to achieve a common denominator we multiply the top and bottom of the first fraction by *d*, and the top and bottom of the second fraction by *b*. This converts both fractions into fractions with the same denominator, *bd*. Then they're ready to be added together.

Homework

12. Perform the indicated operation:

a. $\frac{1}{2}+\frac{1}{3}$ b. $\frac{2}{5}-\frac{1}{10}$ c. $\frac{1}{4}-\frac{5}{6}$ d. $\frac{1}{3}-\frac{9}{2}$

e. $\frac{a}{b}+\frac{w}{x}$ f. $\frac{c}{d}-\frac{g}{h}$ g. $\frac{m}{n}+\frac{m}{q}$ h. $\frac{a}{b}-\frac{a}{c}$

i. $\frac{6}{a}+\frac{3}{b}$ j. $\frac{a}{R}-\frac{3}{T}$ k. $a+\frac{b}{c}$ l. $w-\frac{x}{y}$

m. $\frac{k}{j}+n$ n. $\frac{w}{x}-z$ o. $\frac{x^2}{a}+\frac{y^2}{a}$ p. $\frac{w}{x}+\frac{y}{z}$

☐ MULTIPLYING AND DIVIDING FRACTIONS

$\frac{7}{10}\cdot\frac{11}{15} = \frac{77}{150}$ ⇨ $\frac{m}{n}\cdot\frac{p}{q} = \frac{mp}{nq}$

To **multiply** fractions

i) the numerator of the answer is the product of the numerators

ii) the denominator of the answer is the product of the denominators

---

$\frac{2}{7}\div\frac{5}{9} = \frac{2}{7}\times\frac{9}{5} = \frac{18}{35}$ ⇨ $\frac{c}{d}\div\frac{u}{w} = \frac{c}{d}\times\frac{w}{u} = \frac{cw}{du}$

To **divide** fractions, multiply the first fraction by the reciprocal of the second fraction.

$\frac{5}{6}\times\frac{13}{5} = \frac{\cancel{5}}{6}\times\frac{13}{\cancel{5}} = \frac{13}{6}$ ⇨ $\frac{x}{y}\times\frac{z}{x} = \frac{\cancel{x}}{y}\times\frac{z}{\cancel{x}} = \frac{z}{y}$

Before actually multiplying the tops and bottoms, sometimes dividing out common factors can simplify the reducing of the final answer.

Homework

13. Perform the indicated operation:

a. $\frac{2}{3}\cdot\frac{5}{9}$ b. $\frac{1}{2}\times\frac{5}{7}$ c. $\left(\frac{8}{9}\right)\left(\frac{9}{10}\right)$ d. $\frac{4}{5}\cdot\frac{5}{4}$

e. $\frac{w}{x}\cdot\frac{w}{z}$ f. $\frac{a}{b}\times\frac{c}{b}$ g. $\frac{x}{y}\cdot\frac{c}{d}$ h. $\left[\frac{a}{b}\right]\left[\frac{b}{a}\right]$

i. $a\times\frac{b}{c}$ j. $\frac{m}{n}\times Q$ k. $z\div\frac{w}{z}$ l. $\frac{u}{w}\div a$

m. $\frac{x}{y}\div\frac{x}{y}$ n. $\frac{a}{b}\div\frac{b}{a}$ o. $\frac{a}{b}\div\frac{a}{c}$ p. $\frac{g}{h}\cdot\frac{g}{h}$

q. $3\cdot\frac{x}{y}$ r. $a\left(\frac{b}{c}\right)$ s. $\left(\frac{w}{7}\right)A$ t. $\frac{w}{7}+A$

14. Perform the indicated operation:

a. $\frac{K}{L}\div\frac{K}{M}$ b. $\frac{n}{b}+\frac{n}{b}$ c. $\frac{x}{a}-\frac{x}{a}$ d. $\frac{a}{b}\cdot\frac{b}{c}$

e. $\frac{G}{H}\div\frac{G}{H}$ f. $\frac{1}{a}+\frac{1}{b}$ g. $\frac{2}{c}-\frac{3}{d}$ h. $\frac{a}{w}-\frac{a}{z}$

i. $\frac{R}{T}\times\frac{T}{R}$ j. $6+\frac{x}{y}$ k. $\frac{u}{m}-n$ l. $\frac{x}{y}\div\frac{x}{z}$

m. $\left(\frac{b}{c}\right)\left(\frac{b}{c}\right)$ n. $\frac{b}{c}\times\frac{c}{b}$ o. $\frac{m}{n}\div\frac{p}{q}$ p. $\frac{p}{q}+\frac{m}{n}$

q. $a\left(\frac{b}{c}\right)$ r. $\frac{w}{n}\div 5$ s. $\frac{p}{m}+E$ t. $\frac{x}{y}+\frac{a}{b}\cdot\frac{b}{a}$

Practice Problems

15. Reduce each fraction to lowest terms:

a. $\frac{5x+20}{10x-45}$ b. $\frac{ax-bx}{cx+dx}$ c. $\frac{am+an}{a}$ d. $\frac{w}{w^2-3w}$

e. $\frac{t^2+t}{t+1}$ f. $\frac{R^2-4R}{R}$ g. $\frac{2x+8}{3x-12}$ h. $\frac{ax+ay}{bx+by}$

16. Reduce each fraction to lowest terms:

a. $\frac{5x+10}{10x+45}$ b. $\frac{a^2-ab}{ac+ad}$ c. $\frac{cm+cn}{c}$ d. $\frac{u}{u^2+7u}$

e. $\frac{t-1}{t^2-t}$ f. $\frac{R^2+99R}{R}$ g. $\frac{2x+8}{6x-12}$ h. $\frac{ax+ay}{bx-by}$

17. Reduce each fraction to lowest terms:

a. $\frac{6a^2-5a-21}{3a^2-4a-7}$ b. $\frac{18x+18}{14x^2+42x+28}$ c. $\frac{x^2-9}{x^2-4}$

d. $\frac{10a^2+29a-21}{5a^2-38a+21}$ e. $\frac{x-3}{x^2-9}$ f. $\frac{n^2+14n+49}{(n+7)^2}$

g. $\frac{x^3-2x^2-3x+6}{x^2-x-2}$ h. $\frac{u^2-14u+49}{u^3-7u^2+6u-42}$ i. $\frac{x^3+x^2+x+1}{1+x+x^2+x^3}$

18. a. What is the reciprocal of $-\frac{2}{9}$?

b. What is the reciprocal of 0?

19. Perform the indicated operation:

a. $\frac{a}{b}+\frac{c}{b}$

b. $\frac{x}{y}+\frac{y}{x}$

c. $a-\frac{w}{u}$

d. $\frac{m}{n}-A$

e. $\frac{x}{y}\cdot\frac{w}{z}$

f. $\frac{x}{z}\times\frac{z}{y}$

g. $\frac{a}{b}\div\frac{c}{d}$

h. $c\div\frac{d}{e}$

i. $\frac{g}{h}\div\pi$

j. $\frac{m}{\pi}\cdot\frac{\pi}{m}$

k. $h+\frac{k}{h}$

l. $\frac{a}{b}\left(\frac{b}{c}\right)$

m. $\left(\frac{a}{t}\right)\left(\frac{t}{a}\right)$

n. $\frac{a}{n^2}+\frac{b}{n^2}$

o. $\frac{8}{ab}+\frac{6}{ab}$

p. $\frac{a}{x^3}-\frac{b}{x^3}$

q. $\frac{w}{4c}+\frac{w}{4c}$

r. $\frac{a}{mn}-\frac{a}{mn}$

s. $\frac{A}{B}\cdot\frac{C}{D}$

t. $\frac{C}{D}\div\frac{B}{A}$

u. $\frac{s}{r}\div s$

v. $Q\div\frac{Q}{R}$

w. $\frac{6}{L}\div\frac{6}{L}$

x. $\frac{M}{Q}\div\frac{Q}{M}$

y. $\frac{ab}{xy}+\frac{c}{xy}$

z. $\frac{abc}{d}-\frac{def}{d}$

Solutions

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1. False; 0 does not have a reciprocal.

2.
a. $\frac{2}{3}$
b. $-\frac{5}{4}$
c. $-\frac{7}{8}$
d. $\frac{1}{7}$
e. $-\frac{1}{14}$
f. -7
g. $\frac{1}{a}$
h. x
i. $\frac{y}{x}$
j. $-\frac{x}{y}$
k. $-\frac{1}{T}$
l. $-w$
m. 1
n. -1
o. Undefined

3.
a. $-\frac{19}{7}$
b. $-\frac{7}{19}$
c. $\left(-\frac{17}{9}\right)\left(-\frac{9}{17}\right) = \left(-\frac{{}^{1}\cancel{17}}{{}_{1}\cancel{9}}\right)\left(-\frac{\cancel{9}^{1}}{\cancel{17}_{1}}\right) = 1$ ✓

4. a. $\frac{20}{22} = \frac{\cancel{2}\cdot 2\cdot 5}{\cancel{2}\cdot 11} = \frac{10}{11}$ b. $\frac{1}{3}$ c. $\frac{14}{29}$ d. $\frac{2}{3}$ e. $\frac{20}{41}$
f. $\frac{3}{5}$ g. $\frac{2}{3}$ h. $\frac{1}{3}$ i. $\frac{2}{5}$ j. $\frac{1}{4}$

5. a. $\frac{x-4}{n+7}$ b. $\frac{x}{y}$ c. $\frac{x+z}{y-z}$ d. $\frac{x+z}{a-b}$
e. $\frac{a}{b}$ f. Not reducible g. $\frac{a+b}{a-c}$ h. $\frac{x}{y}$
i. $\frac{1}{m+n}$ j. $P-Q$ k. $\frac{g+h}{g-h}$ l. Not reducible

6. a. $x+3$ b. n c. $\frac{1}{1+z}$ d. $\frac{1}{R}$
e. $y-1$ f. a g. $\frac{1}{t}$ h. $\frac{1}{n+9}$
i. $\frac{1}{a}$ j. Not reducible k. z l. $Q-10$
m. $\frac{1}{a-c}$ n. Not reducible o. $\frac{a+bx}{y-z^2}$ p. Not reducible

7. a. 1 b. –1 c. 1 d. Not reducible e. –1 f. Not reducible
g. Not reducible h. Not reducible i. –1 j. Not reducible
k. 1 l. –1

8. a. $\frac{n-3}{n+3}$ b. $\frac{2a+1}{a+2}$ c. $\frac{1}{c-7}$ d. $2w+3$
e. $\frac{3x+7}{x+9}$ f. Not reducible g. 1 h. $\frac{10-w}{w}$
i. $\frac{2x+3}{x-2}$ j. $\frac{x+5}{x+1}$ k. $\frac{n-3}{n+1}$ l. $\frac{1}{y-7}$
m. $w-9$ n. $\frac{m+5}{m-5}$ o. Not reducible p. $\frac{2a+5}{3a+5}$
q. 1 r. $\frac{8u-3}{u-1}$

9. a. $\frac{n+2}{n-2}$ b. $\frac{x+3}{3x+6}$ c. 1 d. 2
e. $\frac{x+1}{2}$ f. $\frac{n-6}{n+5}$ g. $\frac{2}{3x-30}$ h. $\frac{1}{2}$

10. a. $\frac{x^2+1}{x-3}$ b. $\frac{n+1}{n^2+3}$ c. $\frac{a^2-5}{a+4}$ d. $\frac{w+5}{w^2+3}$

11. a. $\frac{9}{10}$ b. $\frac{87}{101}$ c. $\frac{1}{2}$ d. $-\frac{2}{5}$
e. $\frac{x+z}{y}$ f. $\frac{m-q}{u}$ g. $\frac{c+d}{a}$ h. $\frac{t-z}{w}$
i. $\frac{b+7}{c}$ j. $\frac{2}{Q}$ k. $-\frac{5}{R}$ l. $\frac{8}{3a}$
m. $\frac{a-d}{bc}$ n. $\frac{a+b}{x^2}$ o. $\frac{7}{10x}$ p. $\frac{u-b}{a^3}$

12. a. $\frac{5}{6}$ b. $\frac{3}{10}$ c. $-\frac{7}{12}$ d. $-\frac{25}{6}$
e. $\frac{ax+bw}{bx}$ f. $\frac{ch-dg}{dh}$ g. $\frac{mq+mn}{nq}$ h. $\frac{ac-ab}{bc}$
i. $\frac{6b+3a}{ab}$ j. $\frac{aT-3R}{RT}$ k. $\frac{ac+b}{c}$ l. $\frac{wy-x}{y}$
m. $\frac{k+jn}{j}$ n. $\frac{w-xz}{x}$ o. $\frac{x^2+y^2}{a}$ p. $\frac{wz+xy}{xz}$

13. a. $\frac{10}{27}$ b. $\frac{5}{14}$ c. $\frac{4}{5}$ d. 1
e. $\frac{w^2}{xz}$ f. $\frac{ac}{b^2}$ g. $\frac{cx}{dy}$ h. 1
i. $\frac{ab}{c}$ j. $\frac{mQ}{n}$ k. $\frac{z^2}{w}$ l. $\frac{u}{aw}$
m. 1 n. $\frac{a^2}{b^2}$ o. $\frac{c}{b}$ p. $\frac{g^2}{h^2}$
q. $\frac{3x}{y}$ r. $\frac{ab}{c}$ s. $\frac{Aw}{7}$ t. $\frac{w+7A}{7}$

14. a. $\frac{M}{L}$ b. $\frac{2n}{b}$ c. 0 d. $\frac{a}{c}$

e. 1 f. $\frac{a+b}{ab}$ g. $\frac{2d-3c}{cd}$ h. $\frac{az-aw}{wz}$

i. 1 j. $\frac{x+6y}{y}$ k. $\frac{u-mn}{m}$ l. $\frac{z}{y}$

m. $\frac{b^2}{c^2}$ n. 1 o. $\frac{mq}{np}$ p. $\frac{np+mq}{qn}$

q. $\frac{ab}{c}$ r. $\frac{w}{5n}$ s. $\frac{p+Em}{m}$ t. $\frac{x+y}{y}$

15. a. $\frac{x+4}{2x-9}$ b. $\frac{a-b}{c+d}$ c. $m + n$ d. $\frac{1}{w-3}$

e. t f. $R - 4$ g. Not reducible h. $\frac{a}{b}$

16. a. $\frac{x+2}{2x+9}$ b. $\frac{a-b}{c+d}$ c. $m + n$ d. $\frac{1}{u+7}$

e. $\frac{1}{t}$ f. $R + 99$ g. $\frac{x+4}{3x-6}$ h. Not reducible

17. a. $\frac{2a+3}{a+1}$ b. $\frac{9}{7x+14}$ c. Not reducible d. $\frac{2a+7}{a-7}$

e. $\frac{1}{x+3}$ f. 1 g. $\frac{x^2-3}{x+1}$ h. $\frac{u-7}{u^2+6}$

i. 1

18. a. $-\frac{9}{2}$ b. 0 does not have a reciprocal.

19. a. $\frac{a+c}{b}$ b. $\frac{x^2+y^2}{xy}$ c. $\frac{au-w}{u}$ d. $\frac{m-An}{n}$

e. $\frac{wx}{yz}$ f. $\frac{x}{y}$ g. $\frac{ad}{bc}$ h. $\frac{ce}{d}$

i. $\frac{g}{\pi h}$ j. 1 k. $\frac{h^2+k}{h}$ l. $\frac{a}{c}$

m. 1 n. $\frac{a+b}{n^2}$ o. $\frac{14}{ab}$ p. $\frac{a-b}{x^3}$

q. $\frac{w}{2c}$

r. 0

s. $\frac{AC}{BD}$

t. $\frac{AC}{BD}$

u. $\frac{1}{r}$

v. R

w. 1

x. $\frac{M^2}{Q^2}$

y. $\frac{ab+c}{xy}$

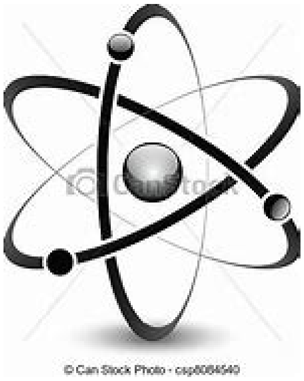
z. $\frac{abc-def}{d}$

"It is far better to grasp the Universe as it really is than to persist in delusion, however satisfying and reassuring."

– Carl Sagan (1934 – 1996)

CH 21 – NEGATIVE EXPONENTS

❑ INTRODUCTION



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The mass of a proton (the positively charged particle in the nucleus of an atom) is written in "scientific notation" as

$$\mathbf{1.67 \times 10^{-27}\ kilograms}.$$

How do we read that? Is that a really big or a really small number? In this chapter we will learn exactly what is meant by a negative exponent.

If n is a natural number ($n = 1, 2, 3, \ldots$), then we know that the meaning of x^n is based upon repeated multiplication:

$$x^n = \underbrace{(x)(x)(x)\cdots(x)}_{n \text{ factors of } x}$$

For instance, we know that x^3 means $x \cdot x \cdot x$. We also learned that $x^0 = 1$ (as long as x itself isn't 0). But what does something like x^{-3} mean?

❑ REVIEW OF THE FIVE LAWS OF EXPONENTS

Before we begin the discussion of negative exponents, it would be beneficial for us to review the five laws of exponents.

$$x^a x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

<u>EXAMPLE 1</u>: **Simplify each expression:**

A. $x^3x^4 = x^7$

When multiplying powers of the same base, add the exponents.

B. $\dfrac{a^{10}}{a^2} = a^8$

When dividing powers of the same base, subtract the exponents.

C. $(z^3)^4 = z^{12}$

When raising a power to a power, multiply the exponents.

D. $(mn)^4 = m^4n^4$

When raising a product to a power, apply the power to all the factors of the product.

E. $\left(\dfrac{x}{y}\right)^7 = \dfrac{x^7}{y^7}$

When raising a quotient to a power, apply the power to both the numerator and denominator.

<u>EXAMPLE 2</u>: **Simplify each expression:**

A. r^4t^5 = As is

Don't do anything. You cannot multiply powers of <u>different</u> bases by adding the exponents.

B. $a^2 + a^4$ = As is

Again, don't try combining these terms. First, the exponent rule regarding adding the exponents applies only when the operation is multiplication, not addition. Second, they can't be added either, since they're not like terms.

C. $w^3 + w^3 = 2w^3$

This sum <u>can</u> be simplified, but <u>not</u> by the first law of exponents, since the powers of w are <u>not</u> being multiplied. But the two terms <u>are</u> like terms, which means we simply add them together to get $2w^3$.

D. $(a + b)^{23}$ = As is (for now)

It does <u>not</u> equal $a^{23} + b^{23}$. The third law of exponents, $(xy)^n = x^n y^n$, does not apply because xy is a single term, but $a + b$ consists of two terms.

Homework

1. Simplify each expression:

a. $n^4 n^5$
b. $\dfrac{b^{18}}{b^6}$
c. $(z^4)^5$
d. $(abc)^4$
e. $\left(\dfrac{k}{n}\right)^4$
f. $a^3 b^7$
g. $x^3 + x^2$
h. $p^4 + p^4$
i. $u^3 - u^3$
j. $(a + b)^2$
k. $(x - y)^3$
l. $(g + h)^{56}$
m. $\dfrac{d^{20}}{d^{30}}$
n. $x^3 x^5 + x^6 x^2$

❑ DEVELOPING THE MEANING OF A NEGATIVE EXPONENT

Consider the following:

$$x^{-3}$$

What could it mean? To find out, let's multiply it by x^3 and see what happens:

$$x^{-3} \cdot x^3 = x^{-3+3} = x^0 = 1$$

which implies that

$$x^{-3} \cdot x^3 = 1 \qquad \text{(using rules we already have)}$$

Now, x^{-3} is what we're analyzing, so let's "solve" for it (or isolate it) by dividing each side of the equation by x^3:

$$\frac{x^{-3} \cdot x^3}{x^3} = \frac{1}{x^3}$$

$$\Rightarrow \quad \boxed{x^{-3} = \frac{1}{x^3}}$$

You could do the same thing to n^{-12}:

$$n^{-12} \cdot n^{12} = n^0 = 1$$

$$\Rightarrow \quad n^{-12} \cdot n^{12} = 1$$

$$\Rightarrow \quad \frac{n^{-12} \cdot n^{12}}{n^{12}} = \frac{1}{n^{12}}$$

$$\Rightarrow \quad \boxed{n^{-12} = \frac{1}{n^{12}}}$$

Homework

2. Use the above logic to rewrite the expression with NO negative exponents:

   a. w^{-4} b. y^{-19} c. z^{-50} d. 2^{-10}

☐ WORKING WITH NEGATIVE EXPONENTS

The previous section should convince you that a negative exponent means reciprocal. In fact, the exponent doesn't have to be a negative whole number. [Even things like $w^{-5/4}$ and $h^{-\sqrt{2\pi}}$ represent reciprocals.]

In general,

$$\boxed{x^{-n} = \frac{1}{x^n}}$$

for ANY number *n*.

This fact, together with our knowledge of fractions and the laws of exponents, is all we need to understand all of the examples in this chapter.

EXAMPLE 2:

A. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B. $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

C. $-3^{-4} = -\dfrac{1}{3^4} = -\dfrac{1}{81}$

See the difference?

D. $(-3)^{-4} = \dfrac{1}{(-3)^4} = \dfrac{1}{81}$

E. $\dfrac{1}{2^{-5}} = \dfrac{1}{\dfrac{1}{2^5}} = \dfrac{1}{\dfrac{1}{32}} = 32$

F. $\dfrac{1}{10^{-3}} = \dfrac{1}{\dfrac{1}{10^3}} = 10^3 = 1000$

G. $\left(\dfrac{2}{3}\right)^{-3} = \dfrac{1}{\left(\dfrac{2}{3}\right)^3} = \dfrac{1}{\dfrac{8}{27}} = \dfrac{27}{8}$

H. $0^{-5} = \dfrac{1}{0^5} = \dfrac{1}{0} = \text{Undefined}$

Homework

3. Evaluate each expression:

a. 3^{-2} b. 2^{-5} c. 10^{-3} d. 5^{-4} e. 1^{-10}

f. -2^{-3} g. $(-3)^{-3}$ h. -10^{-4} i. $(-5)^{-1}$ j. -6^{-2}

k. $\dfrac{1}{3^{-2}}$ l. $\dfrac{1}{10^{-4}}$ m. $\dfrac{1}{5^{-3}}$ n. $\left(\dfrac{3}{4}\right)^{-2}$ o. $\left(\dfrac{1}{6}\right)^{-3}$

4. Explain why $x = 0$ is NOT allowed in the expression x^{-8}.

EXAMPLE 3:

A. $n^{-1} = \frac{1}{n}$

B. $a^{-12} = \frac{1}{a^{12}}$

C. $\frac{x}{y^{-3}} = \frac{x}{\frac{1}{y^3}} = x \cdot \frac{y^3}{1} = xy^3$

D. $\frac{x^{-7}}{a^{12}} = \frac{\frac{1}{x^7}}{a^{12}} = \frac{1}{x^7} \cdot \frac{1}{a^{12}} = \frac{1}{a^{12}x^7}$

E. $a^2b^{-3} = a^2 \cdot \frac{1}{b^3} = \frac{a^2}{b^3}$

F. $R^{-1}T^{-7} = \frac{1}{R} \cdot \frac{1}{T^7} = \frac{1}{RT^7}$

EXAMPLE 4:

A. $x + y^{-5} = x + \frac{1}{y^5} = \left[\frac{y^5}{y^5}\right]\frac{x}{1} + \frac{1}{y^5} = \frac{xy^5}{y^5} + \frac{1}{y^5} = \frac{xy^5 + 1}{y^5}$

B. $a^{-2} + b^{-7} = \frac{1}{a^2} + \frac{1}{b^7} = \frac{1}{a^2}\left[\frac{b^7}{b^7}\right] + \frac{1}{b^7}\left[\frac{a^2}{a^2}\right] = \frac{b^7 + a^2}{a^2b^7}$

Homework

5. Simplify each expression (no negative exponents in the answer):

a. x^{-9}

b. a^{-30}

c. $\dfrac{a}{b^{-2}}$

d. $\dfrac{x^{-4}}{y^{2}}$

e. $\dfrac{w^{3}}{x^{-3}}$

f. $\dfrac{m^{-3}}{n^{-4}}$

g. $a^{-4}b^{-5}$

h. $k^{3}p^{-5}$

i. $h^{-5}z^{7}$

j. $\dfrac{w^{-5}}{w^{-5}}$

k. $x+y^{-1}$

l. $a^{-3}+b^{2}$

m. $p^{-1}+r^{-1}$

n. $w^{-3}-z^{-2}$

o. $a^{-3}-a^{-3}$

p. $a^{-1}b^{-2}c^{-3}$

<u>EXAMPLE 5:</u> **Using the Five Laws of Exponents:**

A. $x^{-6}x^{-8} = x^{-14} = \dfrac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{14}}}$

Add the exponents

B. $\left(N^{-5}\right)^{-7} = \boldsymbol{N}^{\mathbf{35}}$

Multiply the exponents

C. $(abc)^{-4} = a^{-4}b^{-4}c^{-4} = \dfrac{1}{a^4}\cdot\dfrac{1}{b^4}\cdot\dfrac{1}{c^4} = \dfrac{\mathbf{1}}{\boldsymbol{a}^{\mathbf{4}}\boldsymbol{b}^{\mathbf{4}}\boldsymbol{c}^{\mathbf{4}}}$

Apply exponent to
each factor

D. $\dfrac{w^{5}}{w^{15}} = w^{5-15} = w^{-10} = \dfrac{\mathbf{1}}{\boldsymbol{w}^{\mathbf{10}}}$

Subtract the exponents

E. $\dfrac{x^{-17}}{x^{-12}} = x^{-17-(-12)} = x^{-17+12} = x^{-5} = \dfrac{\mathbf{1}}{\boldsymbol{x^5}}$

Subtract the exponents

F. $\left(\dfrac{a}{b}\right)^{-3} = \dfrac{a^{-3}}{b^{-3}} = \dfrac{\dfrac{1}{a^3}}{\dfrac{1}{b^3}} = \dfrac{1}{a^3}\cdot\dfrac{b^3}{1} = \dfrac{\boldsymbol{b^3}}{\boldsymbol{a^3}}$

Apply exponent to
top and bottom

Another way: $\left(\dfrac{a}{b}\right)^{-3} = \dfrac{1}{\left(\dfrac{a}{b}\right)^3} = \dfrac{1}{\dfrac{a^3}{b^3}} = \dfrac{b^3}{a^3}$

G. $(x+y)^{-2}$

Be careful; no law of exponents applies here since the quantity $x + y$ consists of <u>two</u> terms -- it is not a product -- so do NOT apply the exponent to each term of the sum. Begin the problem by dealing with the meaning of the negative exponent.

$$(x+y)^{-2} = \frac{1}{(x+y)^2} = \frac{1}{(x+y)(x+y)} = \frac{\mathbf{1}}{\boldsymbol{x^2 + 2xy + y^2}}$$

Homework

6. Simplify each expression (no negative exponents in the answer):

a. $a^{-3}a^7$ b. $x^{-5}x^{-10}$ c. $k^{-10}k^{10}$ d. $m^{-5}w^{-3}$

e. $(x^2)^{-5}$ f. $(a^{-5})^3$ g. $(c^{-3})^{-4}$ h. $(b^2)^{-1}$

i. $(ax)^{-5}$ j. $(xyz)^{-3}$ k. $(x+y)^{-1}$ l. $(a-b)^{-2}$

m. $\dfrac{x^{-5}}{x^{3}}$ n. $\dfrac{z^{5}}{z^{-4}}$ o. $\dfrac{a^{-5}}{a^{-8}}$ p. $\dfrac{y^{-12}}{y^{-7}}$

q. $\left(\dfrac{a}{b}\right)^{-4}$ r. $\left(\dfrac{y}{x}\right)^{-1}$ s. $\left(\dfrac{a}{b+c}\right)^{-2}$ t. $\left(\dfrac{a+b}{xy+wz}\right)^{0}$

EXAMPLE 6: **Simplify each expression:**

A. $$(2x^{-8})(-5x^{6}) = (2)(-5)x^{-8}x^{6} = -10x^{-2} = -10\left(\frac{1}{x^{2}}\right) = -\boldsymbol{\frac{10}{x^{2}}}$$

B. $$\left(x^{2}y^{-3}\right)^{4} = \left(x^{2}\right)^{4}\left(y^{-3}\right)^{4} = x^{8}y^{-12} = x^{8}\left(\frac{1}{y^{12}}\right) = \boldsymbol{\frac{x^{8}}{y^{12}}}$$

C. $$\left(\frac{a^{-3}}{b^{4}}\right)^{-5} = \frac{\left(a^{-3}\right)^{-5}}{\left(b^{4}\right)^{-5}} = \frac{a^{15}}{b^{-20}} = \frac{a^{15}}{\frac{1}{b^{20}}} = \frac{a^{15}}{1}\cdot\frac{b^{20}}{1} = \boldsymbol{a^{15}b^{20}}$$

Final Note: In the Introduction we learned about the mass of a proton. We're now in a position to see just what that number is:

$$1.67\times10^{-27} = 1.67\times\frac{1}{10^{27}} = \frac{1.67}{1}\times\frac{1}{10^{27}} = \frac{1.67}{10^{27}}$$

$$= \frac{1.67}{1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$$

$$= \boxed{0.00000000000000000000000000167 \text{ kilograms}}$$

A proton certainly doesn't weigh very much! In addition, I hope you can appreciate why we use scientific notation (1.67×10^{-27}) rather than the number in the box.

Homework

7. Simplify each expression (no negative exponents in the answer):

a. $(2x^{-4})(3x^7)$ b. $(-3y^{-3})(2y^{-5})$ c. $(-4a^3)(a^{-7})$

d. $\left(u^3u^{-5}\right)^{-2}$ e. $\left(w^{-5}w^{-1}\right)^7$ f. $\left(a^{-2}b^{-3}\right)^{-5}$

g. $\left(t^3u^{-4}\right)^3$ h. $\left(\frac{x^2}{x^5}\right)^{-2}$ i. $\left(\frac{y^{-3}}{y^{-4}}\right)^5$

j. $\left(\frac{a^{-2}}{b^5}\right)^{-3}$ k. $\left(\frac{c^3}{d^{-4}}\right)^{-5}$ l. $\left(\frac{a^3b^{-4}}{x^{-12}z^{-2}}\right)^0$

8. Express each scientific notation number as a regular number:

a. 2.3×10^7 b. 7.11×10^{-5} c. 5.09×10^{-10}

Practice Problems

9. Evaluate: a. -2^{-4} b. $(-2)^{-4}$

10. Evaluate: a. $\left(\frac{4}{5}\right)^{-3}$ b. $\left(\frac{1}{9}\right)^{-1}$

11. Simplify: a. $\dfrac{a^{-2}}{b^{-3}}$ b. $\dfrac{x^{-9}}{x^{9}}$

12. Simplify: a. $a^{-2}a^{-3}$ b. $a^{-2} - a^{-3}$

13. Simplify: a. $(x^{-10})^{5}$ b. $(z^{-5})^{-5}$

14. Simplify: $(3x^{-3}y^{-2})^{-1}(-4x^{3}y^{10})^{-2}$

15. Simplify: a. $\left(\dfrac{x^{-4}}{x^{-6}}\right)^{-2}$ b. $\left(\dfrac{a^{3}}{b^{-4}}\right)^{5}$

16. Simplify: a. $(uw)^{-2}$ b. $(u + w)^{-2}$

17. Explain why $x = 0$ is NOT allowed in the expression x^{-2}.

18. Express as a regular number: 4.9×10^{-13}

Solutions

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1. a. n^{9} b. b^{12} c. z^{20} d. $a^{4}b^{4}c^{4}$ e. $\dfrac{k^{4}}{n^{4}}$ f. As is
g. As is h. $2p^{4}$ i. 0 j. $a^{2} + 2ab + b^{2}$
k. $x^{3} - 3x^{2}y + 3xy^{2} - y^{3}$ l. As is (for now) m. $\dfrac{1}{d^{10}}$ n. $2x^{8}$

2. a. $\dfrac{1}{w^{4}}$ b. $\dfrac{1}{y^{19}}$ c. $\dfrac{1}{z^{50}}$ d. $\dfrac{1}{2^{10}} = \dfrac{1}{1{,}024}$

3. a. $\dfrac{1}{9}$ b. $\dfrac{1}{32}$ c. $\dfrac{1}{1000}$ d. $\dfrac{1}{625}$ e. 1 f. $-\dfrac{1}{8}$

g. $-\frac{1}{27}$ h. $-\frac{1}{10{,}000}$ i. $-\frac{1}{5}$ j. $-\frac{1}{36}$ k. 9 l. 10,000
m. 125 n. $\frac{16}{9}$ o. 216

4. If x were 0 in the expression x^{-8}, we would have

$x^{-8} = 0^{-8} = \frac{1}{0^8} = \frac{1}{0}$, which is Undefined.

5. a. $\frac{1}{x^9}$ b. $\frac{1}{a^{30}}$ c. ab^2 d. $\frac{1}{x^4y^2}$ e. x^3w^3 f. $\frac{n^4}{m^3}$
g. $\frac{1}{a^4b^5}$ h. $\frac{k^3}{p^5}$ i. $\frac{z^7}{h^5}$ j. 1 k. $\frac{xy+1}{y}$ l. $\frac{1+a^3b^2}{a^3}$
m. $\frac{r+p}{pr}$ n. $\frac{z^2-w^3}{w^3z^2}$ o. 0 p. $\frac{1}{ab^2c^3}$

6. a. a^4 b. $\frac{1}{x^{15}}$ c. 1 d. $\frac{1}{m^5w^3}$ e. $\frac{1}{x^{10}}$
f. $\frac{1}{a^{15}}$ g. c^{12} h. $\frac{1}{b^2}$ i. $\frac{1}{a^5x^5}$ j. $\frac{1}{x^3y^3z^3}$
k. $\frac{1}{x+y}$ l. $\frac{1}{a^2-2ab+b^2}$ m. $\frac{1}{x^8}$ n. z^9
o. a^3 p. $\frac{1}{y^5}$ q. $\frac{b^4}{a^4}$ r. $\frac{x}{y}$ s. $\frac{b^2+2bc+c^2}{a^2}$
t. 1

7. a. $6x^3$ b. $-\frac{6}{y^8}$ c. $-\frac{4}{a^4}$ d. u^4 e. $\frac{1}{w^{42}}$ f. $a^{10}b^{15}$
g. $\frac{t^9}{u^{12}}$ h. x^6 i. y^5 j. a^6b^{15} k. $\frac{1}{c^{15}d^{20}}$ l. 1

8. a. 23,000,000 b. 0.0000711 c. 0.0000000000509

9. a. $-\frac{1}{16}$ b. $\frac{1}{16}$ **10**. a. $\frac{125}{64}$ b. 9

11. a. $\frac{b^3}{a^2}$ b. $\frac{1}{x^{18}}$ **12**. a. $\frac{1}{a^5}$ b. $\frac{a-1}{a^3}$

13. a. $\frac{1}{x^{50}}$ b. z^{25} **14**. $\frac{1}{48x^3y^{18}}$

15. a. $\frac{1}{x^4}$ b. $a^{15}b^{20}$ **16**. a. $\frac{1}{u^2w^2}$ b. $\frac{1}{u^2+2uw+w^2}$

17. Because $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ which is Undefined.

18. 0.00000000000049

> "A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."
>
> – ***George Bernard Shaw***