CH 0 – PROLOGUE

❑ *THE REAL NUMBERS*

You may have seen many different kinds of numbers in your previous math courses. Consider these examples:

$$7 \qquad 0 \qquad -9 \qquad 2.835 \qquad -\frac{15}{4} \qquad \frac{2}{3} \qquad \frac{7}{33} \qquad \pi \qquad -\sqrt{2}$$

As varied as all these numbers may seem, they actually have one critical <u>common</u> characteristic: They can all be written as ***decimal numbers***, some repeating and some non-repeating:

$7 = 7.0$	Repeating decimal (zeros forever)
$0 = 0.0$	Repeating decimal (zeros forever)
$-9 = -9.0$	Repeating decimal (zeros forever)
2.835	Repeating decimal (zeros forever)
$-\frac{15}{4} = -3.75$	Repeating decimal (zeros forever)
$\frac{2}{3} = 0.6666666\ldots$	Repeating decimal (6's forever)
$\frac{7}{33} = 0.21212121\ldots$	Repeating decimal (21's forever)
$\pi = 3.14159265\ldots$	Non-repeating decimal
$-\sqrt{2} = -1.41421356\ldots$	Non-repeating decimal

> The repeating decimals are called ***rational numbers***, and the non-repeating decimals are called ***irrational numbers.***

Note that some of the decimals repeat a block of digits forever (the ***rational*** numbers), while some don't repeat (the ***irrational*** numbers). Nevertheless, they are all decimals.

But what about a number like $\sqrt{-9}$? This number must be a number whose square is –9. Now, what number do we know which, when squared, would come out –9? Does 3 work? No, since $3^2 = 9$. Does –3 work? No, since $(-3)^2 = 9$, also. We conclude that there is <u>no</u> decimal in the world that can represent the number $\sqrt{-9}$, or for that matter, the square root of any negative number.

To distinguish between the numbers that are decimals and numbers like $\sqrt{-9}$, which can never be written as a decimal, the term ***real number*** was given to the decimals, and the term ***imaginary number*** was given to numbers like $\sqrt{-9}$.

> The ***Real Numbers*** is the combination of the *Rational Numbers* and the *Irrational Numbers*.

Hundreds of years ago, mathematicians thought it was obvious which numbers were real and which were imaginary. But this demonstrates a rather arrogant attitude. After all, to a beginning algebra student, a real number like $\sqrt{2}$ (which is an infinite, non-repeating decimal) may not seem "real" at all. Moreover, imaginary numbers, like $\sqrt{-1}$, seem very real to people (such as electronics engineers) who use them every day. The bottom line is, the terms *real* and *imaginary* are completely arbitrary -- one person's reality is another's imagination. But we're stuck with the terms, so we might as well learn them.

In summary, we call any number that can be written as a decimal a ***real number***, regardless of whether it repeats or not. The set of real numbers is often denoted by writing ℝ.

Homework

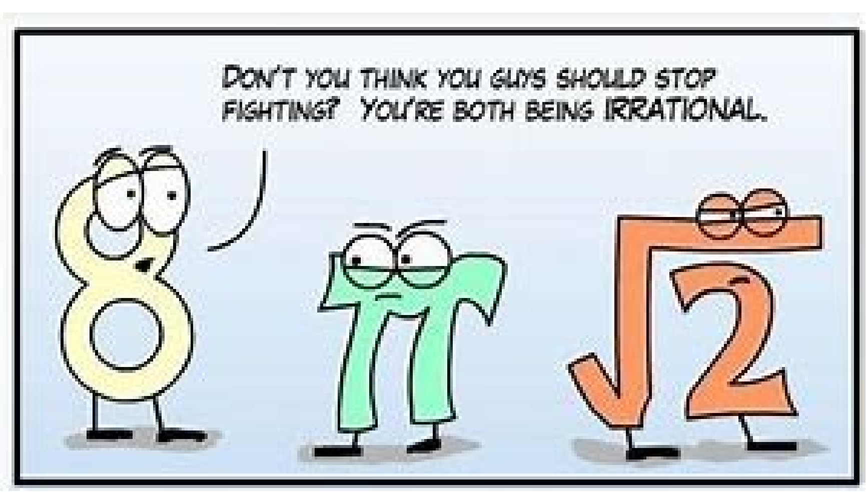
1. Classify each number as **real** or **imaginary**:

a. 123	b. -42	c. 0
d. 2.3	e. $\sqrt{-8}$	f. $\sqrt{144}$
g. $-\sqrt{81}$	h. $\sqrt{10}$	i. -23.78
j. $-\pi$	k. $\sqrt{3}$	l. $\sqrt{-121}$
m. 0.239057	n. $2.787878\ldots$	o. 3.092748526
p. $3.1428669\ldots$	q. $\sqrt{-(-8765)}$	r. $\sqrt{-0.25}$
s. $-\sqrt{-25}$	t. $\sqrt{-(-71)}$	u. 10^6

2. Put the following 13 real numbers in ascending order (smallest to biggest):

$$2\pi, \quad \sqrt{5}, \quad \sqrt{0}, \quad \frac{11}{3}, \quad 3.0808\ldots, \quad -\pi,$$

$$\frac{1}{101}, \quad -\sqrt{3}, \quad 2^3, \quad 3^2, \quad -1, \quad \sqrt{25}, \quad \sqrt{1}$$



❑ THREE OF THE THINGS WE DO TO REAL NUMBERS

Opposite

The ***opposite*** of a real number is found by changing the sign of the number. For example, the opposite of 7 is –7, the opposite of $-\pi$ is π, and the opposite of 0 is 0 (since 0 doesn't really have a sign). The opposite of n is $-n$, and the opposite of $-n$ is n. Also notice that the sum of a number and its opposite is always 0; for example, $17 + (-17) = 0$.

When considering numbers on the real number line, two numbers are opposites of each other if they're the same distance from 0, but on opposite sides of 0. [Note that although 0 is the opposite of 0, it's kind of hard to justify the claim that they're on "opposite" sides of 0.]

Homework

3. What is the **opposite** of each number?

   a. 17 b. 0 c. –3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.

   b. The opposite of 0 is ____.

   c. The opposite of a negative number is always _____.

   d. The opposite of a positive number is always _____.

   e. The sum of a real number and its opposite is always _____.

5. Using the formula $y = -x$, find the y-value for the given x-value:

   a. $x = 9$ b. $x = -3$ c. $x = 0$ d. $x = \pi$ e. $x = -\sqrt{2}$

Reciprocal

The ***reciprocal*** of a real number is found by dividing the number into 1. Equivalently, the reciprocal of x is $\frac{1}{x}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. Every real number has a reciprocal <u>except</u> 0; the reciprocal of 0 would be $\frac{1}{0}$, which is undefined, as explained later in the Prologue.

Notice that the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative. In addition, **the product of any real number with its reciprocal is always 1**; for example, $\frac{2}{7}\cdot\frac{7}{2} = 1$.

Homework

6. Find the reciprocal of each real number:

   a. 5 b. $\frac{2}{9}$ c. $-\frac{7}{3}$ d. 1 e. 0 f. $\frac{1}{\pi}$ g. $-\sqrt{3}$

7. a. T/F: Every number has a reciprocal.

   b. The reciprocal of 0 is ____.

   c. The reciprocal of a negative number is always _____.

   d. The reciprocal of a positive number is always _____.

   e. The product of a real number and its reciprocal is always ____.

8. Using the formula $y = \frac{1}{x}$, answer each question:

   a. If $x = 14$, then $y =$ ____.

   b. If $x = \frac{2}{3}$, then $y =$ ____.

   c. If $x = -99$, then $y =$ ____.

   d. If $x = -\frac{5}{4}$, then $y =$ ____.

   e. If $x = 0$, then $y =$ ____.

Absolute Value

The ***absolute value*** of a real number is its distance to 0 on the number line. The absolute value of 9 is 9, because the number 9 is 9 units from 0 on the number line. The absolute value of –5 is 5, because the number –5 is 5 units from 0 on the number line. As for the real number 0, its absolute value is 0, since the number 0 is 0 units away from 0.

The notation for absolute value is two vertical bars around the number. So, for example, the absolute value of –12 is written $|-12|$, and equals 12. Here are three more examples:

$|35| = 35$ $\qquad |-2.7| = 2.7$ $\qquad |0| = 0$

> In computer programming, $|-12|$ would be written
> **abs(12)**

Homework

9. Evaluate each expression:

   a. $|5\pi|$ b. $|-23.9|$ c. $|0|$ d. $|7-9|$ e. $|-\sqrt{7}|$

10. a. T/F: Every number has an absolute value.
    b. The absolute value of 0 is ____.
    c. The absolute value of a negative number is always _____.
    d. The absolute value of a positive number is always _____.

11. Which two of the following operations can be applied to all real number?

    Opposite Reciprocal Absolute Value

12. Find the **absolute value** of each number:

    a. 72 b. –99 c. 0 d. π e. $-\pi$ f. $-\sqrt{2}$

13. Evaluate each expression:

a. $|17 - 7|$ b. $|3 - 25|$ c. $|2(3) - 6(1)|$ d. $|2\pi + 3\pi|$

14. Using the formula $y = |x|$, answer each question:

a. If $x = 33$, then $y =$ ____.

b. If $x = 0$, then $y =$ ____.

c. If $x = -25$, then $y =$ ____.

d. If $y = 17$, then $x =$ ____ or ____.

e. If $y = 0$ then $x =$ ____.

f. If $y = -5$, then $x =$ ____.

❑ *FRACTIONS*

EXAMPLE 1:

A. **Express $\frac{7x}{2}$ as the product of two quantities:**

Solution: The easiest way to see this process is to just do it and then check that it's right. Here's what I claim:

$$\frac{7x}{2} = \frac{7}{2}x$$

And here's the reason:

$$\frac{7}{2}x = \frac{7}{2} \cdot \frac{x}{1} = \frac{7 \cdot x}{2 \cdot 1} = \frac{7x}{2} \checkmark$$

B. **Express $\frac{7x+9}{-5}$ as the sum or difference of two fractions.**

Solution: What two fractions have a sum of $\frac{7x+9}{-5}$? The -5 tells us that we could use fractions with a denominator of -5. Since the numerator is $7x + 9$, we can make one of the numerators $7x$ and the other one 9. That is

$$\frac{7x+9}{-5} = \frac{7x}{-5} + \frac{9}{-5} = -\frac{7}{5}x - \frac{9}{5}$$

C. **Combine the ideas of parts A and B to split up $\frac{8x-5}{7}$.**

Solution:

$$\frac{8x-5}{7} = \frac{8x}{7} - \frac{5}{7} = \boldsymbol{\frac{8}{7}x - \frac{5}{7}}$$

D. **Express $\frac{8x+16}{16}$ as the sum of two quantities.**

Solution:

$$\frac{8x+16}{16} = \frac{8x}{16} + \frac{16}{16} = \frac{x}{2} + 1 = \boldsymbol{\frac{1}{2}x + 1}$$

Homework

15. Express each fraction as the product of two quantities:

a. $\frac{9x}{4}$ b. $\frac{17x}{2}$ c. $\frac{3y}{16}$ d. $\frac{-33a}{17}$ e. $\frac{5n}{-23}$

f. $\frac{3x}{6}$ g. $\frac{-6x}{-2}$ h. $\frac{22m}{33}$ i. $\frac{-9x}{-15}$ j. $\frac{-39z}{52}$

16. Express each fraction as the sum or difference of two quantities, using parts C and D of Example 1 as a guide:

a. $\dfrac{3x+8}{4}$ b. $\dfrac{-9x+18}{6}$ c. $\dfrac{y-1}{8}$ d. $\dfrac{3n+15}{-5}$

e. $\dfrac{-3w-24}{2}$ f. $\dfrac{45x-75}{15}$ g. $\dfrac{33x+44}{55}$ h. $\dfrac{-31x-17}{-20}$

Operations with Fractions

$$-\frac{2}{3}-\frac{1}{2} = -\frac{4}{6}-\frac{3}{6} = -\mathbf{\frac{7}{6}}$$

$$\frac{4}{5}-\left(-\frac{2}{3}\right) = \frac{4}{5}+\frac{2}{3} = \frac{12}{15}+\frac{10}{15} = \mathbf{\frac{22}{15}}$$

$$\frac{2}{9}-7 = \frac{2}{9}-\frac{63}{9} = -\mathbf{\frac{61}{9}}$$

$$\left(\frac{2}{3}\right)\left(-\frac{5}{7}\right) = -\mathbf{\frac{10}{21}}$$

$$-\frac{4}{7}\div -2 = -\frac{4}{7}\times -\frac{1}{2} = \frac{4}{14} = \mathbf{\frac{2}{7}}$$

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{5}\right) = \mathbf{\frac{1}{120}}$$

$$\frac{-8}{\frac{1}{3}} = -\frac{8}{1}\div\frac{1}{3} = -\frac{8}{1}\times\frac{3}{1} = -\mathbf{24}$$

$$\frac{-\frac{9}{4}}{-2} = -\frac{9}{4}\div -2 = -\frac{9}{4}\div -\frac{2}{1} = -\frac{9}{4}\times -\frac{1}{2} = \mathbf{\frac{9}{8}}$$

Note: A negative sign can "float." For instance,

$$\frac{-30}{6} = \frac{30}{-6} = -\frac{30}{6}$$

since all of these fractions have the value –5.

Powers and Square Roots of Fractions

An **exponent** still means what it always has, so these next examples should be clear.

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3}\cdot\frac{2}{3} = \mathbf{\frac{4}{9}} \qquad \left(-\frac{1}{4}\right)^3 = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) = -\mathbf{\frac{1}{64}}$$

$$\left(-\frac{9}{4}\right)^1 = -\mathbf{\frac{9}{4}} \qquad \left(\frac{1}{2}\right)^8 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \mathbf{\frac{1}{256}}$$

As for the **square root sign**, we still ask the question: What number (that's not negative) times itself gives the number in the radical sign?

$\sqrt{\frac{9}{25}} = \mathbf{\frac{3}{5}}$ This is true because $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

$\sqrt{\frac{1}{144}} = \mathbf{\frac{1}{12}}$ This is because $\frac{1}{12}\times\frac{1}{12} = \frac{1}{144}$.

$\sqrt{-\frac{4}{49}}$ does <u>not</u> exist as a real number, because $-\frac{4}{49}$ is a negative number, and square roots of negative numbers are outside the real numbers. It's an imaginary number.

$\sqrt{\frac{-4}{-49}}$ <u>does</u> exist as a real number, because the fraction is actually a positive number: $\sqrt{\frac{-4}{-49}} = \sqrt{\frac{4}{49}} = \mathbf{\frac{2}{7}}$.

Homework

Perform the indicated operation:

17. a. $-\frac{1}{2}-\frac{4}{5}$ b. $-\frac{1}{3}-\left(-\frac{1}{3}\right)$ c. $\frac{2}{3}-\left(-\frac{5}{6}\right)$

d. $-\frac{4}{5}+\frac{2}{3}$ e. $9-\frac{4}{5}$ f. $-1-\frac{2}{3}$

g. $\frac{8}{3}-5$ h. $-\frac{2}{3}-(-1)$ i. $-\frac{1}{4}-\frac{2}{7}$

18. a. $\left(-\frac{1}{2}\right)\left(-\frac{5}{6}\right)$ b. $\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)$ c. $-\frac{5}{6}\cdot-\frac{6}{5}$

d. $-\frac{2}{3}\div-\frac{3}{2}$ e. $\frac{1}{2}\div-9$ f. $7\div-\frac{3}{4}$

g. $\dfrac{-\frac{2}{3}}{-\frac{1}{9}}$ h. $\dfrac{\frac{4}{5}}{-8}$ i. $\dfrac{-\frac{4}{5}}{\frac{5}{8}}$

19. True/False: $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$ [assuming $b \neq 0$]

20. a. $\left(-\frac{2}{3}\right)^2$ b. $\left(-\frac{1}{2}\right)^3$ c. $\left(-\frac{1}{3}\right)^4$

d. $\left(-\frac{14}{19}\right)^1$ e. $\left(-\frac{1}{2}\right)^5$ f. $\left(-\frac{2}{3}\right)^6$

g. $\left(\frac{99}{-99}\right)^{99}$

21. a. $\sqrt{\frac{81}{100}}$ b. $\sqrt{\frac{36}{64}}$ c. $\sqrt{\frac{1}{4}}$

d. $\sqrt{\frac{1}{9}}$ e. $\sqrt{\frac{121}{144}}$ f. $\sqrt{-\frac{25}{81}}$

g. $\sqrt{\frac{-256}{-289}}$ h. $\sqrt{-\frac{14}{17}-\left(-\frac{14}{17}\right)}$

❑ *ORDER OF OPERATIONS*

Order of Operations
Parentheses and Brackets [( )]
Exponents
Multiply & Divide (left to right)
Add & Subtract (left to right)

Note: Certainly $(-5)^2 = 25$, since both the 5 and the minus sign are being squared [i.e., $(-5)^2 = (-5)(-5) = 25$]. However, consider the expression

$$-5^2$$

Do we square the –5? The answer is NO; the exponent attaches to the 5 <u>only</u>. The justification is the Order of Operations, which states that exponents (near the top of the chart) are to be done before we deal with negative signs (which are at the bottom of the chart). So, although $(-5)^2 = 25$, we must agree that

$$-5^2 = -25$$

Homework

22. Evaluate (simplify) each expression:

a. $3 \cdot 10^2 - (8 - 4)^3 - 3 \times 2$

b. $(5 - 3)^2 + (10 - 7)^3$

c. $[3 + 2(5)] - 1 + 3 \cdot 10$

d. $2(10 - 5)^2 - 12 \div 3$

e. $[2(10 - 5)]^3 \div (10 \cdot 10^2)$

f. $(1 + 4)^2 - (4 + 1)^2$

g. $[(3^2 - 2^2)^3 - 80] \div (36 / 4)$

h. $3 \cdot 4^2 - (13 - 12)^3$

i. $10 + 8(8 - 1)^2 - 3 - 2 - 1$

j. $[8^2 - 2^3 + 3 \cdot 4 - 2(7)]^2$

k. $[20 - (5 - 2)^2]^2 - 2 \cdot 3 \cdot 4$

l. $[13 - (8 - 3) + (10 - 2)]^3$

23. Evaluate each expression for the given values:

a. $(x + y)^2$ for $x = 2$ and $y = 1$

b. $x^2 + y^2$ for $x = 10$ and $y = 5$

c. $x^2 + xy + y^2$ for $x = 3$ and $y = 6$

d. $(x + y)(x - y)$ for $x = 10$ and $y = 2$

e. $x^2 - y^2$ for $x = 12$ and $y = 10$

❑ TERMS

If the final operation in an expression is multiplication, then the expression consists of one term. The expression $A(B + C - D)$ consists of one term.

If the final operation consists of additions and/or subtractions, then the expression consists of at least two terms – the number of terms is found by counting the things being added or subtracted. The expression $ax - b + L(R + Q)$ consists of three terms: the ax, the b, and the $L(R + Q)$.

Homework

24. Determine the number of **terms** in each expression:

a. abc

b. $a + b - c$

c. $xyz - w$

d. $(x - 3)^2$

e. $x^2 + 25$

f. $ab + ac - xy$

g. $(a + b)(x - y)$

h. $rst - qrw$

i. $(rst)(qrw)$

j. $(rst)(qrw) - 1$

k. $[(rst)(qrw) - 1]^5$

l. $25 - (x + y - z)^2$

m. $a - b$

n. $x + y^2 + z$

o. $20 - mnpq$

p. $a + x - c + QT$

q. $(y + xm)^2A + B$

r. $(a - b)^3 - (c - d)^3 - w$

s. $abcd + x - y + wxyz$

t. $[a(b - c)ed - 7]^4$

u. $w(u - x)^3 - abc(def - mnpq)$

❑ DIVISION WITH ZEROS

This is the result of dividing by zero.

It's a mathematical fact of life that the only number that's never allowed to be in the denominator (bottom) of a fraction is zero. Sometimes this is phrased

"Never divide by zero."

Why the big deal?

Recall from elementary school that

$$\frac{56}{7} = 8 \text{ because } 8 \times 7 = 56.$$

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work.

Zero on the Bottom

Now let's put a zero on the bottom and see what happens:

$$\frac{9}{0} = ???$$

Let's try an answer of 0; unfortunately $0 \cdot 0 = 0$, not 9.

How about we try an answer of 9? Then $9 \cdot 0$ is also 0, not 9.

Could the answer be π? No; $\pi \cdot 0 = 0$, not 9.

In fact, any number we surmise as the answer will have to multiply with 0 to make a product of 9. But this is impossible, since any number times 0 is always 0, never 9. In short, no number in the whole world will work in this problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0; in fact, since $0 \cdot 0 = 0$, a possible answer is 0.

Let's try an answer of 5; because $5 \cdot 0 = 0$, another possible answer is 5.

Could π possibly work? Since $\pi \cdot 0 = 0$, another possible answer is π.

Is there any end to this madness? Apparently not, since any number we conjure up will multiply with 0 to make a product of 0. In short, every number in the whole world will work in this problem.

Summary:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is no answer to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are infinitely many answers to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

Thus,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

> "Black holes are where God divided by zero."
>
> *Steven Wright*

Homework

25. Evaluate each expression, and explain your conclusion:

a. $\frac{0}{15}$ b. $\frac{32}{0}$ c. $\frac{0}{0}$

26. Evaluate each expression:

a. $\frac{0}{17}$ b. $\frac{0}{-9}$ c. $\frac{6-6}{17+3}$ d. $\frac{3^2-8-1}{100}$

e. $\frac{98}{0}$ f. $\frac{-44}{0}$ g. $\frac{7+8}{2^3-8}$ h. $\frac{7^2-40}{-23+23}$

i. $\frac{0}{0}$ j. $\frac{-9+9}{10-10}$ k. $\frac{5^2-25}{0^2+0^3}$ l. $\frac{4\cdot 5-2\cdot 10}{3^3-9}$

27. $\frac{0}{\pi} = 0$ because
   a. 0 is the only number multiplied by π that will produce 0.
   b. no number times π equals 0.
   c. every number times π equals 0.

28. $\frac{0}{0}$ is undefined because
   a. no number times 0 equals 0.
   b. every number times 0 equals 0.
   c. any number divided by itself is 1.

29. $\frac{7}{0}$ is undefined because
   a. 0 is the only number multiplied by 0 that will produce 7.
   b. no number times 0 equals 7.
   c. every number times 0 equals 7.

30. a. The numerator of a fraction is 0. What can you conclude?
    b. The denominator of a fraction is 0. What can you conclude?

❑ *LINEAR EQUATIONS AND FORMULAS*

Solve for x: $\mathbf{2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)}$

Solution: The steps are

1) Distribute
2) Combine like terms
3) Solve the simplified equation

$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$

$\Rightarrow$ $6x - 14 - 5 + 15x = 4x - 1 + x + 7$ (distribute)

$\Rightarrow$ $21x - 19 = 5x + 6$ (combine like terms)

$\Rightarrow$ $21x \mathbf{- 5x} - 19 = 5x \mathbf{- 5x} + 6$ (subtract $5x$ from each side)

$\Rightarrow$ $16x - 19 = 6$ (simplify)

$$\Rightarrow \quad 16x - 19 \mathbf{+ 19} = 6 \mathbf{+ 19} \qquad \text{(add 19 to each side)}$$

$$\Rightarrow \quad 16x = 25 \qquad \text{(simplify)}$$

$$\Rightarrow \quad \frac{16x}{\mathbf{16}} = \frac{25}{\mathbf{16}} \qquad \text{(divide each side by 16)}$$

$$\Rightarrow \quad \boxed{x = \frac{25}{16}} \qquad \text{(simplify)}$$

Solve for x: $\quad \dfrac{\boldsymbol{nx - w}}{\boldsymbol{y + z}} = \boldsymbol{e - f}$

Solution: Notice the use of parentheses in the solution.

$$\frac{nx - w}{y + z} = e - f \qquad \text{(original formula)}$$

$$\Rightarrow \quad \frac{nx - w}{y + z}\left(y + z\right) = \left(e - f\right)\left(y + z\right) \qquad \text{(multiply each side by } y + z\text{)}$$

$$\Rightarrow \quad nx - w = (e - f)(y + z) \qquad \text{(simplify)}$$

$$\Rightarrow \quad nx = (e - f)(y + z) + w \qquad \text{(add } w \text{ to each side)}$$

$$\Rightarrow \quad \boxed{x = \frac{(e - f)(y + z) + w}{n}} \qquad \text{(divide each side by } n\text{)}$$

Homework

31. Solve each equation:

a. $-4(a - 6) + (-5a - 3) = 6(2a + 1) - (5a + 4)$

b. $2(-8e - 6) - 8(-3e - 2) = 3(-8e - 7) - 4(-2e + 9)$

c. $5(-9r - 5) + 3(8r + 3) = -2(8r - 3) - 3(7r + 7)$

d. $9(-7j - 6) - 7(-5j + 3) = 6(8j + 1) + 5(5j - 8)$

e. $-6(-9d + 6) + 3(-3d - 9) = -8(-d - 9) - 8(-3d - 4)$

32. Solve each formula for x:

a. $x - c = d$ b. $2x + b = R$ c. $abx = c$

d. $\frac{x}{u} = N$ e. $x(y + z) = a$ f. $\frac{x}{n} = c - d$

g. $\frac{x}{a+b} = m - n$ h. $\frac{x}{c-Q} = c + Q$ i. $\frac{x}{R} = a - b + c$

j. $x(b_1 + b_2) = A$ k. $\frac{x}{a} - e = m$ l. $\frac{x+a}{b} = y$

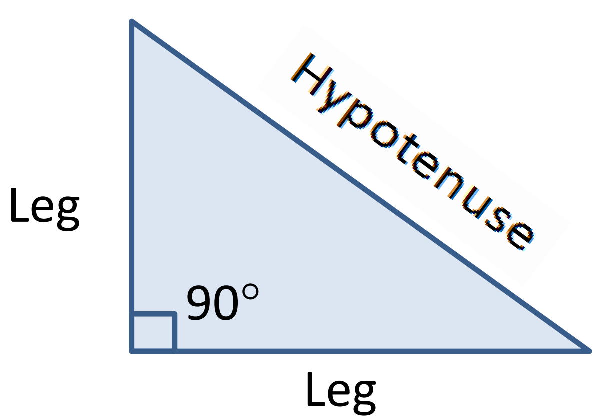
m. $\frac{ax - by}{c} = z$ n. $\frac{cx - a}{y + z} = h - g$ o. $\frac{ax + b}{c} - d = Q$

p. $\frac{9x + u - w}{Q + R} = m + n$ q. $9x - 7y + 13 = 0$

❑ The Pythagorean Theorem

The Right Triangle

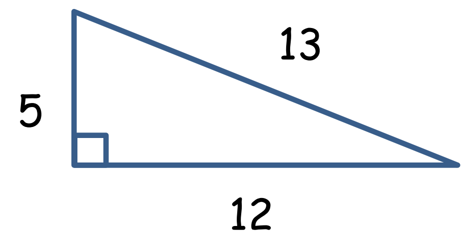
An angle of 90° is called a ***right angle***, and when two things meet at a right angle, we say they are ***perpendicular***. For example, the angle between the floor and the wall is 90°, and so the floor is perpendicular to the wall. And in Manhattan, 5th Avenue is perpendicular to 42nd Street.



If we have a triangle with a 90° angle in it, we call the triangle a ***right triangle***. The two sides which form the right angle (90°) are called the ***legs*** of the right triangle, and the side opposite the right angle is called the ***hypotenuse*** (accent on the 2nd syllable). It also turns out that the hypotenuse is always the longest side of a right triangle.

The Pythagorean Theorem

Ancient civilizations discovered that a triangle with sides 5, 12, and 13 would actually be a right triangle -- that is, a triangle with a 90° angle in it.
[By the way, is it obvious that the hypotenuse must be the side of length 13?]



A Classic Right Triangle

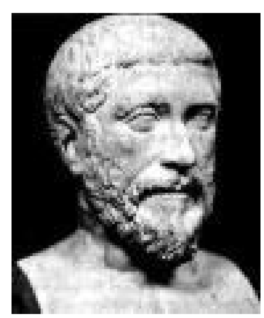
But what if just the two legs are known? Is there a way to calculate the length of the hypotenuse? The answer is yes, and the formula dates back to 600 BC, the time of Pythagoras and his faithful followers.

To discover this formula, let's rewrite the three sides of the above triangle:

leg = 5 leg = 12 hypotenuse = 13

Here's the secret: Use the idea of squaring. If we square the 5, the 12, and the 13, we get 25, 144, and 169; that is,

$5^2 = 25 \qquad 12^2 = 144 \qquad 13^2 = 169$



and we notice that the sum of 25 and 144 is 169:

$$25 + 144 = 169$$

In other words, a triangle with sides 5, 12, and 13 forms a right triangle precisely because

$$5^2 + 12^2 = 13^2$$

Now let's try to express this relationship in words -- it appears that

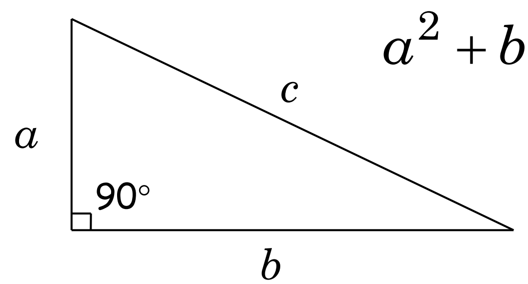
When you square the legs of a right triangle and add them together, you get the square of the hypotenuse.

As a formula, we can state it this way:

If a and b are the legs of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

The Pythagorean Theorem



$$a^2 + b^2 = c^2$$

The sum of the squares of the legs is equal to the square of the hypotenuse

Solving Right Triangles

EXAMPLE 2: **The legs of a right triangle are 6 and 8. Find the hypotenuse.**

Solution: We begin by writing the Pythagorean Theorem. Then we plug in the known values, and finally determine the hypotenuse of the triangle.

$$a^2 + b^2 = c^2 \qquad \text{(the Pythagorean Theorem)}$$
$$\mathbf{6}^2 + \mathbf{8}^2 = c^2 \qquad \text{(substitute the known values)}$$
$$36 + 64 = c^2 \qquad \text{(square each leg)}$$
$$100 = c^2 \qquad \text{(simplify)}$$

What number, when squared, results in 100? A little experimentation yields the solution 10 (since $10^2 = 100$). Our conclusion is that

$$\boxed{\text{The hypotenuse is 10}}$$

Note: The equation $100 = c^2$ also has the solution $c = -10$ [since $(-10)^2 = 100$]. But a negative length makes no sense, so we stick with the positive solution, $c = 10$.

EXAMPLE 3: **Find the hypotenuse of a right triangle whose legs are 5 and 7.**

Solution: This is very similar to Example 2.

$$a^2 + b^2 = c^2 \qquad \text{(the Pythagorean Theorem)}$$
$$\mathbf{5}^2 + \mathbf{7}^2 = c^2 \qquad \text{(substitute the known values)}$$
$$25 + 49 = c^2 \qquad \text{(square each leg)}$$
$$74 = c^2 \qquad \text{(simplify)}$$

Is there a whole number whose square is 74? No, there's not, because $8^2 = 64$, which is too small, while $9^2 = 81$, which is too big. We see that the solution for c is somewhere between 8 and 9. But where between 8 and 9?

Finding a number whose square is 74 is the kind of problem that has plagued and enticed mathematicians, scientists, and philosophers for literally thousands of years. They'd really be irked if they knew that we can find an excellent approximation of this number using a cheap calculator. Enter the number 74 followed by the ***square root*** key, which is labeled something like $\sqrt{x}$. Thus, the hypotenuse is $\sqrt{74}$ (read: the positive square root of 74), and your calculator should have the result 8.602325267, or something close. [With fancier calculators, enter the square root key first, then the 74.]

But your calculator doesn't tell the whole story. The fact is, the square root of 74 has an <u>infinite</u> number of digits following the decimal point, and they <u>never</u> have a repeating pattern. Thus, we'll have to round off the answer to whatever's appropriate for the problem. For this problem, we'll round to the third digit past the point.

> This means that $\sqrt{74}$ is an **irrational** number.

The hypotenuse is 8.602

Homework

33. In each problem, the two legs of a right triangle are given. Find the **hypotenuse**.

a. 3, 4	b. 5, 12	c. 10, 24	d. 30, 16
e. 7, 24	f. 12, 16	g. 30, 40	h. 9, 40
i. 12, 35	j. 20, 21	k. 48, 55	l. 13, 84
m. 17, 144	n. 11, 60	o. 51, 140	p. 24, 143

34. Find the **hypotenuse** of the triangle with the given legs. Use your calculator and round your answers to the hundredths place.

a. 2, 5	b. 4, 6	c. 1, 7	d. 5, 8
e. 2, 6	f. 3, 5	g. 4, 7	h. 7, 8

Solutions

1. a. Real b. Real c. Real d. Real
e. Imaginary f. Real g. Real h. Real
i. Real j. Real k. Real l. Imaginary
m. Real n. Real o. Real p. Real
q. Real r. Imaginary s. Imaginary t. Real
u. Real

2. $-\pi, -\sqrt{3}, -1, \sqrt{0}, \frac{1}{101}, \sqrt{1}, \sqrt{5}, 3.0808\ldots, \frac{11}{3}, \sqrt{25}, 2\pi, 2^3, 3^2$

3. a. -17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative e. 0

5. a. -9 b. 3 c. 0 d. $-\pi$ e. $\sqrt{2}$

6. a. $\frac{1}{5}$ b. $\frac{9}{2}$ c. $-\frac{3}{7}$ d. 1 e. Undefined f. π g. $\sqrt{3}$

7. a. False; 0 has no reciprocal. b. Undefined c. negative
d. positive e. 1

8. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

9. a. 5π b. 23.9 c. 0 d. 2 e. $\sqrt{7}$

10. a. T b. 0 c. positive d. positive

11. Opposite & Absolute Value (the reciprocal of 0 does not exist)

12. a. 72 b. 99 c. 0 d. π e. π f. $\sqrt{2}$

13. a. 10 b. 22 c. 0 d. 5π

14. a. 33 b. 0 c. 25 d. 17 or -17 e. 0 f. No solution

15. a. $\frac{9}{4}x$ b. $\frac{17}{2}x$ c. $\frac{3}{16}y$ d. $-\frac{33}{17}a$

e. $-\frac{5}{23}n$ f. $\frac{1}{2}x$ g. $3x$ h. $\frac{2}{3}m$

i. $\frac{3}{5}x$ j. $-\frac{3}{4}z$

16. a. $\frac{3}{4}x+2$ b. $-\frac{3}{2}x+3$ c. $\frac{1}{8}y-\frac{1}{8}$ d. $-\frac{3}{5}n-3$

e. $-\frac{3}{2}w-12$ f. $3x-5$ g. $\frac{3}{5}x+\frac{4}{5}$ h. $\frac{31}{20}x+\frac{17}{20}$

17. a. $-\frac{13}{10}$ b. 0 c. $\frac{3}{2}$ d. $-\frac{2}{15}$ e. $\frac{41}{5}$

f. $-\frac{5}{3}$ g. $-\frac{7}{3}$ h. $\frac{1}{3}$ i. $-\frac{15}{28}$

18. a. $\frac{5}{12}$ b. -1 c. 1 d. $\frac{4}{9}$ e. $-\frac{1}{18}$

f. $-\frac{28}{3}$ g. 6 h. $-\frac{1}{10}$ i. $-\frac{32}{25}$

19. True

20. a. $\frac{4}{9}$ b. $-\frac{1}{8}$ c. $\frac{1}{81}$ d. $-\frac{14}{19}$

e. $-\frac{1}{32}$ f. $\frac{64}{729}$ g. -1

21. a. $\frac{9}{10}$ b. $\frac{3}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{11}{12}$

f. Not a real number g. $\frac{16}{17}$ h. 0

22. a. 230 b. 31 c. 42 d. 46 e. 1 f. 0
g. 5 h. 47 i. 396 j. 2916 k. 97 l. 4096

23. a. $(x+y)^2 = (2+1)^2 = 3^2 = 9$
b. $x^2+y^2 = 10^2+5^2 = 100+25 = 125$
c. 63 d. 96 e. 44

24. a. 1 b. 3 c. 2 d. 1 e. 2 f. 3 g. 1 h. 2
i. 1 j. 2 k. 1 l. 2 m. 2 n. 3 o. 2 p. 4
q. 2 r. 3 s. 4 t. 1 u. 2

25. a. $\frac{0}{15} = 0$ since $0 \times 15 = 0$, and 0 is the only number that accomplishes this.

b. $\frac{32}{0}$ is undefined because any number times 0 is 0, <u>never</u> 32; thus NO number works.

c. $\frac{0}{0}$ is undefined because any number times 0 <u>is</u> 0; thus EVERY number works.

26. a. 0 b. 0 c. 0 d. 0 e. Undefined f. Undefined
g. Undefined h. Undefined i. Undefined j. Undefined
k. Undefined l. 0

27. a. **28**. b. **29**. b.

30. a. You can't conclude anything -- it depends on what's on the bottom. If the bottom is a non-zero number (like 7), then $\frac{0}{7} = 0$. If the bottom is zero, then $\frac{0}{0}$ is undefined.

b. This time we <u>can</u> conclude that the fraction is undefined, since division by 0 is undefined, no matter what's on the top of the fraction.

31. a. $a = \frac{19}{16}$ b. $e = -\frac{61}{24}$ c. $r = \frac{1}{16}$
d. $j = -\frac{41}{101}$ e. $d = \frac{167}{13}$

32. a. $x = d + c$ b. $x = \frac{R-b}{2}$ c. $x = \frac{c}{ab}$
d. $x = Nu$ e. $x = \frac{a}{y+z}$ f. $x = n(c-d)$
g. $x = (m-n)(a+b)$ h. $x = (c+Q)(c-Q)$ i. $x = R(a-b+c)$

j. $x = \dfrac{A}{b_1 + b_2}$ k. $x = a(m + e)$ l. $x = by - a$

m. $x = \dfrac{cz + by}{a}$ n. $x = \dfrac{(h - g)(y + z) + a}{c}$

o. $x = \dfrac{c(Q + d) - b}{a}$ p. $x = \dfrac{(m + n)(Q + R) + w - u}{9}$

q. $x = \dfrac{7y - 13}{9}$

33. a. 5 b. 13 c. 26 d. 34 e. 25 f. 20
g. 50 h. 41 i. 37 j. 29 k. 73 l. 85
m. 145 n. 61 o. 149 p. 145

34. a. 5.39 b. 7.21 c. 7.07 d. 9.43
e. 6.32 f. 5.83 g. 8.06 h. 10.63

"The greatest mistake you can make in life is to be continually fearing you will make one."

– Elbert Hubbard (1856–1915)