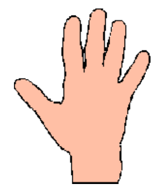
CH 1 – WE BEGIN

❑ Place Value



Our system of numbers, the **decimal** system, is based on the number 10. This may seem completely obvious, but do you realize that the only reason we use base 10 is because we have 10 fingers on our hands? Our number system is based on genetics!

Indeed, the Yuki people near Mendocino used to use base 8 because they counted the spaces between the fingers. The Babylonians used base 60 simply because the number 60 has lots of divisors (factors), which is why there are 60 seconds in a minute and 60 minutes in an hour. And computers use base 2 because electronic components prefer to be either on or off.

Consider the number 3,857:

⇐	1,000,000	100,000	10,000	1,000	100	10	1
				3	**8**	**5**	**7**

The 7 is in the ones place (sometimes called the *units* place), the 5 is in the tens place, the 8 is in the hundreds place, and the 3 is in the thousands place. In other words, we can expand the number 3,857 into

$$3{,}857 \;=\; (3 \times 1{,}000) + (8 \times 100) + (5 \times 10) + (7 \times 1)$$

which could also be written as

$$3{,}857 \;=\; 3000 + 800 + 50 + 7$$

This is a good place to introduce some other notations for multiplication. The × we used above is good, but many of you know that the letter "x" is used a lot in algebra, so sometimes we express multiplication with a dot:

$$2 \cdot 100 = 200$$

But, you know what? Even the dot's not too good -- if you write it a little too low, it might look like a decimal point. So we also indicate multiplication using parentheses:

The answer to a multiplication problem is called the **product**.

$$2(100) = (2)100 = (2)(100) = 200$$

Mathematicians (who are very lazy) have simplified it even more. When variables (letters) are involved, we don't need any symbols at all to denote multiplication:

$3n$ means "3 times n"

xy means "x times y"

❑ Exponents

The place values in the decimal system (1, 10, 100, 1000, 10000, etc.) occur so often in math that it's nice to have a more concise notation for them. Let's take 1,000 as an example. First note that 1,000 can be written as product of three 10's:

$$1{,}000 = 10 \times 10 \times 10$$

Since all the factors (the 10's) are the same, we can condense the product $10 \times 10 \times 10$ into the notation 10^3. We can read this as "10 to the third power," or "10 *cubed*." However you read it,

$$10^3 = 10 \times 10 \times 10 = 1{,}000$$

In the notation 10^3, we call 10 the **base** and 3 the **exponent** (or power).

In a similar way, we can write 100,000 with a base of 10 and an exponent of 5:

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$$

Next, we need to know what exponent would be needed on the base 10 to get a result of 10? How about 10^1? After all, the exponent tells us how many factors of the base to multiply together, so I suppose it makes sense that 10^1 means exactly one factor of 10, which is, of course, just 10.

$$10^1 = 10$$

So here's what we have so far:

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$$
$$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$$
$$10^3 = 10 \times 10 \times 10 = 1{,}000$$
$$10^2 = 10 \times 10 = 100$$ (by the way, 10^2 is read "10 squared")
$$10^1 = 10$$

But remember, there's a units place value (the ones place), so how do we write the number 1 as a power of 10?

Check out the pattern in the powers of 10 above. Looking down the left column of the powers of 10, it's clear that the next power of 10 in line should be 10^0. And if you look at the answers at the far right of each line, you see that each number is found by dividing the number above it by 10. So if we take the last answer on the right, the 10, and divide it by 10, we get a result of 1. So we declare:

$$\mathbf{10^0 = 1}$$

Algebra Preview:

Any number (except 0) raised to the zero power equals 1.

Our place value table can now be written this way:

⇦	1,000,000	100,000	10,000	1,000	100	10	1
⇦	10^6	10^5	10^4	10^3	10^2	10^1	10^0
				3	**8**	**5**	**7**

Let's now go back to the number 3,857. We first wrote it as

$$3{,}857 = (3 \times 1{,}000) + (8 \times 100) + (5 \times 10) + (7 \times 1)$$

Using our power of 10 (exponent) notation we can write this as

$$3{,}857 = (3 \times 10^3) + (8 \times 10^2) + (5 \times 10^1) + (7 \times 10^0)$$

If we can agree that multiplications are to be done before additions, we can dump the parentheses and make it easier to write:

$$3{,}857 = 3 \times 10^3 + 8 \times 10^2 + 5 \times 10^1 + 7 \times 10^0$$

In fact, we can dispense with the ×'s and write our multiplication the way it's done in algebra, using parentheses around the powers of 10. So now we have

$$3{,}857 = 3(10^3) + 8(10^2) + 5(10^1) + 7(10^0)$$

Our agreement that multiplications are always done before additions pervades all of mathematics and computer science. This is an example of what we call **The Order of Operations**.

Last Example: Be sure the following expansion makes sense:

$$93{,}106 = 9(10^4) + 3(10^3) + 1(10^2) + 0(10^1) + 6(10^0)$$

Homework

1. Write the place value 1,000,000 as a power of 10.

2. What are the three place values following 1,000,000, written as regular numbers and powers of 10?

3. We know that $10^6 = 1{,}000{,}000 = 1$ million.

   a. What is 10^9 called? How many zeros are there in this number?

   b. What is 10^{12} called? How many zeros are there in this number?

4. Expand each number using powers of 10:

a. 23	b. 729	c. 2,788	d. 20,976
e. 456,907	f. 4,761,189	g. 23,007,479	h. 7,345,002,892

❑ The Rearrangment Properties

Now we're going to give a name to ideas that are probably familiar to you.

- The first rearrangement property says that we can "switch" the order in which we add or multiply and still get the same answer:

  $7 + 3 = 10$, and $3 + 7 = 10$

  $9 \times 5 = 45$, and $5 \times 9 = 45$

- The second rearrangement property says that we can "associate" in a couple of different ways and get the same result. Let's work the problem $5 + 7 + 8$ in two ways:

  Associate the first two numbers: $(5 + 7) + 8 = 12 + 8 = 20$

  Associate the last two numbers: $5 + (7 + 8) = 5 + 15 = 20$ ✓

  We can also do this "shifting of parentheses" when we multiply three numbers together, like $3 \times 4 \times 5$:

  Associate the first two numbers: $(3 \times 4) \times 5 = 12 \times 5 = 60$

  Associate the last two numbers: $3 \times (4 \times 5) = 3 \times 20 = 60$ ✓

The rearrangement properties allow us to perform some multiplication tricks: To multiply

$$5 \times 4{,}678 \times 2$$

you could multiply from left to right: $5 \times 4{,}678$, and then multiply that result by 2 -- but that's hard! Here's a shortcut: Since we can rearrange the three numbers any way we like, let's write the problem this way:

$$(5 \times 2) \times 4{,}678$$

The product of 5 and 2 is 10. Now multiply 10 by 4,678 by simply adding a zero: 46,780 – done!

Homework

5. Use the rearrangement properties to calculate the product (2)(34,779)(5) in your head.

6. Use the rearrangement properties to calculate the product (25)(5,037)(4) in your head.

7. Verify the first rearrangement property by muliplying the standard elementary school way:

   $34 \times 98 \qquad 98 \times 34$

8. Verify the second rearrangement property by doing the parentheses first in each addition problem:

   $(32 + 99) + 14 \qquad 32 + (99 + 14)$

Solutions

1. $1{,}000{,}000 = 10^6$

2. $10{,}000{,}000 = 10^7$ $\quad 100{,}000{,}000 = 10^8$ $\quad 1{,}000{,}000{,}000 = 10^9$

3. a. 10^9 = 1 Billion; 9 zeros
b. 10^{12} = 1 Trillion; 12 zeros

4. a. $2(10^1) + 3(10^0)$

b. $7(10^2) + 2(10^1) + 9(10^0)$

c. $2(10^3) + 7(10^2) + 8(10^1) + 8(10^0)$

d $2(10^4) + 0(10^3) + 9(10^2) + 7(10^1) + 6(10^0)$

e. $4(10^5) + 5(10^4) + 6(10^3) + 9(10^2) + 0(10^1) + 7(10^0)$

f. $4(10^6) + 7(10^5) + 6(10^4) + 1(10^3) + 1(10^2) + 8(10^1) + 9(10^0)$

g. $2(10^7) + 3(10^6) + 0(10^5) + 0(10^4) + 7(10^3) + 4(10^2) + 7(10^1) + 9(10^0)$

h. $7(10^9) + 3(10^8) + 4(10^7) + 5(10^6) + 0(10^5) + 0(10^4) + 2(10^3) + 8(10^2) + 9(10^1) + 2(10^0)$

5. Multiply 2 by 5 to get 10; multiply 10 by 34,779 by adding a zero: final answer = 347,790

6. Multiply 25 by 4 to get 100; add two zeros; final answer = 503,700

7. First do $\begin{array}{r} 34 \\ \times \underline{98} \end{array}$ and then do $\begin{array}{r} 98 \\ \times \underline{34} \end{array}$.

8. (32 + 99) + 14 = 131 + 14 = 145   32 + (99 + 14) = 32 + 113 = 145 ✓

$$\begin{array}{ll} (32 + 99) + 14 & 32 + (99 + 14) \\ = 131 + 14 & = 32 + 113 \\ = 145 & = 145 \quad \checkmark \end{array}$$

❑ Algebra Preview

Let a, b, and c represent any numbers in the whole world.

The first rearrangement properties can be written in general like this:

For Addition: $a + b = b + a$

For Multiplication: $ab = ba$

and are called the ***commutative properties***.

The second rearrangement properties can be written in general like this:

For Addition: $(a + b) + c = a + (b + c)$

For Multiplication: $(ab)c = a(bc)$

and are called the ***associative properties***.

This implies that the expression $a + b + c$ can be written either as $(a + b) + c$ or $a + (b + c)$; it's up to you -- same answer either way. Similarly for the expression abc.

❑ To ∞ and Beyond

A. A **googol** is a 1 followed by _______ zeros. Therefore, as a power of 10, a googol is written _________. A 1 following by a googol of zeros is called a ___________________.

B. Use the associative property for multiplication to write the product xyz in two different ways.

To live a creative life, we must lose our fear of being wrong.

- *Joseph Chilton Pearce*