CH 7 – FACTORING NUMBERS

❑ Introduction

Factors are numbers that are multiplied together to create a number called the ***product***. For example, 15 can be written as the *product* of the two *factors* 5 and 3:

$$\underbrace{15}_{\textbf{product}} = \underbrace{5 \times 3}_{\textbf{factors}}$$

Factors are multiplied

We say that 5 is a factor of 15, and that 3 is a factor of 15. But 3 and 5 aren't the only factors of 15. Since $1 \times 15 = 15$, the numbers 1 and 15 are also factors of 15. So the number 15 has 4 factors.

For a second example, we can check that $7 \times 13 = 91$. We say that 91 is the product of the factors 7 and 13. Also, 7 is a factor of 91, and 13 is a factor of 91.

And for a third example, consider the number 17. Its only factors are 1 and 17.

And what about the number 1? What are its factors?

We also want to review the **exponent** idea. If a single factor occurs more than once in a product, we can write the factor <u>once</u> and use an exponent to denote how many times the factor was used. For example,

$$3 \times 3 \times 3 \times 3 = 3^4$$

When the same factor occurs exactly twice, we say that the factor is **squared**. For example, 5 squared $= 5 \times 5 = 5^2$.

And if the same factor occurs three times, we say that the factor is **cubed**. For example, 4 cubed $= 4 \times 4 \times 4 = 4^3$.

Homework

1. In the statement $4 \times 11 = 44$, the 4 and the 11 are ______ and 44 is the ______.

2. In the statement $c = ab$, c is the _____ and the a and b are _____.

3. List all of the factors of each number:

   a. 1 b. 2 c. 3 d. 4 e. 11 f. 16

   g. 25 h. 31 i. 49 j. 50 k. 51 l. 87

4. In the statement $x^2 - 9 = (x + 3)(x - 3)$, what do you think are the factors?

5. Evaluate:

   a. 10^4 b. 12 squared c. 5 cubed d. 2 to the 10th

6. Express the product $nnnnnnnnnn$ in exponent form.

❑ Divisibility Rules

These three divisibility rules can save a lot of time and heartache.

Testing Divisibility by **2**

A number is divisible by 2 if the last digit of the number is even; i.e., 0 or 2 or 4 or 6 or 8.

1,794 is divisible by 2 because it ends in the even number 4.

24,863 is not divisible by 2 because it ends in the odd number 3.

Testing Divisibility by **3**

A number is divisible by 3 if the sum of the digits of the number is divisible by 3. (Don't be concerned if you don't understand why this trick works.)

123,801 is divisible by 3 because the sum of the digits is 15; i.e.,

$$1 + 2 + 3 + 8 + 0 + 1 = 15$$

and 15 is divisible by 3. Divide 123,801 by 3 to prove to yourself that 3 is a factor of 123,801.

33,961 is not divisible by 3 because the sum of the digits is 22, which is not divisible by 3. If you divide 33,961 by 3, you'll get a remainder that's not zero.

Testing Divisibility by **5**

A number is divisible by 5 if the last digit is either 0 or 5.

34,085 is divisible by 5 because it ends in a 5.

239,770 is divisible by 5 because it ends in a 0.

45,087 is not divisible by 5 because it ends in a 7.

Homework

7. For each number below, use the *divisibility rules* to determine
   1. if it's divisible by 2.
   2. if it's divisible by 3.
   3. if it's divisible by 5.

a. 28	b. 39	c. 65	d. 29
e. 123	f. 496	g. 8,000	h. 121
i. 696	j. 6,734	k. 1002	l. 111,111
m. 17,375	n. 2,227	o. 43,710	p. 741,942

❑ The Prime Numbers

When we listed all the factors of some whole numbers (problem 3), you may have noticed three things:

1. The number 1 has exactly one factor (itself).
2. Some of the numbers had exactly two factors (e.g., 31).
3. And some numbers had more than two factors (e.g., 25).

* A ***prime number*** is a whole number which has exactly two factors. *

So let's check out the numbers mentioned above:

1: It has only one factor, so it's not prime.

31: It has exactly two factors, so it is prime.

25: It has three factors, so it is not prime.

1 is not a prime number.

Since 1 is not prime, but 2 is, we reach the conclusion that

2 is the smallest prime number.

Thus, the first few primes are

2, 3, 5, 7, 11, 13, 17, 19

In fact, there are 25 primes less than 100. Finish the following table of prime numbers. Here's a hint: 51 should not be in the table.

2	3	5	7	11

Homework

8. Fill in the table above. Another hint: 93 should not be in the table.
9. T/F: 1 is a prime number.
10. What is the smallest prime number?
11. T/F: 2 is the only even prime number.
12. Prove that 51 is not a prime number.
13. Is 101 prime? Explain.
14. T/F: All prime numbers are odd.
15. T/F: All odd numbers are prime.
16. How many prime numbers are there?

❑ Prime Factorization – What it Means

The **prime factorization** of a number means that the number is written as a product of primes. To explain what this means, consider the following five examples.

EXAMPLE 1:

A. $70 = 2 \times 5 \times 7$

First check that it's a true statement. Is $2 \times 5 \times 7$ really equal to 70? Yes it is, so the statement is true. Second, verify that the numbers 2, 5, and 7 are prime. Our conclusion is that we truly have a prime factorization of 70.

B. $117 = 9 \cdot 13$

Is this a true statement? Yes, 117 is the product of 9 and 13. But notice that the 9 is not prime. We conclude that we may have a partial factorization, but certainly not the prime factorization of 117. But if the problem had read $117 = 3 \cdot 3 \cdot 13$, then this would be the prime factorization of 117.

C. $16 = 11 + 5$

Is this a true statement? Certainly it is. Also, notice that 11 and 5 are indeed primes. So do we have a prime factorization? Look carefully -- 16 has been written as an addition problem, that is, as a sum. This is not a product, so 11 + 5 can't possibly be the factorization of anything at all.

D. $39 = 3 \times 11$

Is it a true statement? NO! It can't possibly be a prime factorization (even though the 3 and the 11 are primes) because it's not even a true statement to begin with.

E. $17 = 17 \cdot 1$

This is certainly a true statement; the problem is, even though 17 is prime, 1 is not.

Let's recap our notion of the ***prime factorization*** of a number: The number is written as a **product** of **primes**.

Before the detailed example in the next section, let's list three more prime factorizations:

$117 = 3 \times 3 \times 13$

$125 = 5 \cdot 5 \cdot 5$

$330 = (2)(3)(5)(11)$

Each of the numbers used in these factorizations – 2, 3, 5, 11, and 13 – are prime numbers.

Homework

17. Explain why the following is a prime factorization:

$$5{,}304 = 2 \times 2 \times 2 \times 3 \times 13 \times 17$$

18. Explain why each of the following is not an example of prime factorization:

    a. $23 = 5 + 7 + 11$

    b. $413 = 17 \cdot 19$

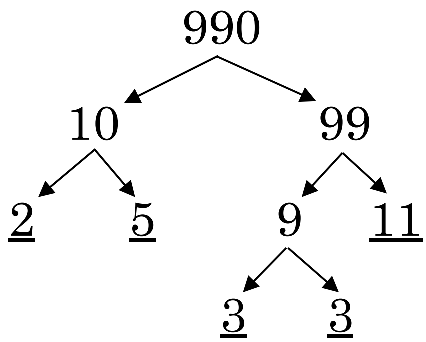
    c. $630 = 2 \cdot 3 \cdot 5 \cdot 21$

❑ Carrying Out the Prime Factorization

Now that we know what a prime factorization should look like, how do we actually create one all by ourselves? Let's do an example -- step by step -- realizing that there are probably many different ways to get the correct answer. So this example is just one way to do it. You are encouraged to come up with your own scheme if it works better for you.

EXAMPLE 2: **Find the prime factorization of 990.**

Solution: Let's review our objective one more time. We need to break down 990 into a *product of primes*. The first step is to find any number which will divide into 990 without remainder (that is, a factor of 990). Any such factor will be fine (it doesn't have to be prime to start with -- we'll eventually make all the factors prime). Even if your buddy starts with a different number, ultimately both of you should end up with exactly the same answer, although the order of the factors may be different.



990 can be written as 10×99

10 can be written as 2×5, and
99 can be written as 9×11

9 can be written as 3×3

The prime factors have been underlined; they are (in ascending order) 2, 3, 3, 5, and 11.

We now have all the factors (the underlined primes) needed for the prime factorization of 990:

$$\mathbf{990 = 2 \times 3 \times 3 \times 5 \times 11}$$

Recalling the exponent ideas in the Introduction, we can write our answer in still another form:

$$\mathbf{990 = 2 \times 3^2 \times 5 \times 11}$$

To verify our result, check that

1) $2 \cdot 3^2 \cdot 5 \cdot 11$ is actually 990.

2) All of the factors are prime.

Let's do a second example, using a different approach.

<u>EXAMPLE 3:</u> **Find the prime factorization of 6,664.**

<u>Solution:</u>

	6,664	
=	$2 \times 3{,}332$	$(6{,}664 \div 2 = 3{,}332)$
=	$2 \times 2 \times 1{,}666$	$(3{,}332 \div 2 = 1{,}666)$
=	$2 \times 2 \times 2 \times 833$	$(1{,}666 \div 2 = 833)$
=	$2 \times 2 \times 2 \times 7 \times 119$	$(833 \div 7 = 119)$
=	$\mathbf{2 \times 2 \times 2 \times 7 \times 7 \times 17}$	$(119 \div 7 = 17)$
or,	$\mathbf{2^3 \times 7^2 \times 17}$	

You may want to ask your teacher if one method of factoring is preferred over another for your quizzes and tests. Also find out if your teacher insists that you use exponents for repeated factors.

Homework

Find the **prime factorization** of each number:

19. 4	20. 6	21. 10	22. 35
23. 26	24. 17	25. 110	26. 36
27. 32	28. 234	29. 210	30. 99
31. 77	32. 245	33. 187	34. 936
35. 143	36. 375	37. 1,377	38. 121
39. 128	40. 169	41. 51	42. 39
43. 138	44. 1,225	45. 507	46. 1,045
47. 391	48. 150	49. 2,431	50. 1,000
51. 539	52. 875	53. 1,573	54. 2,310
55. 80,325	56. 75,600	57. 47	58. 61

❑ An Application of Prime Factoring

Reducing a fraction to lowest terms is sometimes best done by factoring. For example, let's reduce $\frac{210}{231}$:

$$\frac{210}{231} = \frac{2\cdot3\cdot5\cdot7}{3\cdot7\cdot11} = \frac{2\cdot\cancel{3}\cdot5\cdot\cancel{7}}{\cancel{3}\cdot\cancel{7}\cdot11} = \frac{2\cdot5}{11} = \mathbf{\frac{10}{11}}$$

> Algebra Preview: Try to reduce the fraction $\frac{abc}{bcd}$.

Solutions

1. factors; product

2. product; factors

3. a. 1 b. 1, 2 c. 1, 3 d. 1, 2, 4
e. 1, 11 f. 1, 2, 4, 8, 16 g. 1, 5, 25 h. 1, 31
i. 1, 7, 49 j. 1, 2, 5, 10, 25, 50 k. 1, 3, 17, 51
l. 1, 3, 29, 87

4. The factors are $x + 3$ and $x - 3$.

5. a. 10,000 b. 144 c. 125 d. 1,024

6. n^{10}

7.

a.	2	b.	3	c.	5	d.	none	e.	3	f.	2
g.	2, 5	h.	none	i.	2, 3	j.	2	k.	2, 3	l.	3
m.	5	n.	none	o.	2, 3, 5	p.	2, 3				

8. Check with a friend, or Google it.

9. F; a prime number must have exactly two factors. The number 1 has only one factor; hence, it's not prime.

10. 2

11. T; any even number bigger than two would be divisible by 2, and thus would have at least three factors (1, 2, and itself).

12. $51 = 3 \times 17$ (51 therefore has more than two factors.)

13. Yes, it has exactly two factors, 1 and 101.

14. F; 2 is prime and it's not odd.

15. F; 9 is odd but it's not prime.

16. It's hard to prove, but there are infinitely many primes -- that is, there's no end to the supply of prime numbers (though they occur less and less frequently as we move up through the numbers). Euclid proved this fact

23 centuries ago. He did this basically by assuming that there was a largest prime; he then demonstrated that there was always another prime number bigger than that one. Thus, there can be no end to the prime numbers.

17. First: Multiplying 2 by 2 by 2 by 3 by 13 by 17 really and truly gives a product of 5,304.
Second: Each of the numbers 2, 3, 13, and 17 is prime.

18. a. While 5, 7, and 11 are prime, the unfortunate fact is that the product of those three numbers is not 23. Their sum is 23, but factorization involves multiplication, not addition.

b. This time, at least we have multiplication, and the 17 and 19 are prime. So what's the problem? 17 times 19 is only 323, not 413; so the statement $413 = 17 \cdot 19$ is just plain bogus.

c. Now this one looks pretty good. Check that the product of the four factors is 630 -- it is. And the numbers are being multiplied -- that's good. So where's the culprit? It's the 21 -- it's not prime. We may be close, but we do not have a prime factorization of 630. By the way, $630 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7$ is the proper factorization.

19. $2 \cdot 2$	**20**. $2 \cdot 3$	**21**. $2 \cdot 5$
22. $5 \cdot 7$	**23**. $2 \cdot 13$	**24**. 17
25. $2 \cdot 5 \cdot 11$	**26**. $2^2 \cdot 3^2$	**27**. 2^5
28. $2 \cdot 3^2 \cdot 13$	**29**. $2 \cdot 3 \cdot 5 \cdot 7$	**30**. $3 \cdot 3 \cdot 11$
31. $7 \cdot 11$	**32**. $5 \cdot 7^2$	**33**. $11 \cdot 17$
34. $2^3 \cdot 3^2 \cdot 13$	**35**. $11 \cdot 13$	**36**. $3 \cdot 5^3$
37. $3^4 \cdot 17$	**38**. 11^2	**39**. 2^7
40. 13^2	**41**. $3 \cdot 17$	**42**. $3 \cdot 13$
43. $2 \cdot 3 \cdot 23$	**44**. $5^2 \cdot 7^2$	**45**. $3 \cdot 13^2$

46. $5 \cdot 11 \cdot 19$ **47**. $17 \cdot 23$ **48**. $2 \cdot 3 \cdot 5^2$

49. $11 \cdot 13 \cdot 17$ **50**. $2^3 \cdot 5^3$ **51**. $7^2 \cdot 11$

52. $5^3 \cdot 7$ **53**. $11^2 \cdot 13$ **54**. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$

55. $3^3 \cdot 5^2 \cdot 7 \cdot 17$ **56**. $2^4 \cdot 3^3 \cdot 5^2 \cdot 7$ **57**. 47

58. 61

"The aim of education should be to teach us rather how to think, than what to think — rather to improve our minds, so as to enable us to think for ourselves, than to load the memory with thoughts of other men."

– Bill Beattie