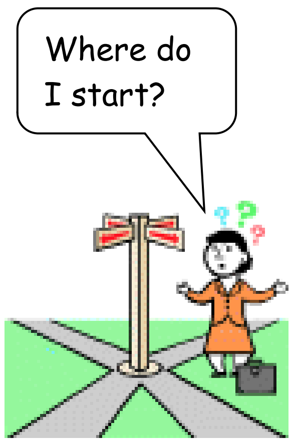
CH 6 – THE ORDER OF OPERATIONS

❑ *INTRODUCTION*

There are a variety of operations we perform in math, including addition, multiplication, and squaring. When we're confronted with an expression that has a variety of these operations, how do we know which operation to perform first? This chapter will show you exactly what steps to take, and the order in which you should take them.



Consider the problem

$$3 + 2 \times 4$$

There are two reasonable methods to work this problem:

$3+2\times4$		
$\downarrow$		
$= \quad 5 \times 4$	(add first)	
$= \quad \mathbf{20}$	(then multiply)	

$3+2\times4$		
$\downarrow$		
$= \quad 3 + 8$	(multiply first)	
$= \quad \mathbf{11}$	(then add)	

The dilemma should now be clear: Which one of these is correct? The answer is determined by the ***Order of Operations***, an agreement that determines the <u>order</u> in which arithmetic <u>operations</u> are to be carried out. This agreement is followed by people in every country of the world, and in almost all computer systems. The above dilemma, $3 + 2 \times 4$, will be resolved in a while.

❑ *ADDITION VS SUBTRACTION*

Adding is the most fundamental operation we have in arithmetic and algebra. Also, since subtracting is merely the reverse of adding, subtracting is considered as fundamental as adding. We say that **adding and subtracting have equal priority**. So, when a problem has only adding and subtracting in it, we do the operations from left to right.

EXAMPLE 1: **Evaluate (or simplify): 20 – 8 + 13 – 22**

Solution: The only operations are adding and subtracting, so we proceed from left to right:

	20 – 8 + 13 – 22	(the original expression)
=	12 + 13 – 22	(1st step: subtract 8 from 20)
=	25 – 22	(2nd step: add 12 and 13)
=	$\boxed{3}$	(3rd step: subtract 22 from 25)

Homework

1. Evaluate (or simplify) each expression:

a. 45 – 3 – 5

b. 50 – 7 – 1 – 2

c. 80 + 31 – 10 – 20

d. 23 – 23 + 18 – 18

e. 10 – 2 – 3 – 5

f. 34 + 17 – 1 – 20 + 2 – 10

g. 32 + 8 – 4 – 5 + 2 – 1

h. 4 + 6 + 10 – 1 – 2 + 1 – 4

i. 12 + 34 – 1 – 34 – 11

j. 23 + 12 – 1 – 2 – 6 + 7 – 10

❑ *MULTIPLICATION VS ADDITION*

Since $4 \times 9 = 9 + 9 + 9 + 9$, we see that multiplication is just repeated addition. Since multiplication is based upon addition, we will declare that **multiplication has a higher priority than addition**. Therefore, the agreed-upon way to work the problem discussed in the Introduction, $3 + 2 \times 4$, is to perform the multiplication first, giving

$$\begin{aligned} & 3 + 2 \times 4 \\ = \; & 3 + 8 \qquad \text{(multiply BEFORE adding)} \\ = \; & \mathbf{11} \end{aligned}$$

Since we multiply before we add, and since addition and subtraction have equal priority, it follows that **multiplication has priority over subtraction**, also.

<u>EXAMPLE 2:</u> **Evaluate: $5 \times 4 - 2 \cdot 3 + 25 - 4(3)$**

<u>Solution:</u> We do all the multiplications first, since multiplication has priority over addition and subtraction:

$$\begin{aligned} & 5 \times 4 - 2 \cdot 3 + 25 - 4(3) \\ = \; & 20 - 6 + 25 - 12 \qquad \text{(multiply first)} \\ = \; & 14 + 25 - 12 \qquad \text{(add/subtract left to right)} \\ = \; & 39 - 12 \\ = \; & \boxed{27} \end{aligned}$$

Homework

2. Evaluate (that is, simplify) each expression:

a. $7 + 2 \cdot 9$

b. $12 - 3 \times 4$

c. $10 \times 3 + 20$

d. $2 \cdot 7 + 9 \cdot 1$

e. $5 \times 7 - 8 \cdot 2$

f. $10 + 3 - 4 + 8 \cdot 5$

g. $23 \cdot 2 + 5 \cdot 1 - 4 \times 3$

h. $14 \cdot 3 - 1 - 2 - 10 \cdot 2 + 1$

i. $3 + 6 - 2 \cdot 4 + 1 \times 12$

j. $4(5) - 3(2) + 7(7) - 49$

❑ *MULTIPLICATION VS DIVISION*

Since dividing is the reverse operation of multiplying, we declare that **multiplication and division have equal priority**. This means that when a problem has only multiplication and division in it, we proceed from left to right (except when the fraction bar is used for division).

Two ways of indicating division are the symbols " ÷ ", which is the symbol usually found on a calculator, and " / ", used in computer programs such as spreadsheets, word processors, and programming languages. (The fraction bar method will be discussed after the next two examples.)

EXAMPLE 3: **Evaluate: $18 / 2 \times 3 \div 9 \cdot 10$**

Solution: There are two divisions and two multiplications in this expression. Since multiplication and division have equal priority, we work the problem from left to right.

	$18 / 2 \times 3 \div 9 \cdot 10$	(the original problem)
=	$9 \times 3 \div 9 \cdot 10$	(divide 18 by 2)
=	$27 \div 9 \cdot 10$	(multiply 9 by 3)

$$= \quad 3 \cdot 10 \qquad \text{(divide 27 by 9)}$$

$$= \quad \boxed{30} \qquad \text{(multiply 3 by 10)}$$

<u>EXAMPLE 4:</u> **Evaluate: 12 – 10 ÷ 2 + 3 + 3 × 4 + 20 / 10**

<u>Solution:</u> Here's the overview: This expression has all four operations in it. Considering the agreement that multiplying and dividing have priority over adding and subtracting, we will first multiply and divide from left to right, and then go back to the front of the problem to add and subtract from left to right. Let's do it:

$$12 - 10 \div 2 + 3 + 3 \times 4 + \underbrace{20/10} \qquad \text{(the original problem)}$$

$$= \quad 12 - 5 + 3 + 12 + 2 \qquad \text{(multiply \& divide left to right)}$$

$$= \quad 7 + 3 + 12 + 2 \qquad \text{(add \& subtract left to right)}$$

$$= \quad \boxed{24}$$

The division symbol usually used in algebra, science, and business is the fraction bar, but the following all have exactly the same meaning:

$$\frac{a}{b} \qquad b\overline{)a} \qquad a \div b \qquad a/b$$

For example,

$$\frac{8}{2} = 2\overline{)8} = 8 \div 2 = 8/2, \text{ all of which} = \mathbf{4}$$

When the fraction bar is used, the division occurs <u>after</u> both the top and the bottom have been completely done.

<u>EXAMPLE 5:</u> **Evaluate:** $\mathbf{8 - \frac{6}{2} + \frac{10-3}{6+1} - \frac{2+5(2)}{12-3(3)}}$

<u>Solution:</u> First we simplify all the numerators and denominators. Notice in the last fraction that we will use the

priorities we learned earlier -- that multiplication has priority over addition and subtraction.

$$8 - \frac{6}{2} + \frac{10-3}{6+1} - \frac{2+5(2)}{12-3(3)} \quad \text{(the original problem)}$$

$$= 8 - \frac{6}{2} + \frac{10-3}{6+1} - \frac{2+10}{12-9} \quad \text{(multiply first)}$$

$$= 8 - \frac{6}{2} + \frac{7}{7} - \frac{12}{3} \quad \text{(simplify tops and bottoms)}$$

$$= 8 - 3 + 1 - 4 \quad \text{(perform each division)}$$

$$= 5 + 1 - 4 \quad \text{(add/subtract left to right)}$$

$$= 6 - 4$$

$$= \boxed{2}$$

Homework

3. Evaluate each expression:

a. $7 \times 9 \div 3$

b. $100 \div 25 \times 10$

c. $8 \cdot 10 \div 2 \div 2 \div 2$

d. $5 \times 6 \div 2 \times 3 \times 1$

e. $12(10) \div 5 \cdot 2$

f. $100 \div 2 \div 5 \times 3 \div 30$

g. $100 - 20 \div 2 + 8/1$

h. $8 \cdot 2 \cdot 6 / 3 + 2 \cdot 10$

i. $50 + 80 \div 2 \div 4$

j. $\frac{15}{3} + \frac{10}{2} - 2 \cdot 1$

k. $20(10) - 3(14) + 10 \div 2$

l. $200 / 10 + 32 \cdot 2 - \frac{10-2}{2 \cdot 2 \cdot 2}$

m. $\frac{14-4}{12-2} + 8 \times 5 - 2 \times 3$

n. $\frac{32}{8} + \frac{1000}{2} - 8/2 + 22 \div 11 - \frac{5+4}{4-1}$

❑ *Exponents vs Multiplication*

Recall that in the expression

$$B^E$$

B is called the **base** and E is called the **exponent**. Since an exponent represents repeated multiplication, we have, for example, $9^3 = 9 \cdot 9 \cdot 9$. (Raising the base to an exponent of 3 is called ***cubing***.)

So we will declare that **an exponent has a higher priority than multiplication**. It thus follows that exponents have a higher priority than any operation analyzed thus far.

> Albert Einstein's famous equation involving the equivalence of mass and energy, $E = mc^2$, where c is the speed of light, was written in this manner because 26-year-old Al knew that anyone reading the formula would understand that squaring is to be done first, followed by the multiplication. That is, only the c is being squared, not the product mc. Do you have an idea what he would have written had he wanted the entire quantity mc to be squared?

A formula you might be familiar with is the area of a circle: $A = \pi r^2$. The answer to the question "What is being squared?" is "just the r." The multiplication by the π is done <u>after</u> the r is squared.

<u>EXAMPLE 6:</u> **Evaluate: 3×5^2**

<u>Solution:</u> This is a small example, but it's an important one! Since exponents have higher priority than multiplication, we perform the exponent operation first:

$$3 \times 5^2 = 3 \times 25 = \boxed{75}$$

Homework

4. Evaluate each expression:

a. $9 \cdot 5^2$

b. 3×10^2

c. 5×4^3

d. $2 + 3 \cdot 4^2$

e. $5^2 - 3^2$

f. $7 \cdot 7 - 7^2$

g. $10 - 3^2$

h. $2 \times 10^2 - 2 \times 10$

i. $3(5)^2$

j. $1000 \div 5^2 - 18$

k. $100 / 2^2 + 8 \cdot 3$

l. $10^2 + 3 \cdot 5^3$

m. $3^2 + 4^2 - 2^3$

n. $5^2 - 4(2)(3)$

o. $10^3 + 3(4) + 1$

p. $4^2 - 4(1)(1)$

q. $8 + 5^2 - 5 \times 4$

r. $1^2 + 2^2 + 3^2 + 4^2$

s. $8^2 \cdot 10 - 3 \times 10^2$

t. $4^3 - 8^2 + 0 \cdot 99$

u. $10^2 \div 5^2 + 3 \times 4$

v. $18 + 2 \cdot 9^2 - 12 \div 2 + 50/2$

w. $32 \div 2^2 + 50/2 \div 5 \cdot 1^3$

x. $10^2 - (4)(2)(10)$

❑ *PARENTHESES HAVE VETO POWER*

Recall the problem which started this whole discussion:

$$3 + 2 \times 4$$

First we showed that if we do the addition first, the answer is 20; but when the multiplication is done first, the result is 11. Then we decided that multiplication has priority over addition -- that multiplication is to be done first -- and therefore, the proper answer is 11. That is,

$3 + 2 \times 4 = 3 + 8 = \mathbf{11}$ This is the proper way to do it.

But what if we really wanted to add the 3 and the 2 first in the expression $3 + 2 \times 4$? Is there a way to override all the rules we've developed so far? Yes -- we just add a new rule: Use parentheses ( ) or brackets [ ] or braces { } to mean "Do me first!" Now the addition will take place first, because it's in the parentheses:

$$(3 + 2) \times 4 = 5 \times 4 = \mathbf{20}$$

For another example of the use of parentheses, recall Example 6, where we decided that $3 \times 5^2 = 3 \times 25 = 75$, because exponents have priority over multiplication. But if we really intended to do the multiplication first, we could use parentheses and get a totally different answer:

$$(3 \times 5)^2 = 15^2 = \mathbf{225}$$

❑ *Parentheses Are Used to Change How Things Are Associated*

Since parentheses are always done first, you should be confident in the following two calculations:

$(3 + 4) + 5 = 7 + 5 = 12$ Here we "associate" the 3 and 4.

$3 + (4 + 5) = 3 + 9 = 12$ Here we "associate" the 4 and 5.

We got the same answer to the problem $3 + 4 + 5$ whether we added the 3 and the 4 first, or added the 4 and the 5 first. Thus, when three quantities are to be added, we can "associate" the first two <u>or</u> the last two by putting parentheses around them. In short, if a, b, and c represent any numbers at all,

$$\boldsymbol{(a + b) + c = a + (b + c)}$$

and we say that **addition is an *associative* operation**.

Similarly, since $(4 \cdot 5) \cdot 3 = 20 \cdot 3 = 60$ and $4 \cdot (5 \cdot 3) = 4 \cdot 15 = 60$, for instance, we conclude that multiplication is also an associative operation:

$$\boldsymbol{(ab)c = a(bc)} \qquad \text{for any numbers } a, b, \text{ and } c.$$

❑ *Parentheses and Brackets*

When expressions are "nested" (inside one another), we can use parentheses for both levels, or we can use parentheses inside brackets. For example, the following two expressions are completely equivalent:

$$(5 + 3(9 - 2))^2 \qquad\qquad \left[5 + 3(9 - 2)\right]^2$$

The first version is appropriate for a formula for a spreadsheet or computer language; the second is easier to read, and is common in math.

❑ *The Order of Operations*

The following chart is a summary of this chapter, and shows the order in which to carry out math operations -- the higher something is in the chart, the higher its priority. In other words, start at the top of the chart and move down as you do the problem.

Order of Operations
(Parentheses) and [Brackets]
Exponents
Multiply & Divide (left to right)
Add & Subtract (left to right)

EXAMPLE 7: $\mathbf{2 \cdot 10^2 - (7-4)^3 - 3 \times 5}$

$= 2 \cdot 10^2 - 3^3 - 3 \times 5$ (parentheses first)

$= 2 \cdot 100 - 27 - 3 \times 5$ (exponents next)

$= 200 - 27 - 15$ (then multiply)

$= 173 - 15$ (subtract from left to right)

$= \boxed{158}$

EXAMPLE 8: $2[12 - 3(5-3)]$

$= 2[12 - 3(2)]$ (innermost grouping first (parens))

$= 2[12 - 6]$ (multiply before subtract)

$= 2[6]$ (now the outer grouping [brackets])

$= \boxed{\mathbf{12}}$

A Note Regarding Fractions

Even though a fraction indicates division, the order of operations does not include an explicit reference to fractions. This is because it's reasonably clear how to handle them: First use the order of operations to simplify the top, then use the order of operations to simplify the bottom, and last do the division.

Homework

5. Evaluate (simplify) each expression:

a. $3 \cdot 10^2 - (8 - 4)^3 - 3 \times 2$

b. $(5 - 3)^2 + (10 - 7)^3$

c. $[3 + 2(5)] - 1 + 3 \cdot 10$

d. $2(10 - 5)^2 - 12 \div 3$

e. $[2(10 - 5)]^3 \div (10 \cdot 10^2)$

f. $(1 + 4)^2 - (4 + 1)^2$

g. $[(3^2 - 2^2)^3 - 80] \div (36 / 4)$

h. $3 \cdot 4^2 - (13 - 12)^3$

i. $10 + 8(8 - 1)^2 - 3 - 2 - 1$

j. $[8^2 - 2^3 + 3 \cdot 4 - 2(7)]^2$

k. $[20 - (5 - 2)^2]^2 - 2 \cdot 3 \cdot 4$

l. $[13 - (8 - 3) + (10 - 2)]^3$

6. Evaluate each expression for the given values:

a. $(x + y)^2$ for $x = 2$ and $y = 1$

b. $x^2 + y^2$ for $x = 10$ and $y = 5$

c. $x^2 + xy + y^2$ for $x = 3$ and $y = 6$

d. $(x + y)(x - y)$ for $x = 10$ and $y = 2$

e. $x^2 - y^2$ for $x = 12$ and $y = 10$

❑ First and Final Operations

Homework

7. In each formula, indicate what operation would be performed <u>first</u>:

a. $P = 2l + 2w$ b. $P = 2(l + w)$

c. $A = \pi r^2$ d. $E = mc^2$

e. $k = a^2 + b^2$ f. $D = b^2 - 4ac$

g. $B = (a + b)(c + d)$ h. $V = (x - h)^2$

i. $r = \frac{d+a}{b}$ j. $N = a + \frac{b}{c}$

k. $A = \frac{c}{x - y}$ l. $a = b + cd$

8. Now, using the same 12 formulas from the previous problem, indicate what operation is the final one (that is, the operation which is performed <u>last</u>).

❑ Number of Terms

Determining the *number of terms* in an expression is one of the most important skills you can acquire in algebra, allowing you to better understand a myriad of concepts coming up in this and future courses. The idea is somewhat subtle, however, so don't get discouraged if it takes a while for it to sink in.

If the final operation in an expression is multiplication, then the expression consists of one ***term***. Each of the following consists of one term:

ab $\qquad x(y+z)$ $\qquad (a-2)(a-3)^2$ $\qquad (z-w+u)(A+B)$

Each expression consists of one term
because the final operation is multiplication.

If the final operation in an expression consists of additions and/or subtractions, then the expression consists of at least two ***terms***. The number of terms is found by counting the number of things being added or subtracted in the final operations. The following examples demonstrate two or more terms:

$a+b$	2 terms
$x-y+z$	3 terms
$2+mnp$	2 terms
$a+b-c+def$	4 terms
$(y-x-m)^2A+B$	2 terms
$(a-b)^2+(c+d)^3+e$	3 terms
$ab-cd+x-y+wxyz$	5 terms

Homework

9. Determine the number of ***terms*** in each expression:

a. abc

b. $a+b-c$

c. $xyz-w$

d. $(x-3)^2$

e. x^2+25

f. $ab+ac-xy$

g. $(a+b)(x-y)$

h. $rst-qrw$

i. $(rst)(qrw)$

j. $(rst)(qrw)-1$

k. $[(rst)(qrw)-1]^5$

l. $25-(x+y-z)^2$

m. $a - b$

n. $x + y^2 + z$

o. $20 - mnpq$

p. $a + x - c + QT$

q. $(y + xm)^2A + B$

r. $(a - b)^3 - (c - d)^3 - w$

s. $abcd + x - y + wxyz$

t. $[a(b - c)ed - 7]^4$

u. $w(u - x)^3 - ab(de - mq)$

v. $a + b - c[wx + y - z]$

Review Problems

10. Evaluate each expression:

a. $3^2 - 2^2 + (5 - 1)^3$

b. $(7-2+1)^2 - \frac{8+2}{12-2}$

c. 3×4^2

d. $2 \cdot 5^3$

e. $2 \cdot 3 + 9 \div 3 - 8 / 8 + 3^2$

f. $\frac{3^3 - 3^2 - 3}{4^3 - 9 \cdot 7}$

g. $(2 \cdot 8)^2 - 2 \cdot 8^2$

h. $20 + 5 \times 3 - 2 \times 1 + 0(99)$

11. How many terms are there in each expression?

a. $xyz + w$

b. $x(yz + w)$

c. $abc - def + 7$

d. $(ab - h)^2$

e. $(x - y)^2 + abc^2 - ab - c$

Solutions

1. a. 37 b. 40 c. 81 d. 0 e. 0 f. 22 g. 32
h. 14 i. 0 j. 23

2. a. 25 b. 0 c. 50 d. 23 e. 19 f. 49 g. 39
h. 20 i. 13 j. 14

3. a. 21 b. 40 c. 10 d. 45 e. 48 f. 1 g. 98
h. 52 i. 60 j. 8 k. 163 l. 83 m. 35 n. 499

4. a. 225 b. 300 c. 320 d. 50 e. 16 f. 0 g. 1
h. 180 i. 75 j. 22 k. 49 l. 475 m. 17 n. 1
o. 1013 p. 12 q. 13 r. 30 s. 340 t. 0 u. 16
v. 199 w. 13 x. 20

5. a. 230 b. 31 c. 42 d. 46 e. 1 f. 0
g. 5 h. 47 i. 396 j. 2916 k. 97 l. 4096

6. a. $(x+y)^2 = (2+1)^2 = 3^2 = 9$
b. $x^2 + y^2 = 10^2 + 5^2 = 100 + 25 = 125$
c. 63 d. 96 e. 44

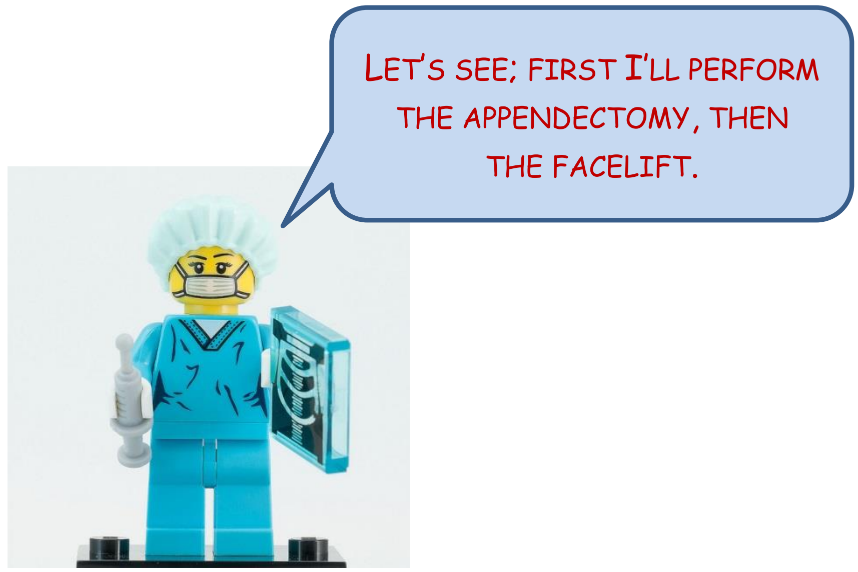
7. a. Multiply b. Add c. Square d. Square
e. Square f. Square g. Add h. Subtract
i. Add j. Divide k. Subtract l. Multiply

8. a. Add b. Multiply c. Multiply d. Multiply
e. Add f. Subtract g. Multiply h. Square
i. Divide j. Add k. Divide l. Add

9. a. 1 b. 3 c. 2 d. 1 e. 2 f. 3 g. 1 h. 2
i. 1 j. 2 k. 1 l. 2 m. 2 n. 3 o. 2 p. 4
q. 2 r. 3 s. 4 t. 1 u. 2 v. 3

10. a. 69 b. 35 c. 48 d. 250 e. 17 f. 15 g. 128 h. 33

11. a. 2 b. 1 c. 3 d. 1 e. 4



What the Order of Operations ISN'T!

"Not with what we have –

but with what we enjoy –

do we find our

true abundance."

Tibetan quote