CH 14 – MORE DIVISION, SIGNED NUMBERS, & EQUATIONS

❑ Division and Those Pesky Zeros

One of the most important facts in all of mathematics is that the denominator (bottom) of a fraction can NEVER be zero.

Sometimes this is phrased

"Never divide by zero."

What's the big deal? Why can't we do it?

This is the result of dividing by zero.

Recall from the discussion of dividing signed numbers that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

We don't have to blindly accept the fact that you should never divide by 0. Let's put zeros in fractions and see what happens.

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work (confirm this yourself).

Zero on the Bottom

Now let's put a zero on the bottom (but not the top) and see what happens:

$$\frac{9}{0} = ???$$

Your first guess might be 0; let's check it out:

$\frac{9}{0} = 0$ would be true only if $0 \cdot 0 = 9$, which it isn't.

How about we try an answer of 9?

$\frac{9}{0} = 9$ which is checked by seeing if $9 \cdot 0 = 9$. Sorry.

Could the answer be π? Nope; $\pi \cdot 0 = 0$, not 9.

In fact, <u>any</u> number we conjure up as a potential answer will have to multiply with 0 to make a product of 9. But this is <u>impossible</u>, since any number times 0 is always 0, never 9. In short, <u>*no number* in the whole world</u> will work as the answer to this division problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0: $\frac{0}{0} = 0$ which seems reasonable, since $0 \cdot 0 = 0$.

Let's try an answer of 5: $\frac{0}{0} = 5$, which -- amazingly! -- works out too.

Could π possibly work? $\frac{0}{0} = \pi$ -- still true, since $\pi \cdot 0$ is surely 0.

That's three answers for this division problem: 0, 5, and π.

Is there any end to this madness? Apparently not, since <u>any</u> number we think up will multiply with 0 to make a product of 0. In short, <u>*every number* in the whole world</u> will work in this division problem.

Summary:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is no answer to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are infinitely many answers to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the two kinds of division problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

Thus,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

"Black holes are where God divided by zero."

Steven Wright

Note: The two division problems with zero in the denominator may both be undefined, but they're undefined for totally different reasons. Your teacher may require you to understand this.

Homework

1. Evaluate each expression, and explain your conclusion:

   a. $\frac{0}{15}$ b. $\frac{32}{0}$ c. $\frac{0}{0}$

2. Evaluate each expression:

   a. $\frac{0}{17}$ b. $\frac{0}{-9}$ c. $\frac{6-6}{17+3}$ d. $\frac{3^2-8-1}{100}$

   e. $\frac{98}{0}$ f. $\frac{-44}{0}$ g. $\frac{7+8}{2^3-8}$ h. $\frac{7^2-40}{-23+23}$

   i. $\frac{0}{0}$ j. $\frac{-9+9}{10-10}$ k. $\frac{5^2-25}{0^2+0^3}$ l. $\frac{4\cdot5-2\cdot10}{3^3-9}$

3. $\frac{0}{\pi} = 0$ because

   a. 0 is the only number that when multiplied by π gives 0.
   b. no number times π equals 0.
   c. every number times π equals 0.

4. $\frac{0}{0}$ is undefined because

   a. no number times 0 equals 0.
   b. every number times 0 equals 0.
   c. any number divided by itself is 1.

5. $\frac{7}{0}$ is undefined because

   a. 0 is the only number multiplied by 0 to get 7.
   b. no number times 0 equals 7.
   c. every number times 0 equals 7.

6. a. The numerator of a fraction is 0. What can you conclude?
   b. The denominator of a fraction is 0. What can you conclude?

❑ Powers of Signed Numbers

Positive Base This is identical to what we've learned before. For example, squaring a positive number results in a positive number:

$$(12)^2 = 12 \times 12 = \mathbf{144}$$

The same holds for cubing and higher powers:

$$(5)^3 = 5 \cdot 5 \cdot 5 = \mathbf{125}$$

$$(10)^4 = 10 \times 10 \times 10 \times 10 = \mathbf{10{,}000}$$

Negative Base Now it gets interesting. Let's square a negative number:

$$(-7)^2 = (-7)(-7) = \mathbf{49}$$ (notice the parentheses around the –7)

The square of a negative number is positive. Now for the cube of a negative number:

$$(-4)^3 = (-4)(-4)(-4) = 16(-4) = \mathbf{-64}$$

This example shows that *the cube of a negative number is negative*.

A Tricky One What is the value of -5^2 ?

Ask yourself: What's being squared? Well, according to the Order of Operations, an exponent has a much higher priority than a lowly minus sign. Therefore, the square applies only to the 5; the minus sign is then attached to the result of squaring the 5. Thus, believe it or not:

$$-5^2 = \mathbf{-25}$$

Summary: $(-5)^2 = 25$, because the –5 is being squared.

$-5^2 = -25$, because only the 5 is being squared.

Moral: The parentheses make all the difference!

Homework

7. For each number below, find two numbers whose square is that number:

   a. 1 b. 9 c. 25 d. 196 e. 225 f. 529

8. Simplify each power:

   a. 13^2 b. 1^2 c. 0^2 d. 6^2

   e. $(-3)^2$ f. $(-12)^2$ g. $(-9)^2$ h. $(-1)^2$

   i. $\left(\frac{2}{5}\right)^2$ j. $\left(-\frac{1}{3}\right)^2$ k. $(1.2)^2$ l. $(-0.5)^2$

9. Simplify each power:

   a. 3^3 b. 10^3 c. $(-5)^3$ d. $(-1)^3$

   e. 0^3 f. $(-12)^3$ g. $\left(-\frac{1}{2}\right)^3$ h. $[-(-4)]^3$

   i. $(2-5)^3$ j. $(0.2)^3$ k. $(-0.4)^3$ l. $(\pi-\pi)^3$

10. Simplify each expression:

    a. $3^2 - 4^2$ b. $0^2 - 7^2$ c. $0 - 9^2$ d. $1 - 10^2$

    e. -8^2 f. -10^2 g. $(-10)^2$ h. -1^2

    i. $(-1)^2$ j. -11^2 k. -12^2 l. -100^2

❑ Square Roots of Signed Numbers

Review:

We saw square roots of positive numbers and zero when we worked with the Pythagorean Theorem. Here's a brief review:

$\sqrt{49} = 7$, since $7^2 = 49$

$\sqrt{144} = 12$, since $12^2 = 144$

$-\sqrt{64} = -8$, since the minus sign signifies the *opposite* of $\sqrt{64}$

$\sqrt{1} = 1$, since $1^2 = 1$

$\sqrt{0} = 0$, since $0^2 = 0$

$\sqrt{21} \approx 4.583$, since $4.583^2 = 21.003889$, which is about 21

$\approx$ is approximately equal to

$-\sqrt{2} \approx -1.414$

The Real Purpose of This Section:

Now we tackle the square root of a negative number. Let's consider

$$\sqrt{-9}$$

Could it be 3? NO, since $3^2 = 9$, not –9. How about –3? Still NO, because $(-3)^2 = 9$, not –9. What's happening here? Why can't we find a number whose square is –9? Because, as we saw in the preceding section, whether a number is positive or negative, its square is positive! There does not exist any number whose square is –9, and therefore there's no square root of –9.

Thus, (in this course) one can never take the square root of a negative number. It simply doesn't exist (yet). What does your calculator say about this situation?

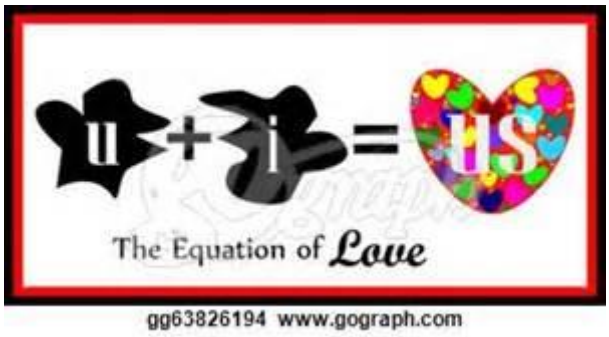
Homework

11. Simplify each square-root expression:

a. $\sqrt{64}$	b. $\sqrt{-64}$	c. $\sqrt{16+9}$	d. $\sqrt{0}$
e. $\sqrt{400}$	f. $\sqrt{-25}$	g. $\sqrt{-1}$	h. $-\sqrt{100}$
i. $\sqrt{1}$	j. $-\sqrt{-2}$	k. $\sqrt{-(-169)}$	l. $-\sqrt{-5}$
m. $-\sqrt{81}$	n. $-\sqrt{-36}$	o. $\sqrt{-(-121)}$	p. $-\sqrt{-0}$
q. $-\sqrt{-(-144)}$	r. $-\sqrt{49}$	s. $-\sqrt{-49}$	t. $\sqrt{225}$

12. Explain the difference between $\sqrt{-3}$ and $-\sqrt{3}$.

❑ Solving Equations



The equation $2x = -14$ presented in the Introduction of the previous chapter can now be solved. Dividing each side by 2 produces the equation $x = \frac{-14}{2}$, which implies that $x = -7$, by the rule that a negative divided by a positive is negative.

EXAMPLE 1: **Solve each equation:**

A. $-3n = 18 \Rightarrow \frac{-3n}{-3} = \frac{18}{-3} \Rightarrow \boldsymbol{n = -6}$

B. $-8y = -34 \Rightarrow \frac{-8y}{-8} = \frac{-34}{-8} \Rightarrow \boldsymbol{y = \frac{17}{4}}$

C. $\frac{x}{2} = -17 \Rightarrow \left[\frac{x}{2}\right]2 = \left[-17\right]2$

$\Rightarrow \left[\frac{x}{\cancel{2}}\right]\cancel{2} = -34 \Rightarrow \boldsymbol{x = -34}$

D. $\frac{a}{-10} = -20 \Rightarrow a = (-20)(-10) \Rightarrow \boldsymbol{a = 200}$

EXAMPLE 2: **Solve each equation:**

A. $2a + 8 = -14$

$2a + 8 - 8 = -14 - 8$ (subtract 8 from each side)

$2a = -22$ (simplify each side)

$\frac{2a}{2} = \frac{-22}{2}$ (divide each side by 2)

$\boldsymbol{a = -11}$ (simplify each side)

B. $-3b - 7 = -10$

$-3b - 7 + 7 = -10 + 7$ (add 7 to each side)

$-3b = -3$ (simplify each side)

$\frac{-3b}{-3} = \frac{-3}{-3}$ (divide each side by –3)

$\boldsymbol{b = 1}$ (simplify each side)

C. $-x + 5 = -2$

$-x + 5 - 5 = -2 - 5$ (subtract 5 from each side)

$-x = -7$ (simplify each side)

At this point, it's handy to write $-x$ as $-1x$.

$-1x = -7$

$\frac{-x}{-1} = \frac{-7}{-1}$ (divide each side by –1)

$\boldsymbol{x = 7}$ (simplify each side)

D. $-6 - x = -4$

$-6 + 6 - x = -4 + 6$ (add 6 to each side)

$-x = 2$ (simplify each side)

$\frac{-x}{-1} = \frac{2}{-1}$ (divide each side by –1)

$\boldsymbol{x = -2}$ (simplify each side)

E. $3y + 7 = 5$

$3y + 7 - 7 = 5 - 7$ (subtract 7 from each side)

$3y = -2$ (simplify)

$\frac{3y}{3} = \frac{-2}{3}$ (divide each side by 3)

$\boldsymbol{y = -\frac{2}{3}}$ (simplify)

Homework

13. Solve each equation:

a. $5x = -15$ b. $-3y = 24$ c. $-2n = -20$

d. $-3z = 14$ e. $8n = -6$ f. $-12Q = -40$

g. $\frac{x}{3} = -23$ h. $\frac{w}{-5} = 40$ i. $\frac{a}{-5} = -6$

j. $-5x = -15$ k. $-3y = -24$ l. $2n = -20$

m. $-3z = 15$ n. $8n = -2$ o. $-14Q = -40$

p. $\frac{x}{3} = -12$ q. $\frac{w}{-5} = 20$ r. $\frac{a}{-6} = -6$

14. Solve each equation:

a. $3x + 7 = 5$ b. $8n - 3 = -9$ c. $4y + 8 = -13$

d. $-5t - 1 = -8$ e. $-3d + 8 = 4$ f. $-4p - 1 = -8$

g. $-x - 8 = 7$ h. $23 - y = -1$ i. $-g - 4 = -5$

j. $12 = 2x + 20$ k. $8 - 3w = 8$ l. $-6 + 2x = -20$

m. $-23 = 7 - 3r$ n. $12 + 12u = 12$ o. $-20 = 23 - 7z$

p. $2x + 7 = 5$ q. $9n - 3 = -9$ r. $3y + 8 = -13$

Review Problems

15. a. Find two numbers whose square is 1,000,000.
b. There's only one number whose square is 0. What is it?
c. There's no number (in this class) whose square is –25. Explain.

16. a. $\frac{0}{0}$ b. $\frac{0}{23}$ c. $\frac{-5}{0}$ d. $(13)^2$

e. $(-12)^2$ f. $(6)^3$ g. -15^2 h. $(-7)^3$

i. $\sqrt{196}$ j. $\sqrt{-81}$ k. $-\sqrt{-144}$

l. Solve for x: $\frac{x}{-5} = -6$ m. Solve for n: $\frac{n}{-6} = 10$

n. Solve for a: $-3a - 9 = 5$ o. Solve for w: $8 - w = 10$

17. a. $-20^2 =$ b. $(-6)^2 =$ c. $-6^2 =$

d. $-0^2 =$ e. $-4^2 =$ f. $-15^2 =$

18. Which statement is true?

a. $\frac{8}{0}$ is undefined AND $\frac{0}{3} = 0$ AND $\frac{0}{0} = 0$.

b. $\frac{8}{0} = 0$ AND $\frac{0}{3} = 0$ AND $\frac{0}{0} = 0$.

c. $\frac{8}{0}$ is undefined AND $\frac{0}{3}$ is undefined AND $\frac{0}{0} = 0$.

d. $\frac{8}{0}$ is undefined AND $\frac{0}{3} = 0$ AND $\frac{0}{0}$ is undefined.

e. $\frac{8}{0} = 0$ AND $\frac{0}{3} = 0$ AND $\frac{0}{0} = 0$.

Solutions

1. a. $\frac{0}{15} = 0$ since $0 \times 15 = 0$, and 0 is the only number that accomplishes this.

b. $\frac{32}{0}$ is undefined because any number times 0 is 0, <u>never</u> 32; thus NO number works.

c. $\frac{0}{0}$ is undefined because any number times 0 is 0; thus EVERY number works.

2. a. 0 b. 0 c. 0 d. 0 e. Undefined f. Undefined
g. Undefined h. Undefined i. Undefined j. Undefined
k. Undefined l. 0

3. a. **4**. b. **5**. b.

6. a. You can't conclude anything -- it depends on what's on the bottom. If the bottom is a non-zero number (like 7), then $\frac{0}{7} = 0$. If the bottom is zero, then $\frac{0}{0}$ is undefined.

b. This time we can conclude that the fraction is undefined, since division by 0 is undefined, no matter what's on the top of the fraction.

7. a. 1 & –1 b. 3 & –3 c. 5 & –5 d. 14 & –14
e. 15 & –15 f. 23 & –23

8. a. 169 b. 1 c. 0 d. 36 e. 9 f. 144
g. 81 h. 1 i. $\frac{4}{25}$ j. $\frac{1}{9}$ k. 1.44 l. 0.25

9. a. 27 b. 1000 c. –125 d. –1 e. 0 f. –1728
g. $-\frac{1}{8}$ h. 64 i. –27 j. 0.008 k. –0.064 l. 0

10. a. –7 b. –49 c. –81 d. –99 e. –64 f. –100
g. 100 h. –1 i. 1 j. –121 k. –144 l. –10,000

11. a. 8 b. Does not exist c. 5 d. 0
e. 20 f. Does not exist g. Does not exist h. –10
i. 1

j. Does not exist -- the square root of –2 does not exist, so the problem is over right there. The minus sign in front has no bearing on the problem.

k. 13 -- This time, the minuses cancel, leaving $\sqrt{169}$.

l. Does not exist m. –9 n. Does not exist
o. 11 p. 0 q. –12
r. –7 s. Does not exist t. 15

12. $\sqrt{-3}$ is the square root of –3, which does not exist in Algebra 1. But $-\sqrt{3}$ does exist. It's simply the opposite of $\sqrt{3}$, and is therefore approximately –1.732.

13. a. –3 b. –8 c. 10 d. $-\frac{14}{3}$ e. $-\frac{3}{4}$ f. $\frac{10}{3}$
g. –69 h. –200 i. 30 j. 3 k. 8 l. –10
m. –5 n. $-\frac{1}{4}$ o. $\frac{20}{7}$ p. –36 q. –100 r. 36

14. a. $-\frac{2}{3}$ b. $-\frac{3}{4}$ c. $-\frac{21}{4}$ d. $\frac{7}{5}$ e. $\frac{4}{3}$ f. $\frac{7}{4}$
g. –15 h. 24 i. 1 j. –4 k. 0 l. –7
m. 10 n. 0 o. $\frac{43}{7}$ p. –1 q. $-\frac{2}{3}$ r. –7

15. a. 1000 & –1000 b. 0 c. The square of any number in this class is 0 or positive, never negative; this is because a positive times a positive is positive, and a negative times a negative is positive.

16. a. Undefined b. 0 c. Undefined d. 169
e. 144 f. 216 g. –225 h. –343
i. 14 j. Does not exist k. Does not exist
l. $x = 30$ m. $n = -60$ n. $a = -\frac{14}{3}$ o. $w = -2$

17. a. –400 b. 36 c. –36 d. 0 e. –16 f. –225

18. d.

"We cannot hold a torch to light another's path without brightening our own."

– Ben Sweetland