CH 39 – CREATING THE EQUATION OF A LINE

❑ Introduction

Some chapters back we played around with straight lines. We graphed a few, and we learned how to find their intercepts and slopes. Now we're ready to formalize the whole concept into a single idea -- the kind of thing you'll need for Intermediate Algebra, statistics, chemistry, economics, and many other disciplines.

Before we tackle these lines, let's review the algebra skills and straight-line concepts we'll need for this chapter.

Homework

1. Solve each formula for y:

a. $-4x + y - 1 = 0$ b. $-3x - y = 7$

c. $-2x + y + 1 = 0$ d. $3x + 2y = 10$

e. $2x - 7y = 14$ f. $-8x - 2y + 7 = 0$

2. Find all the intercepts of the line $3x - 7y = 42$.

3. Find the slope of the line connecting the points $(2, -3)$ and $(-8, 9)$.

❑ Calculating Slope & *y*-intercept

Consider the line

$$y = 2x + 3$$

Let's see what we can discover about this line, using just the ideas from the previous chapters.

First we'll determine the **slope** of the line. To do this, we need a pair of points on the line, which are simply solutions of the line equation. To get points on the line, we'll choose a couple of x-values off the top of our head, and then calculate the corresponding y-values.

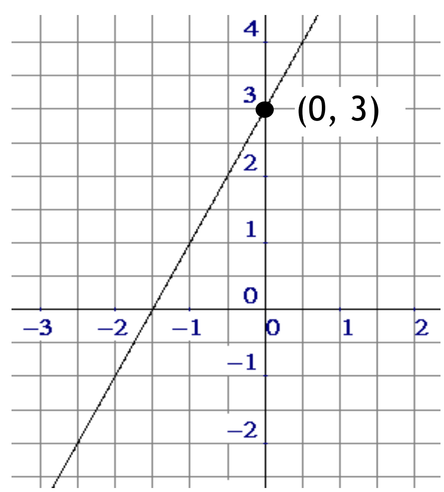
Suppose $x = 5$; then $y = 2(\mathbf{5}) + 3 = 10 + 3 = 13$.
Thus, (5, 13) is a point on the line.

Now let $x = -1$; and so $y = 2(\mathbf{-1}) + 3 = -2 + 3 = 1$.
Therefore, (–1, 1) is a point on the line.

We can now compute the slope, using the points we just calculated, (5, 13) and (–1, 1):

$$m = \frac{\Delta y}{\Delta x} = \frac{13-1}{5-(-1)} = \frac{13-1}{5+1} = \frac{12}{6} = \mathbf{2}$$

Second we'll calculate the ***y*-intercept** of the line. Recall that the y-intercept of any graph is found by setting x to 0. Here's what we get:



$$y = 2(\mathbf{0}) + 3 = 0 + 3 = \mathbf{3}$$

The y-intercept is therefore (0, 3). In summary,

The line

$$\mathbf{y = 2x + 3}$$

has a slope of 2 and a *y*-intercept of (0, 3).

Homework

4. Consider the line $y = -3x + 7$.

   a. Find two points on the line. For instance, let $x = 1$ and then let $x = 3$ (or choose your own x's).

   b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

   c. Find the y-intercept of the line (by letting $x = 0$, of course).

5. Consider the line $y = 5x - 14$.

   a. Find two points on the line. For instance, let $x = -2$ and then let $x = 4$.

   b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

   c. Find the y-intercept of the line by letting $x = 0$.

6. Consider the line $y = -x - 17$.

   a. Find two points on the line.

   b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

   c. Find the y-intercept of the line by letting $x = 0$.

7. Consider the line $y = 9x + 44$.

   a. Find two points on the line.

   b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \frac{\Delta y}{\Delta x}$.

   c. Find the y-intercept of the line by letting $x = 0$.

❑ The $y = mx + b$ Form of a Line

Here's a summary of the four homework problems you just completed (You did complete them, right?):

$y = -3x + 7$	$m = -3$	y-int = $(0, 7)$
$y = 5x - 14$	$m = 5$	y-int = $(0, -14)$
$y = -x - 17$	$m = -1$	y-int = $(0, -17)$
$y = 9x + 44$	$m = 9$	y-int = $(0, 44)$

See a pattern here? The slope of each line is simply the number in front of the x; that is, the *coefficient* of the first term. For example, the slope of the line $y = 5x - 14$ is 5.

Important: Note that the slope of the line is **5**, NOT 5*x*.

Also, the y-intercept is essentially the number hanging off the end of the equation. ("Essentially" means that the y-intercept is the point (0, *something*), and that *something* is the number at the end of the line equation.) For example, for the line $y = 9x + 44$, the y-intercept is $(0, 44)$.

Therefore, if you're asked for the slope and the y-intercept of the line $y = -14x + 17$, for example, you should be able to immediately reply that the slope is -14 and that the y-intercept is $(0, 17)$.

Conversely, if you're asked to come up with the equation of a line whose slope is 23 and whose y-intercept is $(0, \pi)$, be sure you understand that no calculations are required to come up with the equation $y = 23x + \pi$.

Let's generalize these examples. We will, as usual, let m represent the slope of the line, and let b (for some odd reason) stand for the y-coordinate of the y-intercept. Here's what it all boils down to:

$$y \;=\; mx + b$$

SLOPE

y-INTERCEPT

To be precise, *b* is not the *y*-intercept; *b* is the *y*-coordinate of the *y*-intercept. The *y*-intercept is properly written (0, *b*).

EXAMPLE 1:

A. Find the slope and the *y*-intercept of the line $y = -\frac{2}{3}x - 5$.

Answer: The slope is $-\frac{2}{3}$ and the *y*-intercept is (0, –5).

B. Find the equation of the line whose slope is –6 and whose *y*-intercept is $\left(0, \frac{4}{5}\right)$. Answer: $y = -6x + \frac{4}{5}$.

EXAMPLE 2: **Find the slope and the *y*-intercept of the line $3x - 5y + 15 = 0$.**

Solution: The given equation, $3x - 5y + 15 = 0$, doesn't fit the slope-intercept form, $y = mx + b$, of a line. But we can make it fit; we can solve the equation $3x - 5y + 15 = 0$ for *y*:

$$3x - 5y + 15 = 0 \qquad \text{(the original line)}$$

$$\Rightarrow \quad 3x - 5y = -15 \qquad \text{(subtract 15 from each side)}$$

$$\Rightarrow \quad -5y = -3x - 15 \qquad \text{(subtract } 3x \text{ from each side)}$$

$$\Rightarrow \quad \frac{-5y}{-5} = \frac{-3x - 15}{-5} \qquad \text{(divide each side by } -5\text{)}$$

$$\Rightarrow \quad y = \frac{3x}{5} + \frac{15}{5} \qquad \text{(split the right-hand fraction)}$$

$$\Rightarrow \quad y = \frac{3}{5}x + 3 \qquad \text{(rewrite the first fraction)}$$

Now that the line is in the $y = mx + b$ form, we conclude that

$$\boxed{\text{The slope is } \frac{3}{5} \text{ and the } y\text{-intercept is } (0, 3).}$$

Homework

8. a. Find the slope and y-intercept of the line $y = -17x + 13$.

   b. Find the equation of the line whose slope is 99 and whose y-intercept is $(0, -101)$

   c. Find the equation of the line whose slope is $\frac{2}{3}$ and whose y-intercept is $(0, -\pi)$.

   d. Find the equation of the line whose slope is $-\frac{5}{4}$ and whose y-intercept is $(0, 2\pi)$.

9. Find the slope and y-intercept of each line by converting the line to $y = mx + b$ form, if necessary:

   a. $y = 132x - 1000$ b. $y = -\frac{8}{7}x - \frac{13}{9}$

   c. $7x - 9y = 10$ d. $-3x - 5y + 1 = 0$

   e. $2x + 7y = 13$ f. $-5x + 2y + 3 = 0$

   g. $y = \frac{9x - 5}{2}$ h. $-17x - y = 4$

   i. $2x - 6y = 8$ j. $-x + 4y - 2 = 0$

   k. $7y - 2x = 0$ l. $7x + 4y + 5 = 0$

❑ A Proof of the Slope / *y*-Intercept Form of a Line

We've learned the following:

The line with slope *m* and *y*-intercept (0, *b*) has the equation $y = mx + b$.

We did this by viewing some homework results; but examples prove nothing. So let's do it the right way, by proving that $y = mx + b$ really has the properties we've been claiming it has.

Claim #1: The line $y = mx + b$ has a *y*-intercept of (0, *b*).

Proof: To find the *y*-intercept of any graph, we set x to 0 and solve for y:

$$y = m(\mathbf{0}) + b \quad \Rightarrow \quad y = 0 + b \quad \Rightarrow \quad y = b.$$

In other words, when $x = 0$, $y = b$, which means that $(0, b)$ is on the line, and thus $(0, b)$ is precisely the y-intercept.

Claim #2: The line $y = mx + b$ has a slope of *m*.

Proof: Our definition of slope, $m = \dfrac{\Delta y}{\Delta x}$, will be used to calculate the slope of the line. To apply this definition, we need two points on the line. One of them might as well be the y-intercept calculated above: $(0, b)$. For the other point, pick $x = 1$. This yields a y-value of $y = m(\mathbf{1}) + b = m + b$. Therefore, $(1, m + b)$ is a second point on the line. Now we can find the slope, using the two points $(0, b)$ and $(1, m + b)$:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{(m+b)-b}{1-0} = \frac{m+b-b}{1} = \frac{m}{1} = m,$$

and thus the slope of the line $y = mx + b$ is indeed m.
Our two claims have verified our assertion, and the proof is complete.

Review Problems

10. Consider the line $y = 10x - 13$.

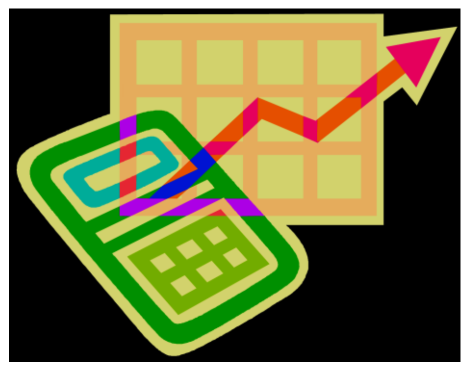
    a. Find two points on the line. For instance, let $x = 1$ and then let $x = -2$.

    b. Find the slope of the line using the two points you found in part a. by applying the definition of slope, $m = \dfrac{\Delta y}{\Delta x}$.

    c. Find the y-intercept of the line by letting $x = 0$.

11. Find the equation of the line whose slope is -17 and whose y-intercept is $(0, 99)$.

12. Find the slope and the y-intercept of the line $-28x + 4y = 16$ by converting the line to $y = mx + b$ form.



Solutions

1. a. $y = 4x + 1$

b. $-3x - y = 7$

$$\Rightarrow \quad -y = 3x + 7$$

$$\Rightarrow \quad \frac{-y}{-1} = \frac{3x+7}{-1} \qquad \text{(you could also multiply each side by } -1\text{)}$$

$$\Rightarrow \quad y = -3x - 7$$

c. $y = 2x - 1$

d. $3x + 2y = 10 \Rightarrow 2y = -3x + 10$

$$\Rightarrow y = \frac{-3x+10}{2}$$

$$\Rightarrow y = -\frac{3}{2}x + 5$$

e. $y = \frac{2}{7}x - 2$

f. $y = -4x + \frac{7}{2}$

2. x-int: (14, 0); y-int: (0, –6)

3. $m = -\frac{6}{5}$

4. a. (1, 4) and (3, –2) b. $m = \frac{\Delta y}{\Delta x} = \frac{4-(-2)}{1-3} = \frac{6}{-2} = -3$

c. $y = -3(\mathbf{0}) + 7 = 7$; so the y-intercept is (0, 7).

5. a. (–2, –24) and (4, 6) b. $m = \frac{\Delta y}{\Delta x} = \frac{-24-6}{-2-4} = \frac{-30}{-6} = 5$

c. $y = 5(\mathbf{0}) - 14 = -14$; so the y-intercept is (0, –14).

6. a. You choose the two points. b. $m = -1$ c. y-int = (0, –17)

7. a. You choose the two points. b. $m = 9$ c. y-int = (0, 44)

8. a. $m = -17$ y-int = (0, 13) b. $y = 99x - 101$

c. $y = \frac{2}{3}x - \pi$ d. $y = -\frac{5}{4}x + 2\pi$

9. a. $m = 132$ y-int = (0, –1000) b. $m = -\frac{8}{7}$ y-int = $(0, -\frac{13}{9})$

c. $m = \frac{7}{9}$ y-int = $(0, -\frac{10}{9})$ d. $m = -\frac{3}{5}$ y-int = $(0, \frac{1}{5})$

e. $m = -\frac{2}{7}$ y-int = $(0, \frac{13}{7})$ f. $m = \frac{5}{2}$ y-int = $(0, -\frac{3}{2})$

g. $m = \frac{9}{2}$ y-int = $(0, -\frac{5}{2})$ h. $m = -17$ y-int = (0, –4)

i. $m = \frac{1}{3}$ y-int = $(0, -\frac{4}{3})$ j. $m = \frac{1}{4}$ y-int = $(0, \frac{1}{2})$

k. $m = \frac{2}{7}$ y-int = (0, 0) l. $m = -\frac{7}{4}$ y-int = $(0, -\frac{5}{4})$

10. a. (1, –3) and (–2, –33)

b. $m = \frac{\Delta y}{\Delta x} = \frac{-3-(-33)}{1-(-2)} = \frac{-3+33}{1+2} = \frac{30}{3} = 10$

c. $y = 10(\mathbf{0}) - 13 = 0 - 13 = -13$; the y-intercept is $(0, -13)$.

11. $y = -17x + 99$

12. $m = 7$; y-int: $(0, 4)$

❑ **To ∞ and Beyond!**

Consider the infinite sequence of numbers:

8, 10, 12, 14, 16, . . .

If 8 is the 1st term, and 10 is the 2nd term, etc., what is the 1,000th term?

"Without education, we are in a horrible and deadly danger of taking educated people seriously."

– G.K. Chesteron