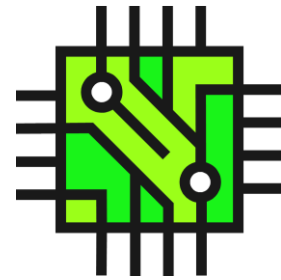

CH 68 – MORE PYTHAGOREAN THEOREM AND MORE RADICALS

□ INTRODUCTION

The last time we studied the Pythagorean Theorem we may have used our calculator to round square roots that didn't come out whole numbers. But rounding off square roots is not always a good idea in the real world.

After we've done some applied mathematics, we may give the results to an engineer. Suppose we've rounded to the hundredths place (good enough for business) but the engineer is designing the Core 9 Quad microprocessor, and she needs all the digits of the numbers out to the billionths



place. We haven't done her any good at all by rounding off for her. The safest thing to do is to give her the exact results and let her round according to her requirements.

We're now going to work with the Pythagorean Theorem again, but this time we're going to leave our answers exact, using the methods of the previous chapter. To this end, we need to review the key concept regarding square roots.

Important Fact About Square Roots

If we square the square root of “something” (that's not negative), we get the “something”:

$$(\sqrt{25})^2 = 5^2 = 25$$

$$(\sqrt{34})^2 \approx (5.830951895)^2 \approx 34$$

And we can write the general formula:

$$\boxed{(\sqrt{x})^2 = x}$$

Squaring Undoes
Square-Rooting

Another skill we'll need for this chapter is the ability to simplify an expression like $(5\sqrt{3})^2$. Recall one of our five laws of exponents: $(ab)^2 = a^2b^2$.

This rule means that we can simplify $(5\sqrt{3})^2$ as follows:

$$(5\sqrt{3})^2 = 5^2(\sqrt{3})^2 = 25 \cdot 3 = \mathbf{75}$$

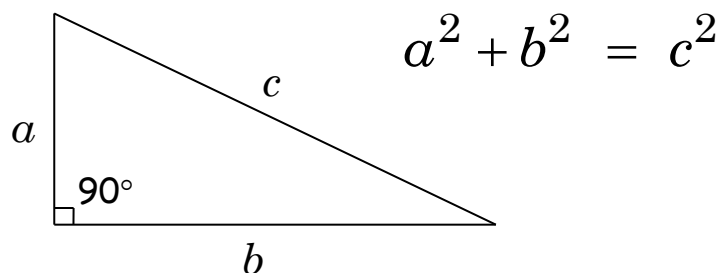
Homework

1. While it's the case that $(ab)^2 = a^2b^2$, is this rule true for addition? That is, does $(a + b)^2 = a^2 + b^2$?
2. Simplify each expression:

a. $(\sqrt{7})^2$	b. $(\sqrt{29})^2$	c. $(\sqrt{233})^2$	d. $(\sqrt{101})^2$
e. $(2\sqrt{3})^2$	f. $(5\sqrt{5})^2$	g. $(9\sqrt{2})^2$	h. $(12\sqrt{7})^2$

□ SOLVING RIGHT TRIANGLES

Recall the Pythagorean Theorem:



EXAMPLE 1: Solve each right triangle problem:

A. The legs are 6 and 10. Find the hypotenuse.

$$\begin{aligned} a^2 + b^2 = c^2 &\Rightarrow 6^2 + 10^2 = c^2 \Rightarrow 36 + 100 = c^2 \\ \Rightarrow c^2 = 136 &\Rightarrow c = \sqrt{136} = \sqrt{4 \cdot 34} = 2\sqrt{34} \end{aligned}$$

Notice that $c^2 = 136$ is a quadratic equation, and therefore probably has two solutions; in fact, the solutions are $\pm\sqrt{136}$, or $\pm 2\sqrt{34}$. But in this problem we're talking about the hypotenuse of a triangle, whose length must be positive. So we immediately discard the negative solution and retain just the positive one.

B. One leg is 20 and the hypotenuse is 30. Find the other leg.

$$\begin{aligned} a^2 + b^2 = c^2 &\Rightarrow 20^2 + b^2 = 30^2 \Rightarrow 400 + b^2 = 900 \\ \Rightarrow b^2 = 500 &\Rightarrow b = \sqrt{500} = \sqrt{100 \cdot 5} = 10\sqrt{5} \end{aligned}$$

C. The legs are $2\sqrt{7}$ and $3\sqrt{13}$. Find the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow (2\sqrt{7})^2 + (3\sqrt{13})^2 = c^2 \\ \Rightarrow 28 + 117 &= c^2 \Rightarrow c^2 = 145 \Rightarrow c = \sqrt{145} \end{aligned}$$

D. The hypotenuse is $\sqrt{72}$ and one of the legs is 8. Find the other leg.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow a^2 + 8^2 = (\sqrt{72})^2 \Rightarrow a^2 + 64 = 72 \\ \Rightarrow a^2 &= 8 \Rightarrow a = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2} \end{aligned}$$

E. One of the legs is $3\sqrt{2}$ and the hypotenuse is $\sqrt{42}$. Find the other leg.

$$\begin{aligned} a^2 + b^2 &= c^2 \Rightarrow a^2 + (3\sqrt{2})^2 = (\sqrt{42})^2 \\ \Rightarrow a^2 + 18 &= 42 \Rightarrow a^2 = 24 \Rightarrow a = \sqrt{24} = 2\sqrt{6} \end{aligned}$$

Homework

Calculate the EXACT answer to each Pythagorean problem:

3. The legs of a right triangle are 8 and 12. Find the hypotenuse.
4. The legs of a right triangle are 6 and 9. Find the hypotenuse.
5. Find the hypotenuse of a right triangle given that its legs have lengths of 5 and 7.
6. One leg of a right triangle is 15 and the hypotenuse is 20. Find the other leg.
7. The hypotenuse of a right triangle is 18 and one of its legs is 10. Find the other leg.

8. One leg of a right triangle is 5 and the hypotenuse is $\sqrt{43}$. Find the other leg.
9. Find the missing leg of a right triangle if the hypotenuse is $\sqrt{18}$ and one of its legs is 3.
10. The legs of a right triangle are $2\sqrt{3}$ and $3\sqrt{2}$. Find the hypotenuse.
11. The hypotenuse of a right triangle is $5\sqrt{7}$ and one of its legs is $2\sqrt{3}$. Find the length of the other leg.
12. A leg of a right triangle is $2\sqrt{8}$ and its hypotenuse is 10. Find the length of the other leg.
13. The leg of a right triangle is 5 and its hypotenuse is 4. Find the other leg.

□ REVIEW OF SIMPLIFYING RADICALS

Recall how we simplify $\sqrt{63}$:

$$\sqrt{63} = \sqrt{9 \cdot 7} = \sqrt{9} \cdot \sqrt{7} = 3\sqrt{7}$$

If we come across $\sqrt{121}$, we simply change it to 11, and we know that $\sqrt{0} = 0$. And when asked to simplify $\sqrt{-12}$, we don't even try because this does not exist in Elementary Algebra.

Also remember that ± 5 means “plus or minus 5.” Thus, to evaluate the expression 10 ± 2 , we write

$$10 + 2 = 12 \quad (\text{using the plus sign})$$

$$10 - 2 = 8 \quad (\text{using the minus sign})$$

So 10 ± 2 represents the two numbers 12 and 8.

Just as $\frac{-8}{12} = -\frac{2}{3}$, if we see the expression $\frac{\pm\sqrt{50}}{3}$, we should probably write it as $\pm\frac{5\sqrt{2}}{3}$.

Last example in this section: To simplify $\frac{-10 \pm \sqrt{243}}{7}$, we notice that $243 = 81 \times 3$, and therefore,

$$\frac{-10 \pm \sqrt{243}}{7} = \frac{-10 \pm \sqrt{81 \cdot 3}}{7} = \frac{-10 \pm 9\sqrt{3}}{7}$$

This final answer can also be written separately as

$$\frac{-10 + 9\sqrt{3}}{7}, \quad \frac{-10 - 9\sqrt{3}}{7}$$

The next section of this chapter will deal with the possibility that after we have simplified the radical, we may need to continue simplifying by reducing the fraction to lowest terms.

Homework

14. Simplify each expression:

- | | | | |
|---------------------------------|----------------------------------|----------------------------------|----------------------------------|
| a. 12 ± 5 | b. 100 ± 144 | c. 32 ± 0 | d. $2 \pm \sqrt{9}$ |
| e. $-0 \pm \sqrt{100}$ | f. $\frac{\pm\sqrt{80}}{7}$ | g. 1 ± 1 | h. $-8 \pm \sqrt{144}$ |
| i. $1 \pm \sqrt{2}$ | j. $-3 \pm \sqrt{8}$ | k. $1 \pm \sqrt{18}$ | l. $-7 \pm \sqrt{50}$ |
| m. $\frac{1 \pm \sqrt{27}}{5}$ | n. $\frac{4 \pm \sqrt{-9}}{2}$ | o. $\frac{-4 \pm \sqrt{75}}{9}$ | p. $\frac{-2 \pm \sqrt{39}}{3}$ |
| q. $\frac{3 \pm \sqrt{147}}{5}$ | r. $\frac{\pm\sqrt{52}}{7}$ | s. $\frac{\pm\sqrt{220}}{3}$ | t. $\frac{10 \pm \sqrt{20}}{7}$ |
| u. $\frac{-4 \pm \sqrt{8}}{3}$ | v. $\frac{-7 \pm \sqrt{245}}{8}$ | w. $\frac{13 \pm \sqrt{196}}{2}$ | x. $\frac{-1 \pm \sqrt{-95}}{2}$ |
| y. $\frac{5 \pm \sqrt{0}}{7}$ | z. $\frac{-8 \pm \sqrt{363}}{5}$ | | |

□ REDUCING FRACTIONS WITH RADICALS

We recall that reducing a fraction to lowest terms involves two main steps:

- 1) **Factor** the top and bottom
- 2) **Divide** out (cancel) any common factors



For example,

$$\frac{\overbrace{8+6x}^{2 \text{ terms}}}{4} = \frac{\overbrace{2(4+3x)}^{1 \text{ term}}}{2(2)} = \frac{\cancel{2}(4+3x)}{\cancel{2}(2)} = \frac{4+3x}{2}$$

EXAMPLE 2: Simplify: $\frac{15 \pm \sqrt{75}}{20}$

Solution: We have all the skills required to simplify this expression:

$$\begin{aligned} & \frac{15 \pm \sqrt{75}}{20} && \text{(the original fraction)} \\ = & \frac{15 \pm \sqrt{25 \cdot 3}}{20} && \text{(factor 75 using a perfect square)} \\ = & \frac{15 \pm \sqrt{25} \sqrt{3}}{20} && \text{(split the radical in two)} \\ = & \frac{15 \pm 5\sqrt{3}}{20} && \text{(simplify the radical)} \\ = & \frac{5(3 \pm \sqrt{3})}{2 \cdot 2 \cdot 5} && \text{(factor the top and bottom)} \\ = & \frac{\cancel{5}(3 \pm \sqrt{3})}{2 \cdot 2 \cdot \cancel{5}} && \text{(divide out the common factor of 5)} \end{aligned}$$

A little arithmetic gives the final reduced fraction

$$\boxed{\frac{3 \pm \sqrt{3}}{4}}$$

EXAMPLE 3: Simplify each radical expression:

$$A. \quad \frac{\pm\sqrt{147}}{7} = \frac{\pm\sqrt{(49)(3)}}{7} = \frac{\pm 7\sqrt{3}}{7} = \frac{\pm \cancel{7}\sqrt{3}}{\cancel{7}} = \boxed{\pm\sqrt{3}}$$

$$B. \quad \frac{\pm\sqrt{300}}{15} = \frac{\pm\sqrt{100 \times 3}}{15} = \frac{\pm 10\sqrt{3}}{15} = \frac{\pm 2 \cdot \cancel{5}\sqrt{3}}{3 \cdot \cancel{5}} = \boxed{\pm \frac{2\sqrt{3}}{3}}$$

Homework

15. Simplify each expression:

a. $\frac{12 + \sqrt{100}}{4}$

b. $\frac{75 - \sqrt{75}}{35}$

c. $\frac{32 \pm \sqrt{40}}{6}$

d. $\frac{-4 \pm \sqrt{8}}{2}$

e. $\frac{18 \pm \sqrt{18}}{6}$

f. $\frac{5 \pm \sqrt{200}}{10}$

g. $\frac{-2 \pm \sqrt{28}}{6}$

h. $\frac{20 \pm \sqrt{50}}{15}$

i. $\frac{-7 \pm \sqrt{98}}{7}$

j. $\frac{3 \pm \sqrt{0}}{10}$

k. $\frac{-0 \pm \sqrt{144}}{6}$

l. $\frac{30 \pm \sqrt{500}}{70}$

m. $\frac{7 \pm \sqrt{-90}}{14}$

n. $\frac{8 \pm \sqrt{0}}{24}$

o. $\frac{-8 \pm \sqrt{-1}}{8}$

p. $\frac{-9 \pm \sqrt{8-8}}{3}$

q. $\frac{\pm\sqrt{72}}{8}$

r. $\frac{\pm\sqrt{242}}{33}$

Review Problems

16. In each problem, two of the sides of a right triangle are given (l = leg and h = hypotenuse). Find the third side.

a. $l = 7; l = 10$

b. $l = 1; l = 3$

c. $l = 4; l = 6$

d. $l = \sqrt{8}; l = 2\sqrt{2}$

e. $l = 4; l = \sqrt{10}$

f. $l = \sqrt{2}; l = \sqrt{14}$

g. $l = 5; h = 12$

h. $l = \sqrt{5}; h = 2\sqrt{3}$

i. $l = 6; h = 5\sqrt{2}$

j. $l = \sqrt{7}; h = \sqrt{31}$

k. $l = \sqrt{2}; h = 2\sqrt{5}$

l. $l = \sqrt{5}; h = 4\sqrt{5}$

m. $l = \sqrt{2}; l = \sqrt{7}$

n. $l = 7; h = 8$

o. $l = \sqrt{5}; h = \sqrt{13}$

p. $l = 5; h = \sqrt{26}$

q. $l = \sqrt{10}; h = \sqrt{41}$

r. $l = \sqrt{11}; h = 8$

s. $l = \sqrt{13}; l = \sqrt{13}$

t. $l = 5; l = \sqrt{6}$

u. $l = 4; l = 5$

v. $l = 1; h = 2$

w. $l = 1; l = 1$

x. $l = 2\sqrt{2}; h = 2\sqrt{5}$

y. $l = \sqrt{5}; h = \sqrt{21}$

z. $l = 3\sqrt{2}; h = 20$

Simplify each expression:

17. $\frac{10 \pm \sqrt{50}}{15}$

18. $\frac{\pm\sqrt{27}}{3}$

19. $\frac{-14 \pm \sqrt{147}}{28}$

20. $\frac{12 \pm \sqrt{-9}}{4}$

21. $\frac{-7 \pm \sqrt{10}}{14}$

22. $\frac{5 \pm \sqrt{8}}{2}$

23. $\frac{14 \pm \sqrt{14}}{14}$

24. $\frac{\pm\sqrt{98}}{21}$

Solutions

1. NO! You should know quite well that

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = \underline{a^2 + 2ab + b^2}$$

2. a. 7 b. 29 c. 233 d. 101
 e. 12 f. 125 g. 162 h. 1008

3. $4\sqrt{13}$ 4. $3\sqrt{13}$ 5. $\sqrt{74}$ 6. $5\sqrt{7}$ 7. $4\sqrt{14}$ 8. $3\sqrt{2}$

9. 3 10. $\sqrt{30}$ 11. $\sqrt{163}$ 12. $2\sqrt{17}$

13. This scenario is impossible, since the hypotenuse of a right triangle must be longer than either of its legs. Here's a mathematical proof:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ \Rightarrow a^2 + 5^2 &= 4^2 \quad (\text{presuming that the leg is 5 and the hypotenuse is 4}) \\ \Rightarrow a^2 + 25 &= 16 \end{aligned}$$

$$\Rightarrow a^2 = -9 \quad (\text{subtract 25 from each side of the equation})$$

$\Rightarrow a = \sqrt{-9}$, which is not a number in this class. And even if it becomes a number in a later class, it still couldn't be considered the length of a side of a triangle.

- 14.** a. 17, 7 b. 244, -44 c. 32 d. 5, -1
 e. ± 10 f. $\pm \frac{4\sqrt{5}}{7}$ g. 2, 0 h. 4, -20
 i. $1 \pm \sqrt{2}$ j. $-3 \pm 2\sqrt{2}$ k. $1 \pm 3\sqrt{2}$ l. $-7 \pm 5\sqrt{2}$
 m. $\frac{1 \pm 3\sqrt{3}}{5}$ n. Does not exist o. $\frac{-4 \pm 5\sqrt{3}}{9}$ p. $\frac{-2 \pm \sqrt{39}}{3}$
 q. $\frac{3 \pm 7\sqrt{3}}{5}$ r. $\pm \frac{2\sqrt{13}}{7}$ s. $\pm \frac{2\sqrt{55}}{3}$ t. $\frac{10 \pm 2\sqrt{5}}{7}$
 u. $\frac{-4 \pm 2\sqrt{2}}{3}$ v. $\frac{-7 \pm 7\sqrt{5}}{8}$ w. $\frac{27}{2}, -\frac{1}{2}$ x. Does not exist
 y. $\frac{5}{7}$ z. $\frac{-8 \pm 11\sqrt{3}}{5}$

- 15.** a. $\frac{11}{2}$ b. $\frac{15 - \sqrt{3}}{7}$ c. $\frac{16 \pm \sqrt{10}}{3}$
 d. $-2 \pm \sqrt{2}$ e. $\frac{6 \pm \sqrt{2}}{2}$ f. $\frac{1 \pm 2\sqrt{2}}{2}$
 g. $\frac{-1 \pm \sqrt{7}}{3}$ h. $\frac{4 \pm \sqrt{2}}{3}$ i. $-1 \pm \sqrt{2}$
 j. $\frac{3}{10}$ k. ± 2 l. $\frac{3 \pm \sqrt{5}}{7}$
 m. Does not exist n. $\frac{1}{3}$ o. Does not exist
 p. -3 q. $\pm \frac{3\sqrt{2}}{4}$ r. $\pm \frac{\sqrt{2}}{3}$

16. a. $\sqrt{149}$ b. $\sqrt{10}$ c. $2\sqrt{13}$ d. 4 e. $\sqrt{26}$
 f. 4 g. $\sqrt{119}$ h. $\sqrt{7}$ i. $\sqrt{14}$ j. $2\sqrt{6}$
 k. $3\sqrt{2}$ l. $5\sqrt{3}$ m. 3 n. $\sqrt{15}$ o. $2\sqrt{2}$
 p. 1 q. $\sqrt{31}$ r. $\sqrt{53}$ s. $\sqrt{26}$ t. $\sqrt{31}$
 u. 3 v. $\sqrt{3}$ w. $\sqrt{2}$ x. $2\sqrt{3}$ y. 4
 z. $\sqrt{382}$

17. $\frac{2 \pm \sqrt{2}}{3}$ 18. $\pm\sqrt{3}$ 19. $\frac{-2 \pm \sqrt{3}}{4}$

20. Does not exist 21. $\frac{-7 \pm \sqrt{10}}{14}$ 22. $\frac{5 \pm 2\sqrt{2}}{2}$

23. $\frac{14 \pm \sqrt{14}}{14}$ 24. $\pm\frac{\sqrt{2}}{3}$

*“He who opens a school door,
 closes a prison.”*

– Victor Hugo