CH 71 – COMPLETING THE SQUARE

❑ *INTRODUCTION*

It's now time to pay our dues regarding the Quadratic Formula. What, you may ask, does this mean? It means that the formula was merely given to you once or twice in this course, but we never talked about where it came from – trust me; it did not merely descend from the heavens above.

So in this chapter we're going to learn another way to solve quadratic equations (other than the Quadratic Formula and the factoring method), a technique called "Completing the Square" that will also allow us to derive the Quadratic Formula from scratch in the next chapter (as well as for other purposes you'll see in Algebra 2, such as finding the center of a circle).

❑ *FACTORING PERFECT SQUARE TRINOMIALS*

Recall the term "perfect square." The number 100 is a perfect square because 100 can be written as the square of a whole number, namely $100 = 10^2$. Also, the expression $(x + 5)^2$ is a perfect square, since it is the square of $x + 5$. One of the steps in solving a quadratic equation by completing the square is factoring a *perfect square trinomial*. Let's look at four examples.

Example 1:

$x^2 + 14x + 49$ is a perfect square trinomial because it's a trinomial that factors into the square of a binomial:

$$x^2 + 14x + 49 = (x + 7)(x + 7) = (x + 7)^2$$

Example 2:

$n^2 - 20n + 100$ is a perfect square trinomial:

$$n^2 - 20n + 100 = (n - 10)(n - 10) = (n - 10)^2$$

Now a couple of examples with fractions:

Example 3:

$$a^2 + 3a + \frac{9}{4} = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = \left(a + \frac{3}{2}\right)^2$$

Check:

$$\left(a + \frac{3}{2}\right)^2 = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = a^2 + \underbrace{\frac{3}{2}a + \frac{3}{2}a}_{\frac{3}{2}+\frac{3}{2}=\frac{6}{2}=3} + \frac{9}{4} = a^2 + 3a + \frac{9}{4} \checkmark$$

Example 4:

$$y^2 - \frac{2}{5}y + \frac{1}{25} = \left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = \left(y - \frac{1}{5}\right)^2$$

Check:

$$\left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = y^2 - \frac{1}{5}y - \frac{1}{5}y + \frac{1}{25} = y^2 - \frac{2}{5}y + \frac{1}{25} \checkmark$$

Homework

1. Factor each perfect square trinomial:

   a. $x^2 + 10x + 25$ b. $y^2 - 18y + 81$

   c. $a^2 + a + \frac{1}{4}$ d. $m^2 - \frac{4}{3}m + \frac{4}{9}$

e. $z^2 + \frac{2}{5}z + \frac{1}{25}$

f. $w^2 - \frac{5}{3}w + \frac{25}{36}$

g. $b^2 + \frac{9}{5}b + \frac{81}{100}$

h. $u^2 + \frac{3}{2}u + \frac{9}{16}$

i. $n^2 - \frac{4}{7}n + \frac{4}{49}$

j. $x^2 + \frac{10}{11}x + \frac{25}{121}$

❑ *THE "MAGIC NUMBER"*

Consider the trinomial $x^2 + 10x + 25$. We've learned that its factorization is

$$x^2 + 10x + 25 = (x + 5)^2$$, which is the square of a binomial.

Let's look carefully at the numbers in this equality. Notice that the 5 is half of the 10, and that the 25 is the square of the 5.

Let's do one more example. Consider the factorization

$$n^2 - 14n + 49 = (n - 7)^2$$, which is the square of a binomial.

We note that the -7 is half of the -14, and that the 49 is the square of the -7.

So now imagine that I give you

$$x^2 + 6x$$

and I ask you to add a third term to this binomial so that the resulting trinomial will factor into the square of a binomial. In other words,

$$x^2 + 6x + ??? = (x + ?)(x + ?)$$

Each single "?" must be 3, since 3 is half of 6. Also, the "???" must be 9, since 9 is the square of the 3. In other words,

$$x^2 + 6x + \underline{\mathbf{9}} = (x + \underline{\mathbf{3}})(x + \underline{\mathbf{3}})$$

We shall call 9 the "magic number." It can be calculated for this problem using the following two-step rule:

1) Calculate <u>half</u> of 6, which is 3.

2) <u>Square</u> the 3, which is 9, the "magic number."

Note: When we convert $x^2 + 6x$ to $x^2 + 6x + 9$ by adding the "magic number" 9, we are <u>not</u> saying that they're equal, but there will always be a way of adding the magic number without violating any of the laws of algebra.

To become proficient in completing the square, we must be really good at finding the magic number. The following chart gives more examples of how this is done. We'll let *b* represent the number in front of the variable (the coefficient of the linear term).

Original Quadratic	Value of *b*	<u>Half</u> of *b*	Half of *b* <u>Squared</u> = **The Magic Number**	New Quadratic	New Quadratic in Factored Form
$x^2 + 22x$	22	11	$\mathbf{121}$	$x^2 + 22x + \mathbf{121}$	$(x + 11)^2$
$A^2 - 14A$	-14	-7	$\mathbf{49}$	$A^2 - 14A + \mathbf{49}$	$(A - 7)^2$
$n^2 + 9n$	9	$\frac{9}{2}$	$\mathbf{\frac{81}{4}}$	$x^2 + 9x + \mathbf{\frac{81}{4}}$	$\left(x + \frac{9}{2}\right)^2$
$y^2 - \frac{2}{5}y$	$-\frac{2}{5}$	$-\frac{1}{5}$	$\mathbf{\frac{1}{25}}$	$y^2 - \frac{2}{5}y + \mathbf{\frac{1}{25}}$	$\left(y - \frac{1}{5}\right)^2$
$3u^2 - 7u$	This problem can't be done yet; the leading coefficient is <u>not</u> 1.				

Homework

2. Find the **magic number** for each quadratic binomial:

a. $a^2 + 8a$ b. $x^2 - 12x$ c. $y^2 + 28y$

d. $b^2 + 3b$ e. $c^2 - 7c$ f. $t^2 + 11t$

g. $x^2 + x$ h. $y^2 - y$ i. $z^2 + 2z$

j. $g^2 + \frac{2}{3}g$ k. $a^2 - \frac{5}{7}a$ l. $x^2 - \frac{7}{13}x$

❑ *THE FIVE STEPS IN COMPLETING THE SQUARE*

We're now ready to combine many of our algebra skills into the solving of quadratic equations without factoring and without using the Quadratic Formula.

We start by assuming that the quadratic equation is in standard form:

$$ax^2 + bx + c = 0 \qquad \text{(where } a \neq 0\text{)}$$

1. Make sure that the leading coefficient (the *a*) is 1. Divide each side of the equation by *a* if necessary.
2. Move the constant to the other side of the equation.
3. Compute the "magic number" and add it to both sides of the equation. This step "*completes the square*."
4. Factor the left side, and then simplify the right side.
5. Solve the resulting equation by taking square roots, remembering that every positive number has two square roots (the Square Root Theorem).

Note: Steps 1 and 2 can be done in either order.

❑ EXAMPLES OF COMPLETING THE SQUARE

EXAMPLE 5: **Solve by Completing the Square:**

$$\boldsymbol{x^2 + 8x - 20 = 0}$$

Solution: Even though this quadratic equation is factorable, we'll solve it by completing the square, a technique which will also work on problems which aren't factorable.

Step 1:

We must make sure that the leading coefficient (the a) is 1. It already is 1 ($x^2 = 1x^2$), so step 1 is done, and the equation remains the same:

$$x^2 + 8x - 20 = 0$$

Step 2:

Move the constant (the -20) to the right side of the equation, by adding 20 to each side of the equation:

$$x^2 + 8x = 20$$

Step 3:

Now we calculate the magic number:

a. Calculate half of 8: $\frac{1}{2}(8) = 4$
b. Square that 4; the magic number is **16**.

Add the magic number to each side of the equation:

$$x^2 + 8x + \boxed{16} = 20 + \boxed{16}$$

Step 4:

Factor the left side, and simplify the right side:

$$(x + 4)^2 = 36$$

Step 5:

Solve by taking square roots:

$$x + 4 = \pm\sqrt{36} \qquad \text{(36 has \underline{two} square roots)}$$

$$x + 4 = \pm 6 \qquad \text{(simplify the radical)}$$

$$x = -4 \pm 6 \qquad \text{(subtract 4 from both sides)}$$

Using the plus sign $\Rightarrow$ $x = -4 + 6 = 2$

Using the minus sign $\Rightarrow$ $x = -4 - 6 = -10$

We have now solved our first equation by completing the square, and its solutions are

$$\boxed{x = 2 \text{ or } -10}$$

Note that we could have solved the quadratic equation using either the quadratic formula, or by factoring. No matter the method, we would get the same solutions.

<u>EXAMPLE 6</u>: **Solve by Completing the Square:**

$$\mathbf{3x^2 - 5x + 1 = 0}$$

<u>Solution</u>: We proceed as we did above with the 5-step plan.

Step 1:

The first requirement for completing the square is to have a leading coefficient of 1. Since the leading coefficient in this problem is 3, we will have to divide both sides of the equation by 3:

$$\frac{3x^2 - 5x + 1}{3} = \frac{0}{3}$$

or, $x^2 - \frac{5}{3}x + \frac{1}{3} = 0$ (divide <u>all</u> terms by 3)

Step 2:

Move the constant to the right side produces the equation

$$x^2 - \frac{5}{3}x = -\frac{1}{3} \qquad \text{(subtract } \tfrac{1}{3} \text{ from both sides)}$$

Step 3:

Now for the magic number, the number that will complete the square. We calculate half of $-\frac{5}{3}$, square that result, and we'll have the "magic number" that will be added to each side of the equation.

Magic Number Calculation:

$\frac{1}{2}\left(-\frac{5}{3}\right) = -\frac{5}{6}$, and then $\left(-\frac{5}{6}\right)^2 = \mathbf{\frac{25}{36}}$

Add this number to each side of the equation:

$$x^2 - \frac{5}{3}x + \boxed{\frac{25}{36}} = -\frac{1}{3} + \boxed{\frac{25}{36}}$$

Step 4:

Factoring the left side and adding the fractions on the right gives:

$$\left(x - \frac{5}{6}\right)^2 = \frac{13}{36} \qquad \left[-\frac{1}{3} + \frac{25}{36} = -\frac{12}{36} + \frac{25}{36} = \frac{13}{36}\right]$$

Step 5:

Now we take the square root of each side of the equation, remembering that the right side has <u>two</u> square roots:

$$x - \frac{5}{6} = \pm\sqrt{\frac{13}{36}}$$

Next we isolate the x by adding $\frac{5}{6}$ to each side of the equation:

$$x = \frac{5}{6} \pm \sqrt{\frac{13}{36}}$$

Split the radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{\sqrt{36}}$$

And simplify the bottom radical:

$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

Combining the fractions into a single fraction (the LCD is 6) produces the final answer:

$$\boxed{x = \frac{5 \pm \sqrt{13}}{6}}$$

Our quadratic has two solutions.

Homework

3. At what point in completing the square could you determine that the quadratic equation you're trying to solve has NO solution?

4. Solve each quadratic equation by completing the square:

a. $y^2 - 6y + 5 = 0$

b. $x^2 + 13x + 30 = 0$

c. $z^2 + 5z - 14 = 0$

d. $2n^2 - n - 1 = 0$

e. $12t^2 - 5t - 3 = 0$

f. $10a^2 + 7a + 1 = 0$

g. $x^2 + 25 = 10x$

h. $4u^2 + 20u + 25 = 0$

i. $w^2 = -w - 5$

j. $2h^2 + 1 = h$

5. Solve each quadratic equation by completing the square:

a. $x^2 + 3x + 1 = 0$

b. $y^2 - 4y + 2 = 0$

c. $2a^2 + 6a + 3 = 0$

d. $n^2 + 8n - 2 = 0$

e. $3u^2 - 4u - 2 = 0$

f. $t^2 + 10t + 3 = 0$

g. $5w^2 + w + 1 = 0$

h. $2x^2 = -5x - 1$

i. $g^2 = 3g - 5$

j. $3m^2 = 1 - 4m$

Review Problems

6. How many solutions does each quadratic equation have?

a. $(x + 3)(x + 4) = 0$ ______

b. $(x + 9)^2 = 0$ ______

c. $(x - 1)^2 = 10$ ______

d. $(x + 6)^2 = -9$ ______

7. Factor each trinomial:

a. $n^2 + 10n + 25$

b. $x^2 - 18x + 81$

c. $a^2 + 5a + \frac{25}{4}$

d. $x^2 - \frac{4}{3}x + \frac{4}{9}$

e. $y^2 + \frac{6}{7}y + \frac{9}{49}$

f. $t^2 - \frac{1}{5}t + \frac{1}{100}$

8. Solve each equation by completing the square:

a. $q^2 + 8q + 15 = 0$ b. $2x^2 - 7x + 1 = 0$

c. $3n^2 + n - 5 = 0$ d. $4a^2 - 5a = -3$

e. $5w^2 = -2 - 7w$ f. $z^2 + 1 = 2z$

Solutions

1. a. $(x+5)^2$ b. $(y-9)^2$ c. $\left(a+\frac{1}{2}\right)^2$ d. $\left(m-\frac{2}{3}\right)^2$

e. $\left(z+\frac{1}{5}\right)^2$ f. $\left(w-\frac{5}{6}\right)^2$ g. $\left(b+\frac{9}{10}\right)^2$ h. $\left(u+\frac{3}{4}\right)^2$

i. $\left(n-\frac{2}{7}\right)^2$ j. $\left(x+\frac{5}{11}\right)^2$

2. a. $\frac{1}{2}\cdot 8 = 4$, and $4^2 = \mathbf{16}$ b. $\frac{1}{2}\cdot -12 = -6$, and $(-6)^2 = \mathbf{36}$

c. $\frac{1}{2}\cdot 28 = 14$, and $14^2 = \mathbf{196}$ d. $\frac{1}{2}\cdot 3 = \frac{3}{2}$, and $\left(\frac{3}{2}\right)^2 = \mathbf{\frac{9}{4}}$

e. $\frac{1}{2}\cdot -7 = -\frac{7}{2}$, and $\left(-\frac{7}{2}\right)^2 = \mathbf{\frac{49}{4}}$ f. $\frac{1}{2}\cdot 11 = \frac{11}{2}$, and $\left(\frac{11}{2}\right)^2 = \mathbf{\frac{121}{4}}$

g $\frac{1}{2}\cdot 1 = \frac{1}{2}$, and $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$ h. $\frac{1}{2}\cdot -1 = -\frac{1}{2}$, and $\left(-\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

i. $\frac{1}{2}\cdot 2 = 1$, and $1^2 = \mathbf{1}$

j. $\frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$, and $\left(\frac{1}{3}\right)^2 = \mathbf{\frac{1}{9}}$

k. $\frac{1}{2}\cdot -\frac{5}{7} = -\frac{5}{14}$, and $\left(-\frac{5}{14}\right)^2 = \mathbf{\frac{25}{196}}$

l. $\frac{1}{2}\cdot -\frac{7}{13} = -\frac{7}{26}$, and $\left(-\frac{7}{26}\right)^2 = \mathbf{\frac{49}{676}}$

3. As soon as you notice that you'll be taking the square roots of a negative number, you can stop and conclude that the quadratic equation has no solution (in Algebra 1).

4. a. $y = 1, 5$ b. $x = -3, -10$ c. $z = 2, -7$
d. $n = 1, -\frac{1}{2}$ e. $t = \frac{3}{4}, -\frac{1}{3}$ f. $a = -\frac{1}{2}, -\frac{1}{5}$
g. $x = 5$ h. $u = -\frac{5}{2}$ i. No solution
j. No solution

5. a. $x = \frac{-3 \pm \sqrt{5}}{2}$ b. $y = 2 \pm \sqrt{2}$ c. $a = \frac{-3 \pm \sqrt{3}}{2}$
d. $n = -4 \pm 3\sqrt{2}$ e. $u = \frac{2 \pm \sqrt{10}}{3}$ f. $t = -5 \pm \sqrt{22}$
g. No solution h. $x = \frac{-5 \pm \sqrt{17}}{4}$ i. No solution
j. $m = \frac{-2 \pm \sqrt{7}}{3}$

6. a. 2 b. 1 c. 2 d. 0

7. a. $(n + 5)^2$ b. $(x - 9)^2$ c. $\left(a + \frac{5}{2}\right)^2$
d. $\left(x - \frac{2}{3}\right)^2$ e. $\left(y + \frac{3}{7}\right)^2$ f. $\left(t - \frac{1}{10}\right)^2$

8. a. $q = -5, -3$ b. $x = \frac{7 \pm \sqrt{41}}{4}$ c. $n = \frac{-1 \pm \sqrt{61}}{6}$
d. No solution e. $w = -\frac{2}{5}, -1$ f. $z = 1$