CH 72 – DERIVING THE QUADRATIC FORMULA

INTRODUCTION

We recently finished solving quadratic equations by completing the square. Unlike the factoring method, completing the square always works, whether the solutions are integers, fractions, or radicals; it even works to tell us that an equation has no solutions at all in Algebra 1. But it's quite boring and not the easiest thing in the world to do.

There must be some way to complete the square on a generic quadratic equation, $ax^2 + bx + c = 0$, and end up with a formula that can then be used to more easily solve any quadratic equation.

THE QUADRATIC FORMULA

Start with the original quadratic equation, and try to complete the square . . . here we go:

$$ax^2 + bx + c = 0$$

The technique of completing the square requires that we always have a leading coefficient of 1, and so the first step is to divide each side of the equation by a (which we know is not zero, for otherwise we wouldn't have a quadratic equation):

Divide each side of the equation by a:

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

To simplify the left side of the equation, just be sure you divide <u>each</u> term by a:

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a} = \frac{0}{a}$$

And then simplify each side:

$$x^2+\frac{b}{a}x+\frac{c}{a} = 0$$

Now bring the constant to the right side of the equation; that is, subtract $\frac{c}{a}$ from each side of the equation (we could have done this step first):

$$x^2+\frac{b}{a}x = -\frac{c}{a}$$

It's now time to compute the "magic number," the quantity which will "complete the square." We start with the coefficient of the linear term, in this case $\frac{b}{a}$, take half of it, and then square that result:

$$\frac{b}{a}\times\frac{1}{2} = \frac{b}{2a}, \text{ and then } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \frac{b^2}{4a^2}$$ ⟵ The magic number

We now add the magic number to each side of the equation:

$$x^2+\frac{b}{a}x+\boxed{\frac{b^2}{4a^2}} = -\frac{c}{a}+\boxed{\frac{b^2}{4a^2}}$$

We now need to do two things: factor on the left and combine the fractions on the right. Let's factor first:

$$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}+\frac{b^2}{4a^2}$$

Notice that the variable x occurs just once, buried amidst the parentheses on the left side of the equation.

Now we'll combine the fractions on the right side of the equation into a single fraction. Since the LCD of the denominators is $4a^2$, we need to multiply the top and the bottom of the first fraction by $4a$:

$$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}\left[\frac{\mathbf{4a}}{\mathbf{4a}}\right]+\frac{b^2}{4a^2}$$

or,
$$\left(x+\frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2}+\frac{b^2}{4a^2}$$

Adding the fractions yields

$$\left(x+\frac{b}{2a}\right)^2 = \frac{-4ac+b^2}{4a^2}$$

Using the commutative property for addition, we can reverse the two terms in the numerator of the right side of the equation:

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

The next step is to take the square root of each side of the equation, remembering that every non-negative number has <u>two</u> square roots, denoted $\pm\sqrt{}$:

$$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

To isolate the *x* (which is the whole point of this effort), we'll bring the term $\frac{b}{2a}$ to the right side of the equation, and put it in front:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$$

Now it's appropriate to split the radical into two separate radicals:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

Simplify the radical in the denominator:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

Finally, we combine the two fractions into a single fraction (can you believe the two fractions already have the same denominator?):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We've isolated the x in the quadratic equation $ax^2 + bx + c = 0$ by completing the square!

We summarize this procedure in the following statement:

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by **The Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

❑ *NOTES ON THE QUADRATIC FORMULA*

- The Quadratic Formula can be used to solve any quadratic equation, whether or not it's factorable, and whether or not its solutions are integers (whole numbers and their opposites), fractions, or even radicals (like $\sqrt{2}$). And if the quadratic equation has no solution in the real numbers, the Quadratic Formula will tell us that, too.

- Because of the ± sign, there are potentially two solutions. In fact, the two solutions of the quadratic equation $ax^2 + bx + c = 0$ can be written separately:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- The Quadratic Formula is a single fraction:

  It is **NOT** $x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$

  If you don't write it as a single fraction at every stage of your solution, either you or your instructor will misread it and you'll be the one to lose points on the problem.

Homework

1. Which symbol in the Quadratic Formula is responsible for possibly giving us two solutions to a quadratic equation?

2. How many solutions of the quadratic equation will there be if the quantity inside the square root sign (called the *radicand*) is 100?

3. How many solutions of the quadratic equation will there be if the radicand is 30?

4. How many solutions of the quadratic equation will there be if the radicand is 0?

5. How many solutions of the quadratic equation will there be if the radicand is –9?

6. Consider the possibility that the value of a is 0 in the quadratic equation $ax^2 + bx + c = 0$. Does the Quadratic Formula apply? First, consider what happens to the quadratic equation if $a = 0$. Second, analyze what happens if we actually let $a = 0$ in the Quadratic Formula.

7. Consider the quantity "$b^2 - 4ac$," the radicand inside the square root sign of the Quadratic Formula.

   a. If $b^2 - 4ac > 0$, then the equation has ____ solution(s).
   b. If $b^2 - 4ac = 0$, then the equation has ____ solution(s).
   c. If $b^2 - 4ac < 0$, then the equation has ____ solution(s).

   The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic expression $ax^2 + bx + c$.

Solutions

1. The plus/minus sign: ±

2. Since $\sqrt{100} = 10$, and since there's a plus/minus sign in front this, there will be two solutions.

3. Even though $\sqrt{30}$ doesn't result in a whole number, there's still a plus/minus sign in front of it, so there will also be two solutions.

4. Since $\sqrt{0} = 0$, and since ±0 is just the single number 0, the plus/minus sign basically disappears, leaving a single solution.

5. Since $\sqrt{-9}$ does not exist in this class, the quadratic formula has no meaning, and so there are no solutions.

6. First, if $a = 0$, the quadratic equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$, which is NOT quadratic anymore. Indeed, we learned months ago how to solve this equation:

$$bx + c = 0 \Rightarrow bx = -c \Rightarrow x = -\frac{c}{b}$$

Second, if we ignore the above fact and actually use $a = 0$ in the Quadratic Formula, the denominator becomes $2a = 2(0) = 0$. That is, we're dividing by zero, which is undefined. However you look at it, the Quadratic Formula with $a = 0$ simply makes no sense.

7. a. 2 b. 1 c. 0

Postscript: The derivation of the Quadratic Formula in this chapter was the classic method, using the method of Completing the Square. This was a fine method to use, since Completing the Square will be used in your next Algebra class for studying circles, parabola, and other shapes.

But there's another way to derive the Quadratic Formula, perhaps a way that may be easier for you to follow. Let's do it -- beginning, of course, with our quadratic equation in standard form:

$$ax^2 + bx + c \;=\; 0$$

Next, multiply each side of the equation by $4a$:

$$4a(ax^2 + bx + c) \;=\; 4a(0)$$

And distribute:

$$4a^2x^2 + 4abx + 4ac \;=\; 0$$

Bring the $4ac$ to the right side:

$$4a^2x^2 + 4abx \;=\; -4ac$$

Next, add b^2 to each side of the equation:

$$4a^2x^2 + 4abx + b^2 \;=\; b^2 - 4ac$$

Now factor the left side of the equation:

$$(2ax + b)^2 \;=\; b^2 - 4ac$$

Then take the square roots of each side of the equation:

$$2ax + b \;=\; \pm\sqrt{b^2 - 4ac}$$

Subtract b from each side of the equation:

$$2ax \;=\; -b \pm \sqrt{b^2 - 4ac}$$

And last, divide each side of the equation by 2a:

$$x \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

WE DID IT!!


miStAkEs
are proof
that you are
TRYING