CH 13 – DOMAIN

Have you ever gotten an ERROR message on your calculator? You may have tried to divide by zero, or perhaps you attempted to compute the square root of a negative number. In either case, you tried to use a number that was outside the domain of the function you were trying to calculate.

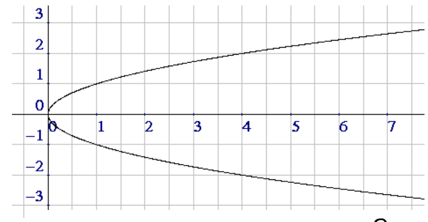
❑ Introduction

The ***domain*** of a function is the set of all inputs to the function. There are two ways we determine the domain of a function.

Sometimes the domain is explicitly given. For instance, let h be the function defined on the set of numbers [0, 3] by the formula $h(x) = x^2$. The domain of this function is the set [0, 3]. Why? Because the definition says so. Thus, while $h(0) = 0$ and $h(2.5) = 6.25$ and $h(3) = 9$, $h(-1)$ is undefined and $h(4)$ is undefined. It's not that -1 and 4 can't be squared -- they simply are not in the explicitly given domain [0, 3].

Other times the domain is not explicitly given, so we agree to abide by the ***rule of maximum domain***. In this scenario, the domain of the function is every real number that's legal to use in the formula. For example, the function $g(x) = x^2$ would have a domain of $\mathbb{R}$, but in the formula $y = \dfrac{3}{x-7}$, x can be any real number except 7, and therefore the domain is $\mathbb{R} - \{7\}$.

The concept of domain also applies to formulas that are not functions. Consider the formula $x = y^2$. We know that this is not a function because an input of 25, for instance, produces two outputs, ±5. Nevertheless, we can still ask what inputs are allowed. Since x is the square of y, it should be clear that x must be greater than or equal to zero. That is, the domain is $[0, \infty)$.



The non-function $x = y^2$ has domain $[0, \infty)$

Homework

1. Consider the function $f(x) = \dfrac{6x-6}{\sqrt{2x-8}}$. Calculate each functional value: a. $f(12)$ b. $f(36)$ c. $f(4.5)$ d. $f(4)$ e. $f(3)$
From these results we see that 12, 36, and 4.5 are in the ________ of the function, while 4 and 3 are not.

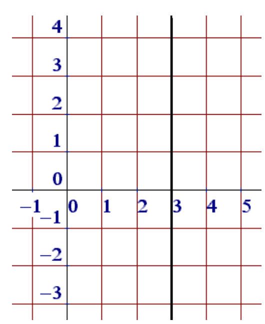
2. What is the domain of the function defined by $y = x^3$, $x \in [3, 5]$?

3. Consider the function given by the formula $g(x) = x^3 - x^2$. What is the domain of g?

4. What is the domain of the function $y = \dfrac{2}{x+3}$?

❑ Finding the Domain

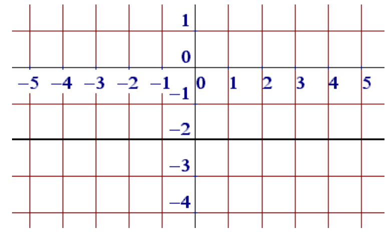
I. $x = 3$



What can x be in this equation? x must be 3, since that's precisely what the equation says. So the domain is

$$\{3\}$$

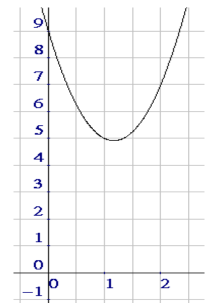
II. $y = -2$



In this equation, the x isn't even mentioned. There can, therefore, be no restrictions on x. That is, x can be any real number. Thus, the domain is

$$\boxed{\mathbb{R}}$$

III. $y = 3x^2 - 7x + 9$

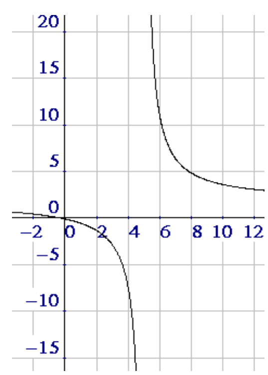


We ask ourselves: What are the legal x's? Well, x can be anything, since the operations in the formula could not possibly be a cause for concern. The domain is therefore

$$\boxed{\mathbb{R}}$$

IV. $y = \dfrac{7x+2}{4x-20}$

Now we've got something interesting to look at. Question: What can go wrong in a division problem? Answer: The possibility of dividing by zero. We must make sure that x is <u>never</u> allowed to be a number which would make the denominator zero. So we find out what value(s) of x <u>would</u> make the bottom zero, and then <u>don't allow those x's to be in the domain</u>. Setting the bottom to zero gives



$$4x - 20 = 0$$
$$\Rightarrow \quad 4x = 20$$
$$\Rightarrow \quad x = 5$$

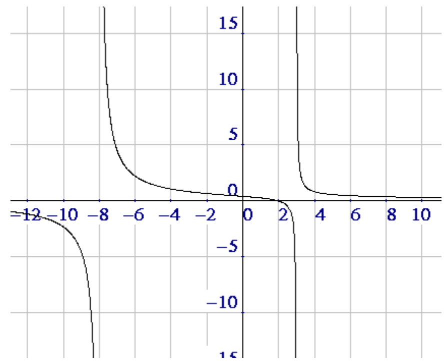
Thus, if $x = 5$, the denominator is zero, which is absolutely forbidden! So, the domain of this function is the set of <u>all</u> real numbers <u>except</u> 5:

$$\boxed{\mathbb{R} - \{5\}}$$

V. $g(x) = \dfrac{5x-10}{x^2+5x-24}$

As in the previous example, we must make certain that the bottom of the fraction is <u>never</u> zero; so we'll see what value(s) of x make it zero, and then <u>exclude</u> such values from our domain:

$$x^2 + 5x - 24 = 0$$
$$\Rightarrow \quad (x+8)(x-3) = 0$$
$$\Rightarrow \quad x + 8 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow \quad x = -8 \text{ or } x = 3$$



We conclude that the domain is all real numbers <u>except</u> −8 and 3:

$$\boxed{\mathbb{R} - \{-8, 3\}}$$

VI. $f(x) = \dfrac{x+2}{2x^2-5x+1}$

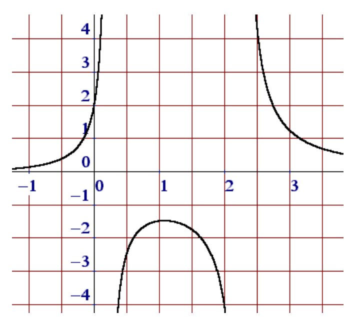
Again, we set the bottom to zero to see what is <u>not</u> allowed in the domain.

$$2x^2 - 5x + 1 = 0 \Rightarrow x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)}$$

$$\Rightarrow x = \frac{5 \pm \sqrt{25-8}}{4} \Rightarrow x = \frac{5 \pm \sqrt{17}}{4} \approx 2.3 \,\&\, 0.2$$

The domain of f is therefore the set of real numbers <u>except</u> for the two solutions of the quadratic:

$$\boxed{\mathbb{R} - \left\{\frac{5 \pm \sqrt{17}}{4}\right\}}$$

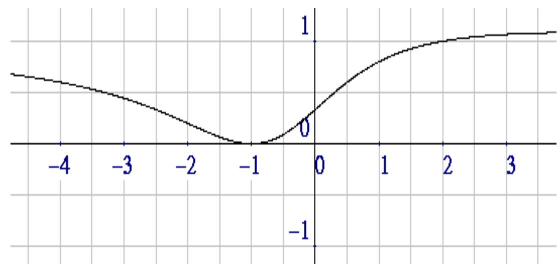


VII. $h(x) = \dfrac{x^2 + 2x + 1}{x^2 + x + 3}$

Set the denominator to zero and solve for x:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2(1)} = \frac{-1 \pm \sqrt{-11}}{2}$$

But these values of x are not real numbers. So there's no number in this class that would make the bottom zero -- no value of x could mess up this problem. So every x is a good x, and thus the domain is



$$\boxed{\mathbb{R}}$$

Homework

Find the domain:

5. $f(x) = \pi$

6. $x = -\sqrt{2}$

7. $y = |x^2 - x - 10|$

8. $y = \dfrac{x^2 - 9}{9x - 7}$

9. $g(x) = \dfrac{2x + 1}{x^2 - 100}$

10. $h(x) = \dfrac{3}{x^2 + 6x + 1}$

11. $y = \dfrac{x^2 - 25}{x^2 + 49}$

12. $y = \dfrac{x - 1}{2x^2 - x + 1}$

13. $x + \pi = 7$

14. $y = \sqrt{2} + \sqrt{3}$

15. $k(x) = \dfrac{4x - 10}{x^2 - 99}$

16. $y = \sqrt{3}$

17. $x = 0$

18. $y = |x^3 - 8|$

19. $y = \dfrac{9x-7}{2x+9}$

20. $f(x) = \dfrac{5x+25}{x^2-144}$

21. $g(x) = \dfrac{2x}{2x^2+3x-10}$

22. $y = \dfrac{x}{x^2+3}$

23. $y = \dfrac{x+1}{x^2+x+1}$

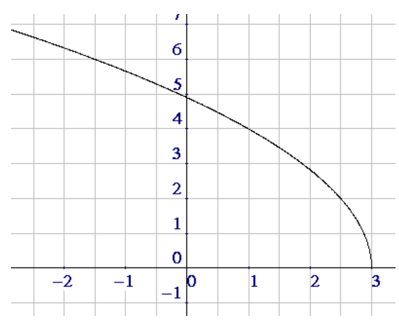
24. $L(x) = \dfrac{x^2-1}{x^2-60}$

❑ Square Roots of Linear Functions

EXAMPLE 1: **Find the domain of $f(x) = \sqrt{24-8x}$.**

Solution: In this formula, we ask ourselves if anything could possibly go wrong using certain values of x. Well, the square root of a negative number doesn't exist in the real numbers, so certainly something could go wrong -- for instance, if $x = 4$, then we have $\sqrt{-8}$.

Now we need a statement which describes the legal x's. How about this: The square root of a quantity is defined (as a real number) only when the quantity is greater than or equal to zero. So, in this example, the radicand $24 - 8x$ must be ≥ 0.



$24 - 8x \geq 0$ (the radicand must be 0 or positive)

$\Rightarrow -8x \geq -24$ (subtract 24 from each side)

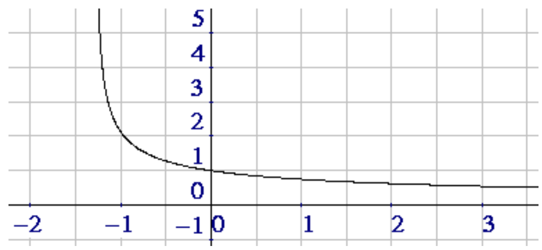
$\Rightarrow x \leq 3$ (reverse the inequality sign when dividing by a negative number)

And therefore, the domain is the set of all real numbers which are less than or equal to 3: $\{x \in \mathbb{R} \mid x \leq 3\}$, which we write as

$$(-\infty, 3]$$

EXAMPLE 2: **Find the domain of** $g(x) = \dfrac{3}{\sqrt{7x+9}}$.

Solution: This one is kind of challenging. In order for the square root to be defined, the radicand $7x + 9$ must be ≥ 0. But if $7x + 9$ were actually equal to 0, we would have 0 in



the bottom of the fraction, and we certainly can't allow that. Thus, in order that nothing goes wrong, $7x + 9$ must be strictly greater than 0:

$$7x + 9 > 0 \qquad \text{(not } \geq 0\text{, but } > 0\text{)}$$

$$\Rightarrow \quad 7x > -9 \qquad \text{(subtract 9 from each side)}$$

$$\Rightarrow \quad x > -\frac{9}{7} \qquad \text{(don't flip the inequality sign)}$$

and so the domain is $\left\{x \in \mathbb{R} \mid x > -\frac{9}{7}\right\}$, or

$$\boxed{\left(-\frac{9}{7}, \infty\right)}$$

Homework

Find the domain of each function:

25. $f(x) = \sqrt{3x+12}$

26. $g(x) = \sqrt{8-2x}$

27. $y = \dfrac{2x}{\sqrt{x-7}}$

28. $h(x) = \dfrac{x-3}{\sqrt{10-5x}}$

29. $y = \sqrt{7x-15}$

30. $y = \sqrt{-4x+1}$

31. $f(x) = \dfrac{x-3}{\sqrt{2x-3}}$

32. $g(x) = \dfrac{1-x}{4-x}$

❑ Square Roots of Quadratic Functions

EXAMPLE 3: **Find the domain of** $y = \sqrt{x^2 - 2x - 35}$**.**

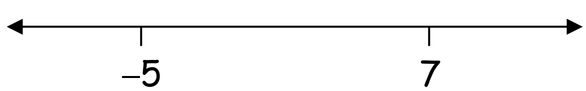
Solution: As we've done before, we note that the radicand in a square root must be greater than or equal to zero so that the square root of the radicand will be a real number. This leads to the inequality

$$x^2 - 2x - 35 \geq 0$$

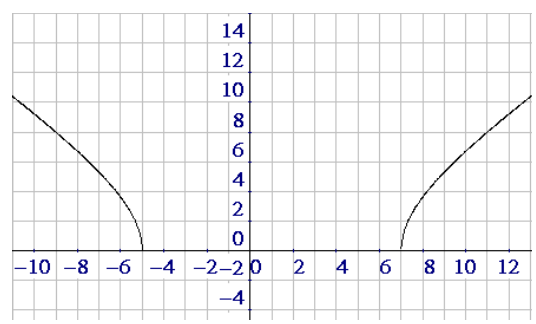
which we solve using boundary points.

Convert the inequality into an equation:	$x^2 - 2x - 35 = 0$
Factor the quadratic:	$(x + 5)(x - 7) = 0$
Set each factor to zero:	$x + 5 = 0$ or $x - 7 = 0$
Solve each linear equation:	$x = -5$ or $x = 7$

Now we know there are two boundary points: –5 and 7. So we mark them on a number line:



Pick a test point in each interval, check them in the original inequality, and you'll see that the intervals $(-\infty, -5)$ and $(7, \infty)$ work, but the middle interval does not.



Last, check the boundary points -- they work in the original inequality. Our final answer is the combination of the left interval and the right interval:
$\{x \in \mathbb{R} \mid x \leq -5 \text{ or } x \geq 7\}$, which is the interval

$$\boxed{(-\infty, -5] \cup [7, \infty)}$$

EXAMPLE 4: **Find the domain of $g(x) = \sqrt{11x - x^2 - 30}$.**

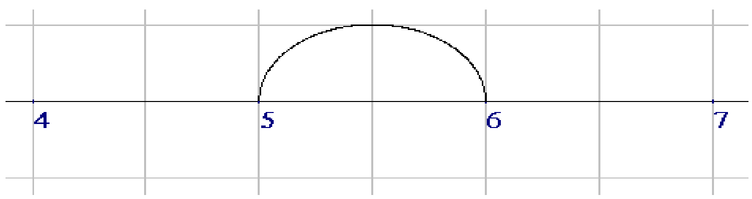
Solution: The radicand must be greater than or equal to zero:

$$11x - x^2 - 30 \geq 0.$$

Convert to an equation:

$$11x - x^2 - 30 = 0 \Rightarrow -x^2 + 11x - 30 = 0 \Rightarrow x^2 - 11x + 30 = 0.$$

The two solutions of this equation (and thus our boundary points) are 5 and 6, so we get three intervals to check: $(-\infty, 5)$, $(5, 6)$, and $(6, \infty)$. Choose a test point in each interval and substitute that test point into the original inequality. You'll find that the only interval that works is the center one: $(5, 6)$.



Finally, check the boundary points themselves. They both work in the original inequality. The final answer is $\{ x \in \mathbb{R} \mid 5 \leq x \leq 6 \}$, which we write as

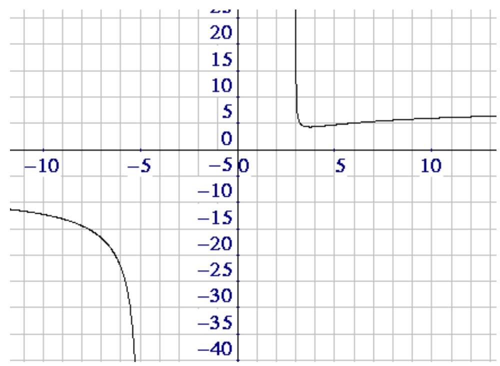
$$\boxed{[5, 6]}$$

EXAMPLE 5: **Find the domain of $h(x) = \dfrac{8x - 19}{\sqrt{x^2 + 2x - 15}}$.**

Solution: What can go wrong in this function? At the very least, the radicand, $x^2 + 2x - 15$, must be ≥ 0, so that the square root is well-defined. But since the radical's on the bottom, it must also never equal 0. Therefore, the radicand must be strictly greater than 0:

$$x^2 + 2x - 15 > 0$$

To solve this inequality, change it to an equation, and factor to find the boundary points –5 and 3. Pick some test points and see that the intervals (–∞, –5) and (3, ∞) work, but the interval (–5, 3) does not. Also, the boundary points themselves will not work. The domain is therefore



$$\boxed{(-\infty, -5) \cup (3, \infty)}$$

Homework

Find the domain of each function:

33. $y = \sqrt{x^2 + 3x - 10}$

34. $y = \sqrt{-x^2 - x + 20}$

35. $y = \sqrt{1 - x^2}$

36. $y = \sqrt{x^2 - 11}$

37. $y = \dfrac{3}{\sqrt{x^2 - 100}}$

38. $y = \dfrac{x + 3}{\sqrt{x^2 - 2x - 80}}$

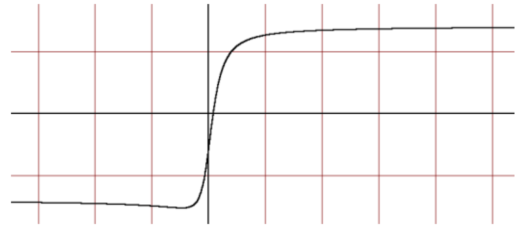
39. $y = \dfrac{x^2 + 9}{\sqrt{-x^2 + 7x - 10}}$

40. $y = \dfrac{88}{\sqrt{36 - x^2}}$

41. Let f be defined by the formula $f(x) = \sqrt{-x^2 - 4}$. Prove that the domain of f is empty. This means that there's no graph whatsoever for this function. Philosophical question: Is f a function?

42. Let g be defined by $g(x) = \dfrac{7x-3}{\sqrt{x^2+1}}$.

Prove that the domain of g is $\mathbb{R}$.



43. [Challenging] Find the domain: $y = \dfrac{\sqrt{2x-10}}{x^2-144}$

Practice Problems

Find the domain:

44. $y = \sqrt{2}$

45. $y = \dfrac{2x-6}{x^2+4x+3}$

46. $y = \dfrac{2}{x^2+x-1}$

47. $y = \dfrac{2x+1}{\sqrt{12-3x}}$

48. $y = \dfrac{2x^2-1}{\sqrt{x^2+x-20}}$

49. $y = \sqrt{-2x^2+11x-5}$

50. $y = \sqrt{x^2+10}$

51. $x = -99$

52. $y = \pi x^3 - 17x + 1$

53. $y = \dfrac{x+2}{x-3}$

54. $f(x) = \dfrac{3}{x^2-x-20}$

55. $g(x) = \sqrt{18-9x}$

56. $h(x) = \dfrac{x^2-9}{\sqrt{7x-2}}$

57. $y = \sqrt{x^2+2x+1}$

58. $f(x) = \dfrac{4x-1}{\sqrt{x^2-25}}$

59. $h(x) = \dfrac{1}{\sqrt{2+x^2}}$

60. $y = \dfrac{5}{\sqrt{49 - x^2}}$

61. $j(x) = \dfrac{6x}{\sqrt{9x^2 + 6x + 1}}$

62. True/False:

a. f is defined on the set $[0, 1]$ by $f(x) = 3x + 4$. Then the domain of f is $\mathbb{R}$.

b. The domain of $y = (x - 7)^2$ is $\mathbb{R} - \{7\}$.

c. The domain of $x = y^2$ is $[0, \infty)$.

d. The domain of $y = 9$ is $\mathbb{R}$.

e. The domain of $x = \pi$ is $\mathbb{R}$.

f. The domain of $y = \dfrac{3x - 9}{5x - 20}$ is $\mathbb{R} - \{4\}$.

g. There are exactly two real numbers that are not in the domain of $y = \dfrac{5x - 2}{x^2 + 10x + 1}$.

h. The domain of $g(x) = \dfrac{1}{x^2 + 9}$ is $\varnothing$, the empty set.

i. The domain of $f(x) = \sqrt{7x - 63}$ is $[9, \infty)$.

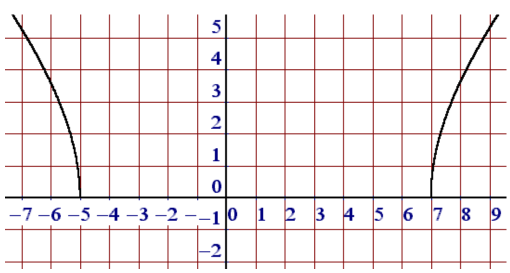
j. The domain of $G(x) = \dfrac{4}{\sqrt{2x - 6}}$ is $[3, \infty)$.

k. The domain of $y = \sqrt{x^2 - 144}$ is $(-\infty, -12] \cup [12, \infty)$.

l. The domain of $y = \dfrac{4x + 9}{\sqrt{x^2 + 25}}$ is $\mathbb{R}$.

m. The domain of $f(x) = \dfrac{x^2 + 17x - \pi}{1000}$ is $\mathbb{R}$.

n. The domain of the function below is $[-5, 7]$.



Solutions

1. a. 33/2 b. 105/4 c. 21 d. Undefined
e. Undefined <u>Domain</u>

2. $[3, 5]$ **3.** $\mathbb{R}$ **4.** $\mathbb{R} - \{-3\}$ **5.** $\mathbb{R}$ **6.** $\left\{-\sqrt{2}\right\}$

7. $\mathbb{R}$ **8.** $\mathbb{R} - \left\{\frac{7}{9}\right\}$ **9.** $\mathbb{R} - \{\pm 10\}$ **10.** $\mathbb{R} - \left\{-3 \pm 2\sqrt{2}\right\}$

11. $\mathbb{R}$ **12.** $\mathbb{R}$ **13.** $\{7 - \pi\}$ **14.** $\mathbb{R}$

15. $\mathbb{R} - \left\{\pm 3\sqrt{11}\right\}$ **16.** $\mathbb{R}$ **17.** $\{0\}$ **18.** $\mathbb{R}$ **19.** $\mathbb{R} - \left\{-\frac{9}{2}\right\}$

20. $\mathbb{R} - \{\pm 12\}$ **21.** $\mathbb{R} - \left\{\frac{-3 \pm \sqrt{89}}{4}\right\}$ **22.** $\mathbb{R}$ **23.** $\mathbb{R}$

24. $\mathbb{R} - \left\{\pm 2\sqrt{15}\right\}$ **25.** $3x + 12 \ge 0 \Rightarrow [-4, \infty)$

26. $8 - 2x \ge 0 \Rightarrow (-\infty, 4]$ **27.** $x - 7 > 0 \Rightarrow (7, \infty)$

28. $10 - 5x > 0 \Rightarrow (-\infty, 2)$ **29.** $\left[\frac{15}{7}, \infty\right)$ **30.** $\left(-\infty, \frac{1}{4}\right]$

31. $\left(\frac{3}{2}, \infty\right)$ **32.** $\mathbb{R} - \{4\}$ **33.** $x^2 + 3x - 10 \ge 0 \Rightarrow (-\infty, -5] \cup [2, \infty)$

34. $-x^2 - x + 20 \ge 0 \Rightarrow [-5, 4]$ **35.** $[-1, 1]$

36. $\left(-\infty, -\sqrt{11}\right] \cup \left[\sqrt{11}, \infty\right)$ **37.** $x^2 - 100 > 0 \Rightarrow (-\infty, -10) \cup (10, \infty)$

38. $x^2 - 2x - 80 > 0 \Rightarrow (-\infty, -8) \cup (10, \infty)$

39. $-x^2 + 7x - 10 > 0 \Rightarrow (2, 5)$

40. $36 - x^2 > 0 \Rightarrow (-6, 6)$

41. Show that the solution set of $-x^2 - 4 \ge 0$ is $\varnothing$ (no solution).

42. Show that the solution set of $x^2 + 1 > 0$ is $\mathbb{R}$ (all numbers work).

43. $[5, \infty) - \{12\}$

44. $\mathbb{R}$

45. $\mathbb{R} - \{-1, -3\}$

46. $\mathbb{R} - \left\{\frac{-1 \pm \sqrt{5}}{2}\right\}$

47. $(-\infty, 4)$

48. $(-\infty, -5) \cup (4, \infty)$

49. $\left[\frac{1}{2}, 5\right]$

50. $\mathbb{R}$

51. $\{-99\}$

52. $\mathbb{R}$

53. $\mathbb{R} - \{3\}$

54. $\mathbb{R} - \{5, -4\}$

55. $(-\infty, 2]$

56. $\left(\frac{2}{7}, \infty\right)$

57. $\mathbb{R}$

58. $(-\infty, -5) \cup (5, \infty)$

59. $\mathbb{R}$

60. $(-7, 7)$

61. $\mathbb{R} - \{-1/3\}$

62. a. F b. F c. T d. T e. F f. T g. T h. F
i. T j. F k. T l. T m. T n. F

Try not to become a person of success,
but rather try to become a person of value.

Albert Einstein