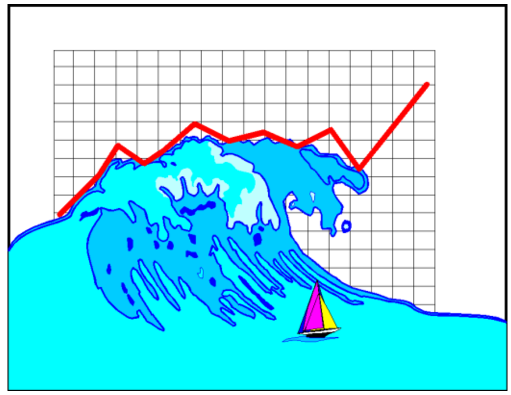
CH 15 – GRAPHING

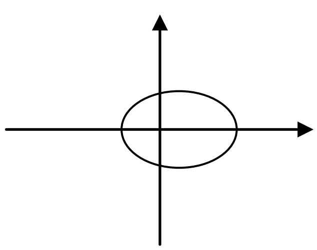


Now we tie together some of the ideas from previous chapters – symmetry, domain, intercepts – into one concept: the transformation of an equation with one or two variables into a picture in 2-dimensional space.

❑ Symmetry

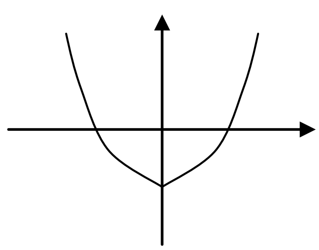
In the previous chapter, The Real Plane, we discussed what it meant for a pair of points to have some kind of symmetry. Now we extend this concept for any graph in the real plane.

***x*-axis symmetry**



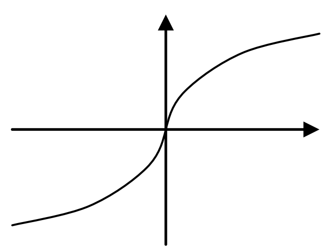
Given any point on the graph, there's an equidistant point on the other side of the *x*-axis.

***y*-axis symmetry**



Given any point on the graph, there's an equidistant point on the other side of the *y*-axis.

origin symmetry



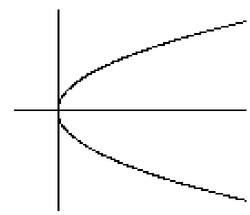
Given any point on the graph, there's an equidistant point on the other side of the origin.

The Litmus Tests for Symmetry

Now that we can visualize the three types of symmetry, we need a way to test a given equation for symmetry without having to graph it.

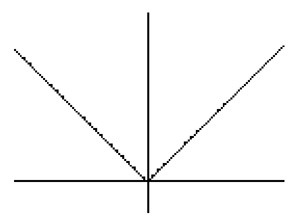
x-axis Replace y with $-y$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has x-axis symmetry.

For example, consider the sideways-opening parabola $x = y^2$. Replacing y with $-y$ gives the equation $x = (-y)^2$, which is the same as $x = y^2$. Since the replacement didn't really change the equation, we have x-axis symmetry.



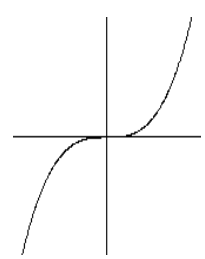
y-axis Replace x with $-x$ everywhere in the given equation. If the new equation is equivalent to the old one, then the graph of the equation has y-axis symmetry.

For instance, look at the absolute value function $y = |x|$. Replacing x with $-x$ gives the equation $y = |-x|$, which is actually the same as $y = |x|$ (since the absolute values of a number and its opposite are the same). Thus, the graph has y-axis symmetry.



origin Replace x with $-x$ and replace y with $-y$ everywhere in the equation. If the new equation is equivalent to the old one, then the graph of the equation has origin symmetry.

Consider the cubic function $y = x^3$. Replacing x with $-x$ and y with $-y$ gives the equation $-y = (-x)^3$, which implies that $-y = -x^3$, which implies that $y = x^3$, so there was no net change in the equation. Therefore, the graph has origin symmetry.



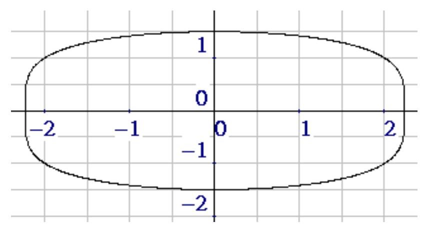
<u>EXAMPLE 1:</u> **Find all the symmetries of $x^2 + y^4 = 5$.**

<u>Solution:</u>

<u>x-axis</u> Replace y with $-y$: $x^2 + (-y)^4 = 5 \Rightarrow$ $x^2 + y^4 = 5$, the same as the original equation; therefore the graph has x-axis symmetry.

<u>y-axis</u> Replacing x with $-x$ will again result in the same equation; thus the graph has y-axis symmetry.

<u>origin</u> Replace <u>both</u> variables with their negatives: $(-x)^2 + (-y)^4 = 5 \Rightarrow x^2 + y^4 = 5$, the original equation; we conclude that the graph has origin symmetry as well.



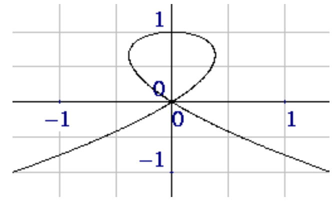
<u>EXAMPLE 2:</u> **Find all the symmetries of $y^2 = x^2 + y^3$.**

<u>Solution:</u>

<u>x-axis</u> Replace y with $-y$: $(-y)^2 = x^2 + (-y)^3$ $\Rightarrow y^2 = x^2 - y^3$, which is <u>not</u> the same as the original equation, nor can it be made to be the same. Thus, the graph does <u>not</u> have x-axis symmetry.

<u>y-axis</u> Replace x with $-x$: $y^2 = (-x)^2 + y^3$ $\Rightarrow y^2 = x^2 + y^3$, which is the same as the original equation. Therefore, the graph has y-axis symmetry.

<u>origin</u> Replace x with $-x$ <u>and</u> y with $-y$: $(-y)^2 = (-x)^2 + (-y)^3 \Rightarrow$ $y^2 = x^2 - y^3$, which is <u>not</u> the same as the original equation -- and will never be -- no matter what we do to each side of the



equation. We conclude that the graph does not have origin symmetry.

In summary, the graph has y-axis symmetry only.

Homework

1. Sketch a line with x-axis symmetry, a line with y-axis symmetry, and a line with origin symmetry.

2. a. Sketch a parabola with y-axis symmetry.
   b. Sketch a parabola with x-axis symmetry.

3. Sketch a circle which has x-axis symmetry, but neither of the other two.

4. Sketch a circle which has y-axis symmetry, but neither of the other two.

5. Sketch a circle which has origin symmetry, but neither of the other two.

6. A graph with origin symmetry has some points in the 2nd quadrant. Which other quadrant must contain some points of the graph?

7. Use the litmus tests to find the symmetries of each graph:

   a. $x^2 - y^2 = 1$ b. $y = x^2 + x - 1$

   c. $y = x^2 - 8$ d. $x = |y|$

   e. $y = \dfrac{1}{x}$ f. $y = x^6 - x^4 + x^2$

8. If a graph has x-axis symmetry and y-axis symmetry, does that imply that it has origin symmetry? What about the converse?

❑ Absolute Values and Square Roots

EXAMPLE 3: **Graph:** $y = |x - 3| + 2$

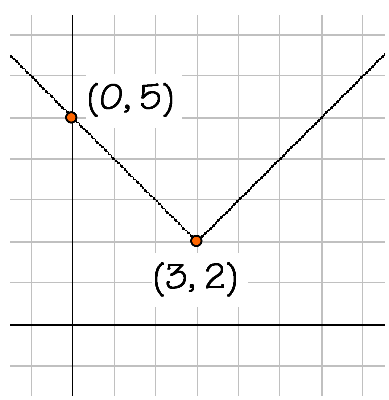
Solution: First, we'll analyze the **domain**, so that we'll know what values of x we'll be allowed to use. Could anything go wrong with the absolute value formula? No matter what real number x is, you can always subtract 3 from it, take the absolute value, and then add 2. In other words, the domain is all real numbers, $\mathbb{R}$.

Second, it appears that we have a **function**, since a given x will produce just one y-value.

Third, we seek the intercepts. Setting x to 0 gives $y = |0 - 3| + 2 = 5$. The y-intercept is therefore (0, 5). Now we set $y = 0$, which gives us $0 = |x - 3| + 2 \Rightarrow |x - 3| = -2$, which is impossible. Thus, there are no x-intercepts.

Now we'll plot lots of points, and see what we end up with.

x	y
–1	6
0	5
1	4
2	3
3	2
4	3
5	4
6	5
7	6



$y = |x - 3| + 2$

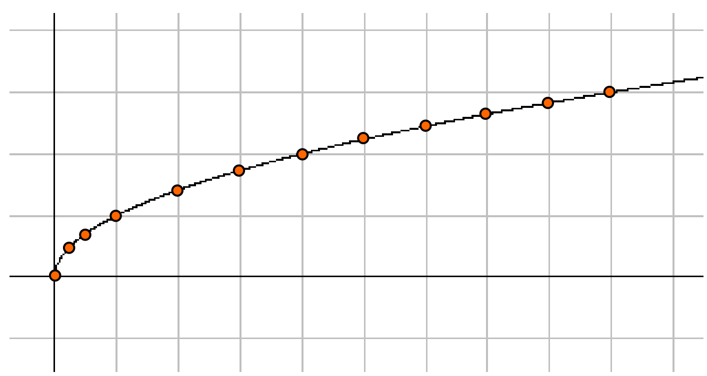
Notice that the graph is in the shape of a "V." It is sharp at its bottom point (3, 2), not smooth and curvy like a parabola. It appears from the graph that it's a **function**. Note also that the graph appears to have no ***x*-intercept**, as we showed even before we plotted points.

EXAMPLE 4: **Graph:** $f(x) = \sqrt{x}$

Solution: First we note that f is really a **function**. Since the radical sign, $\sqrt{}$, represents the non-negative square root only, there can be only one y-value for a given x in the domain.

We now determine the **domain** of this function. Then we'll know which real numbers we can use for x and which we can't. The square root of a quantity is a real number only when the quantity is greater than or equal to zero. That is, the domain is $\{x \in \mathbb{R} \mid x \geq 0\}$, or $[0, \infty)$. So we can use 0 or any positive number for x, but we can't use any negative values for x.

x	y
0	0
0.25	0.5
0.5	0.707
1	1
2	1.41
3	1.73
4	2
5	2.24
6	2.45
7	2.65
8	2.83
9	3



$y = \sqrt{x}$

It may be hard to tell, but the graph we've just drawn is actually the top half of a sideways parabola. The origin is the only **intercept** of this graph; it's both an x-intercept and a y-intercept.

Homework

9. Graph $f(x) = |x|$, $g(x) = |x - 3|$, and $h(x) = |x + 2|$ on the same grid. Explain how the –3 and the 2 affect the graph of f.

10. Graph $f(x) = |x|$, $g(x) = |x| + 4$, and $h(x) = |x| - 2$ on the same grid. Explain how the 4 and the –2 affect the graph of f.

11. Graph $f(x) = |x|$, $g(x) = 3|x|$, and $h(x) = \frac{1}{2}|x|$ on the same grid. Explain how the 3 and the $\frac{1}{2}$ affect the graph of f.

12. Graph $f(x) = -|x|$ and compare it to the graph of $|x|$.

13. Graph $f(x) = |x + 2| - 3$.

14. Graph $g(x) = -|1 - x| + 2$.

15. Graph $h(x) = \sqrt{x+4}$.

16. Graph $y = \sqrt{x+2} - 3$.

17. Graph $y = \sqrt{-x}$.

18. Graph $y = -\sqrt{3-x}$.

19. We've seen the graph that is the top half of a sideways parabola. Can you find an equation whose graph will be the bottom half of the sideways parabola? What about an equation that will represent the entire sideways parabola?

❑ Cube Roots and Other Graphs

The graph of the square root function, $y = \sqrt{x}$, is limited by the fact that the domain of this function is [0, ∞), since $\sqrt{x}$ results in a real number only if $x \ge 0$. The cube root doesn't have this restriction, so it makes for an interesting graph.

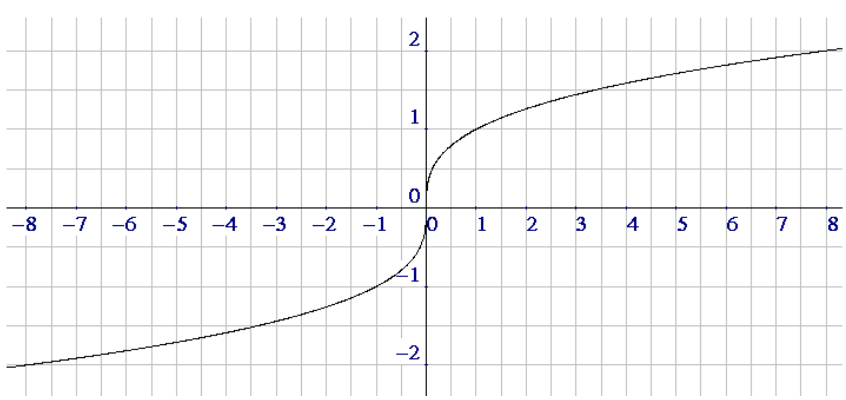
EXAMPLE 5: **Analyze and graph $y = \sqrt[3]{x}$.**

Solution: First we look at the **domain**. The cube root of a positive number is positive (e.g., $\sqrt[3]{8} = 2$), the cube root of 0 is 0, and the cube root of a negative number is negative (e.g., $\sqrt[3]{-1} = -1$). There doesn't seem to be any number whose cube root we couldn't legally take. It appears that the domain of this function is $\mathbb{R}$.

As for **intercepts**, letting $x = 0$ produces 0 for y, and vice versa. Thus, the only intercept is (0, 0), the origin.

Here are a few **ordered pairs** for this function. Plotting them will give the graph that follows.

(27, 3) (8, 2) (4, 1.59) (1, 1) (0, 0) (–1, –1)
(–6, –1.82) (–8, –2) (–27, – 3) (1/8, 1/2) (–1/8, –1/2)



Notice that at the origin the graph seems to be practically vertical. We could say that the tangent line at the origin has infinite slope.

This graph has origin **symmetry**. First of all, it looks like it does. Second, changing x to $-x$ and y to $-y$ produces the equation

$$-y = \sqrt[3]{-x} \quad \Rightarrow \quad -y = -\sqrt[3]{x} \quad \Rightarrow \quad y = \sqrt[3]{x}\,.$$

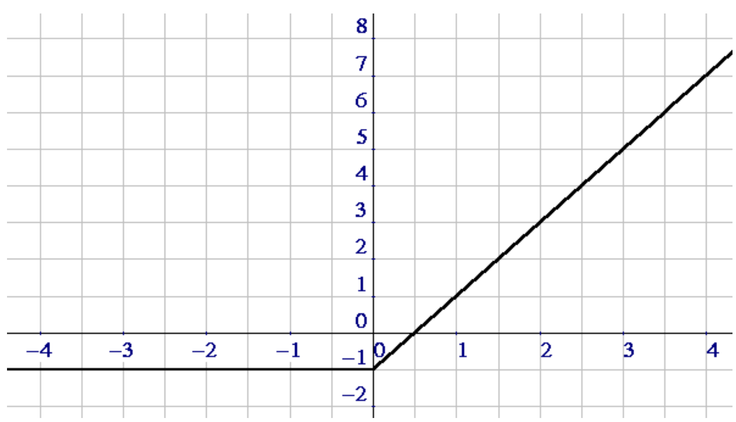
Third, notice that for each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. Anyway you look at it, the graph possesses origin symmetry.

EXAMPLE 6: **Graph $y = x + |x| - 1$.**

Solution: Noting that the domain of the function is $\mathbb{R}$, let's plot a lot of points and see what we get.

x	−4	−3	−2	−1	0	1/2	1	2	3	4
y	−1	−1	−1	−1	−1	0	1	3	5	7

Before we graph the function, notice that if $x \le 0$, y has the constant value −1. Thus, part of the graph will be horizontal. We can also see that there's an x-intercept at (1/2, 0) and a y-intercept at (0, −1). Let's put it all together:



Notice the sharp turn at the y-intercept (0, −1).

Homework

Graph each function:

20. $y = \sqrt[3]{x+2}$

21. $y = \sqrt[3]{x-3}$

22. $y = \sqrt[3]{x}+3$

23. $y = \sqrt[3]{x}-4$

24. $y = \sqrt[3]{x+1}+2$

25. $y = x + |x|$

26. $y = x - |x|$

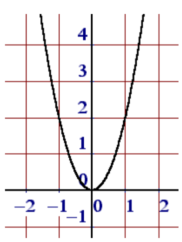
27. $y = 2x - |x|$

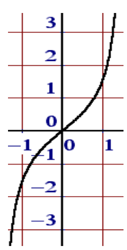
28. $y = x^2 - |4x|$

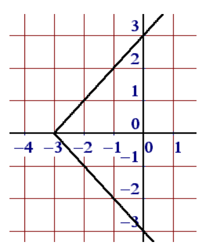
29.** Graph the function $y = \dfrac{x^2-4}{x-2}$.

Practice Problems

30. Find all the symmetries:

a. 

b. 

c. 

d. $x = |y| + 5$

e. $y = x^8 + x^4 - x^2$

f. $y = \dfrac{3}{x}$

31. Find all the symmetries of $x^2 + y^2 = 99$.

32. Find all the symmetries of $x - y^2 = 1$.

33. Graph $y = \sqrt{x-4}+3$ and state its domain.

34. Graph $y = |x+2|-1$ and state its domain.

35. Graph $y = \sqrt[3]{x+3}$ and state its domain.

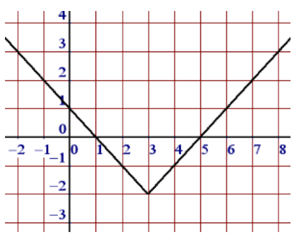
36. Graph $y = -\sqrt[3]{2-x}$ and state its domain.

37. Graph $y = -\sqrt{1-x}+2$ and state its domain.

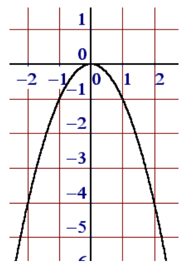
38. Graph $y = \dfrac{|x|}{x}$ and state its domain.

39. Find the equation of each function:

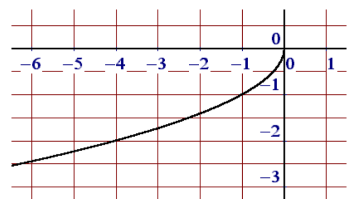
a.



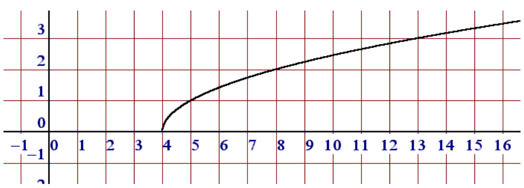
b.



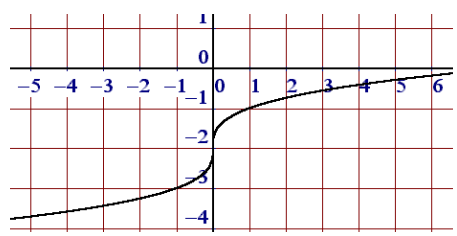
c.



d.



e.



40. True/False:

a. The graph of $y = x^4 - 7x^2 + 1$ has y-axis symmetry.

b. The graph of $x^2 + y^2 = 2$ has all three symmetries.

c. The graph of $y^2 - y^3 = x^2$ has origin symmetry.

d. A circle may have x-axis symmetry only.

e. The graph of $y = \dfrac{1}{x^5}$ has origin symmetry.

f. The "turning point" of the graph of $y = |x-3| + 2$ is (2, 3).

g. The graph of $g(x) = \sqrt{x}$ is the top-half of a sideways parabola.

h. The graph of $y = \sqrt{-x}$ is just the origin.

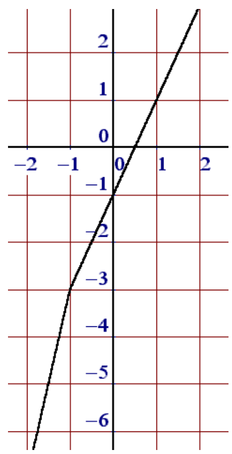
i. The graph of $y = \frac{1}{3}|3x - 10|$ is "skinnier" than the graph of $y = |3x - 10|$.

j. The graphs of $f(x) = |x - 2|$ and $g(x) = |2 - x|$ are identical.

k. The graph of $y = \sqrt[3]{x}$ has an infinite slope at the origin.

l. The graph of $y = \sqrt[3]{x + 2}$ passes through all the quadrants except the second.

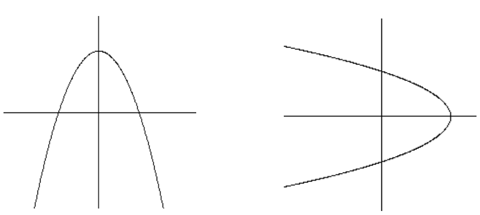
m. The graph of the function $y = 3x - |x + 1|$ is



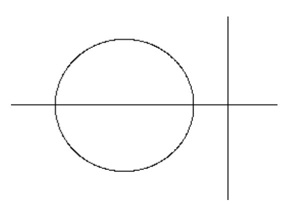
Solutions

1. any vertical line; any horizontal line; any line through the origin

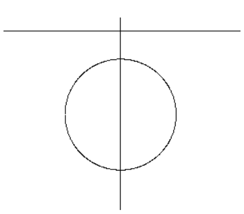
2.



3.



4.



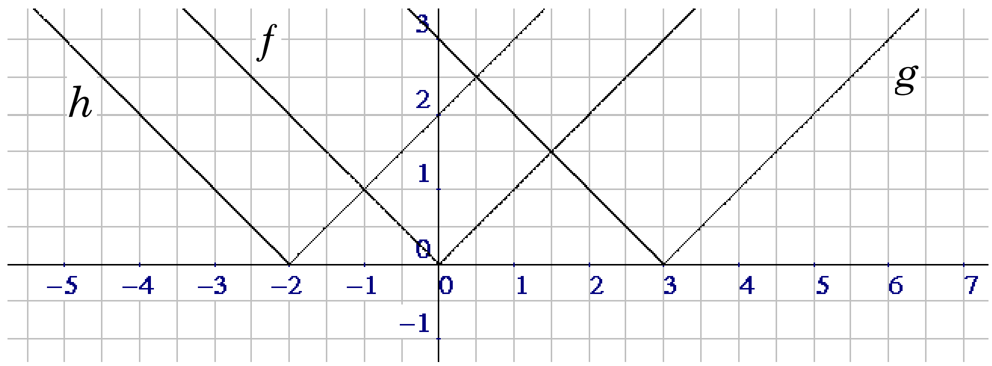
5. Impossible

6. Quadrant IV

7. a. all b. none c. y-axis d. x-axis e. origin f. y-axis

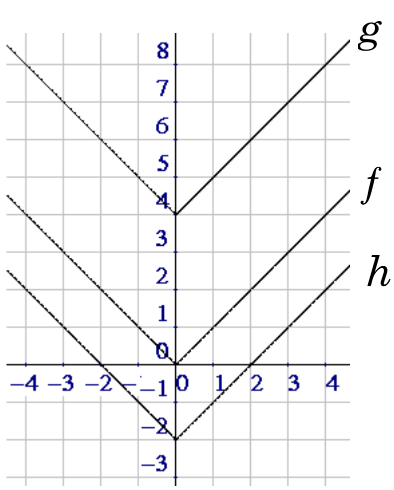
8. Yes; but the converse is false -- consider $y = 2x$. It has origin symmetry but neither of the other two.

9.



g is f shifted 3 units to the right. h is f shifted 2 units to the left. Generalization: The graph of $|x + a|$ is the graph of $|x|$ shifted a units to the right if $a < 0$ and to the left if $a > 0$.

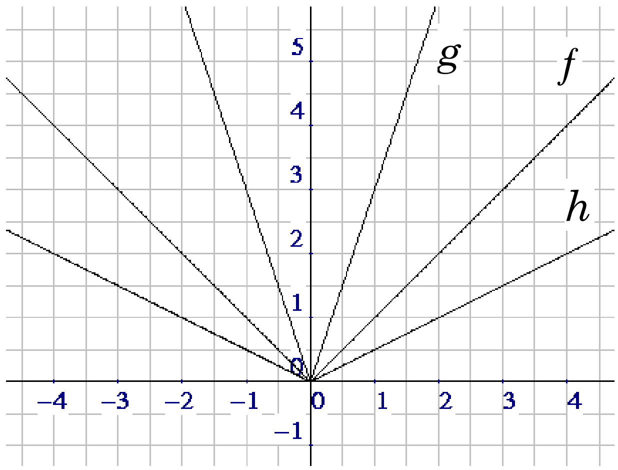
10.



g is shifted up 4 units; h is shifted down 2 units.

Generalization: The graph of $|x| + a$ is the graph of $|x|$ shifted a units up if $a > 0$ and a units down if $a < 0$.

11.

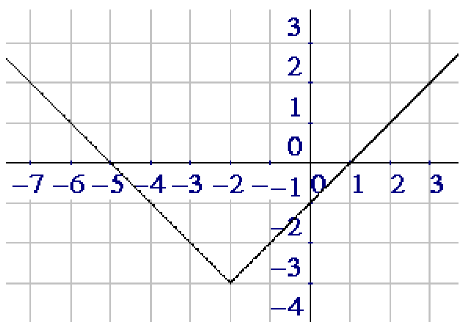


The 3 makes the graph skinnier;

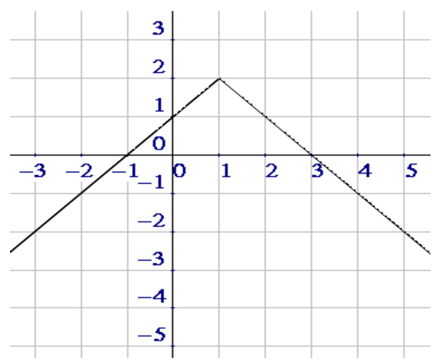
the $\frac{1}{2}$ makes the graph wider.

12. The graph of f is the graph of $|x|$ but flipped over the x-axis.

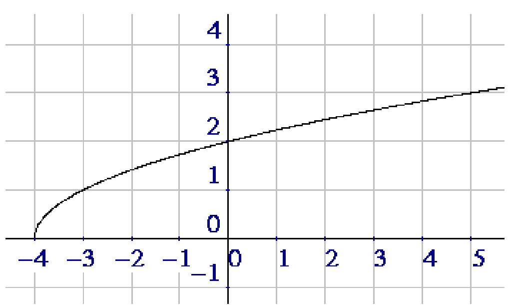
13.



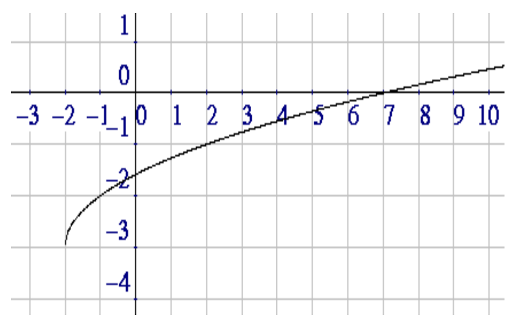
14.



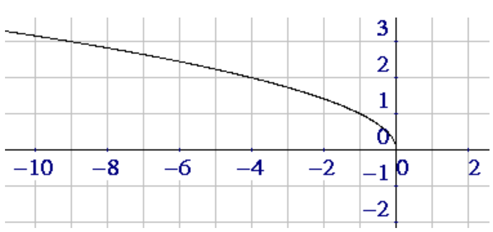
15.



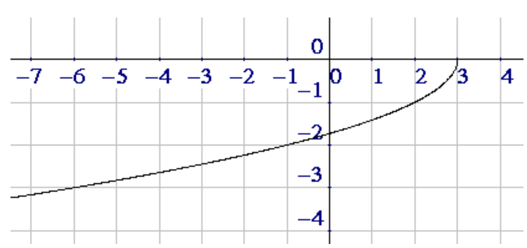
16.



17.

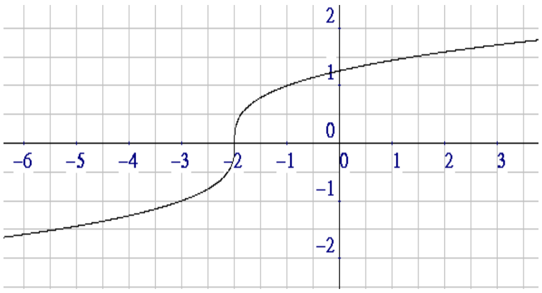


18.

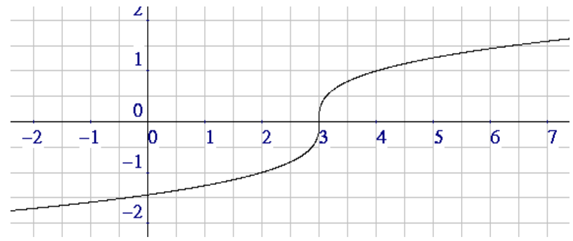


19. What equations did you come up with?

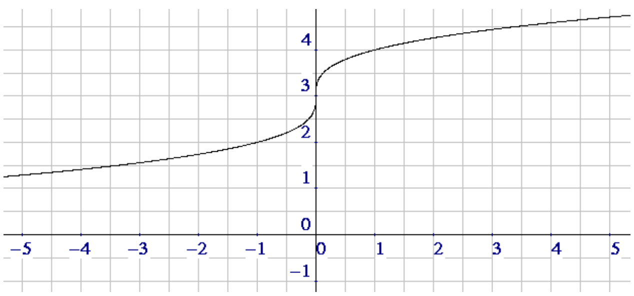
20.



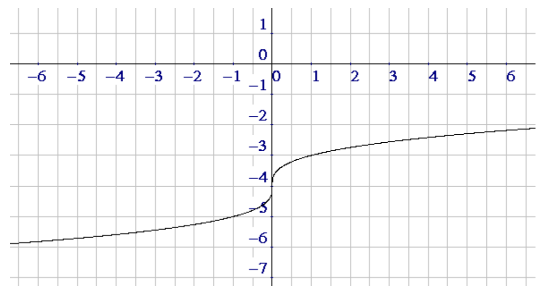
21.



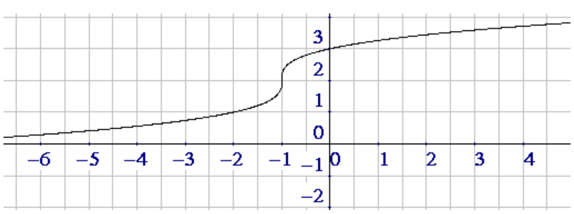
22.



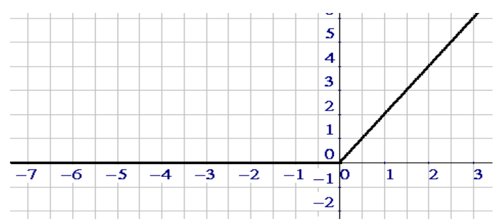
23.



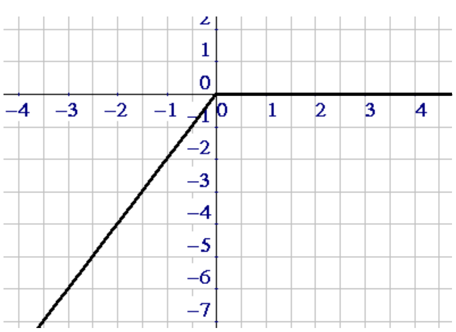
24.



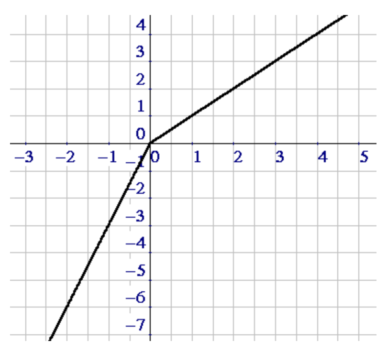
25.



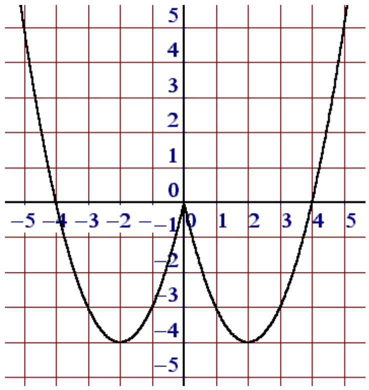
26.



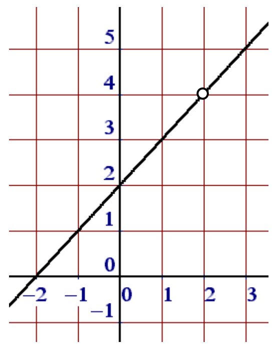
27.



28.



29. The graph is the line $y = x + 2$ with a hole at (2, 4).

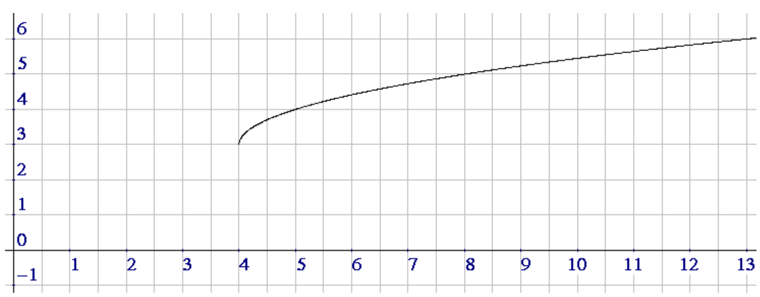


30. a. y-axis b. origin c. x-axis d. x-axis e. y-axis f. origin

31. x-axis, y-axis, and origin

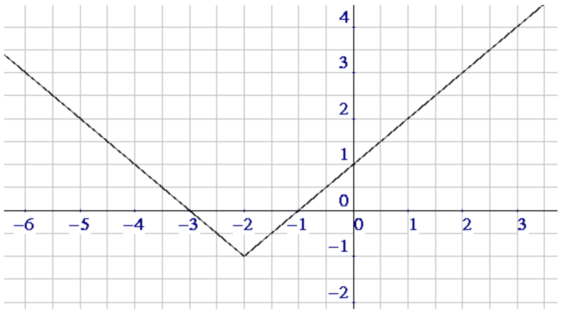
32. x-axis

33.



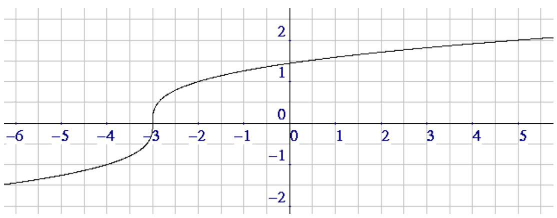
Domain = $[4, \infty)$

34.



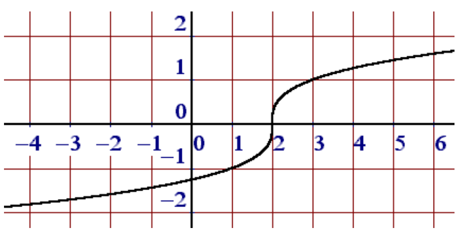
Domain = $\mathbb{R}$

35.



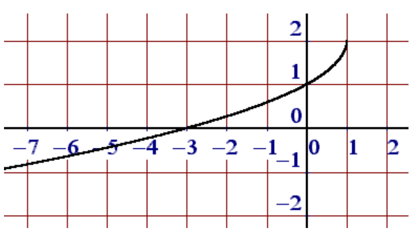
Domain = $\mathbb{R}$

36.



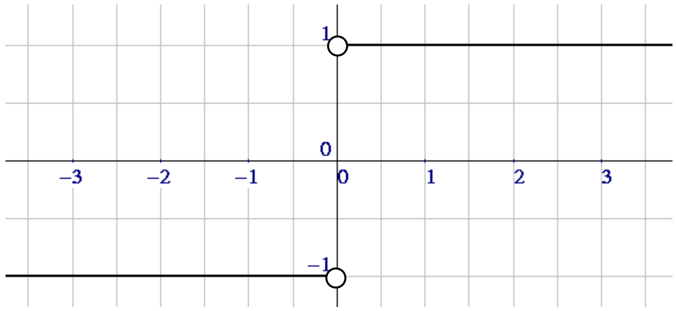
Domain = $\mathbb{R}$

37.



Domain = $(-\infty, 1]$

38.



Domain = $\mathbb{R} - \{0\}$

39. a. $y = |x-3| - 2$ b. $y = -x^2$ c. $y = -\sqrt{-x}$

d. $y = \sqrt{x-4}$ e. $y = \sqrt[3]{x} - 2$

40. a. T b. T c. F d. T e. T f. F g. T h. F i. T
j. T k. T l. F m. T

I have no particular talent.

I am merely inquisitive.

Albert Einstein