CH 19 – FACTORING, PART II

❑ INTRODUCTION

We can now factor lots of quadratic binomials and trinomials. Sorry to have to tell you this, but we're not done with factoring just yet. In this chapter, we learn how to factor expressions with the exponent 4 in them, expressions containing four terms, and expressions containing GCFs you might never have seen before.

❑ FACTORING QUARTICS

EXAMPLE 1: **Factor each quartic (4th degree) polynomial:**

A. $\boldsymbol{c^4 - 256}$

$= (c^2 + 16)(c^2 - 16)$ (difference of squares)

$= \boxed{(c^2 + 16)(c + 4)(c - 4)}$ (difference of squares again)

Note: $c^2 + 16$ can<u>not</u> be factored any further.

B. $\boldsymbol{9a^4 - 37a^2 + 4}$

$= (9a^2 - 1)(a^2 - 4)$ (factor trinomial)

Now we notice that each factor is quadratic and is the difference of two squares. Therefore, each factor can be factored further to get a final answer of

$\boxed{(3a + 1)(3a - 1)(a + 2)(a - 2)}$

Homework

1. Factor each quartic polynomial:

a. $x^4 - 1$

b. $x^4 - x^2 - 6$

c. $n^4 - 10n^2 + 9$

d. $a^4 - 81$

e. $36w^4 - 25w^2 + 4$

f. $9x^4 - 34x^2 + 25$

g. $c^4 - 16$

h. $x^4 - 8x^2 - 9$

i. $x^4 - 3x^2 - 10$

j. $g^4 - 256$

k. $36u^4 - 85u^2 + 9$

l. $y^4 + 81$

❑ *THE GCF REVISITED*

<u>EXAMPLE 2:</u> **Factor: $(a + b)^2 + 4(a + b)$**

<u>Solution:</u> There are two terms in this expression: $(a + b)^2$ and $4(a + b)$. Notice that each of these two terms contains the same factor, namely $a + b$. In other words, the GCF of the two terms is $a + b$. Factoring out this GCF gives us the final factored form, a single term:

$$\boxed{(a + b)(a + b + 4)}$$

The thing <u>not</u> to do in this kind of problem is to distribute the original expression; if you do, you'll be going in the wrong direction. Check it out:

$(a + b)^2 + 4(a + b) = a^2 + 2ab + b^2 + 4a + 4b$
Do you really want to try to factor that last expression?

So, when you see an expression, like $a + b$ in this problem, occurring multiple times in an expression, it's usually best to leave it intact. Also notice that we have converted a 2-termed expression into 1 term -- we have factored.

Alternate Method: Let's try a substitution method. We might be able to better see the essence of the problem if we replace $a + b$ with a simpler symbol -- for example, x will represent $a + b$. Then the original expression

$$(a + b)^2 + 4(a + b)$$

is transformed into

$$x^2 + 4x$$

The GCF in this form is clearly x, so we pull it out in front:

$$x(x + 4)$$

Now substitute in the reverse direction, to get $a + b$ back in the problem:

$$(a + b)(a + b + 4) \qquad \text{(the same answer as before)}$$

<u>EXAMPLE 3:</u> **Factor:** $\boldsymbol{x^2(u - w) - 100(u - w)}$

<u>Solution:</u> The two given terms have a GCF of $u - w$. Factoring this GCF out gives

$$(u - w)(x^2 - 100)$$

But we're not done yet. The second factor is a difference of squares. Factoring that part gives us our final factorization:

$$\boxed{(u - w)(x + 10)(x - 10)}$$

EXAMPLE 4: **Factor: $w^2(x+z) - 4w(x+z) + 3(x+z)$**

Solution: Let's use substitution to make this expression appear a little less intimidating; we'll convert every occurrence of $x + z$ to the symbol A:

$$w^2A - 4wA + 3A$$

Pulling out the GCF of A, we get

$$A(w^2 - 4w + 3)$$

Factor the trinomial in the usual way:

$$A(w - 3)(w - 1)$$

Last, replace the A with its original definition of $x + z$:

$$\boxed{(x+z)(w-3)(w-1)}$$

Homework

2. Factor each expression:

a. $(x+y)^2 + 7(x+y)$

b. $(a-b)^2 - c(a-b)$

c. $x^2(c+d) + 5(c+d)$

d. $n^2(a-b) - 9(a-b)$

e. $x^2(a+4) + 5x(a+4) + 6(a+4)$

f. $y^2(m+n) + 7y(m+n)$

g. $2x^2(a+b) + 3x(a+b) - 5(a+b)$

h. $4x^2(w+z) - 9(w+z)$

i. $(u-w)^2 - 9(u-w)$

j. $n^2(a+b) - 9n(a+b)$

k. $(t+r)y^2 - 100(t+r)$

l. $3ax^2 - 20ax - 7a$

❑ GROUPING WITH FOUR TERMS

EXAMPLE 5: **Factor:** $\boldsymbol{a^2 + ac + ab + bc}$

Solution: Group the first two terms and the last two terms:

$$(a^2 + ac) + (ab + bc)$$

Now factor each pair of grouped terms separately (using the GCF) :

$$a(a + c) + b(a + c)$$

Even though we've grouped and factored, we can't be done because there are still two terms, and we need one term in the final answer to a factoring question. So we continue -- using our knowledge of the previous section -- and factor out the GCF, which is $a + c$:

$$\boxed{(a + c)(a + b)}$$

By the commutative property of multiplication, the final answer could also be written $(a + b)(a + c)$. Also, to check our answer, just double distribute the answer and you should get the original expression.

EXAMPLE 6: **Factor:** $\boldsymbol{x^3 - 7x^2 - 9x + 63}$

Solution: Group the first two terms and the last two terms:

$$(x^3 - 7x^2) + (-9x + 63)$$

Now factor the GCF in each pair of grouped terms. The first GCF is obvious: x^2. Choosing the GCF in the second grouping is a little trickier -- should we choose 9 or -9? Ultimately, it's a trial-and-error process. Watch what happens if we choose -9 for the GCF:

$x^2(x - 7) - 9(x - 7)$ (check the signs carefully)

We now see two terms whose GCF is $x - 7$:

$(x - 7)(x^2 - 9)$

All this, and we're still not done. The second factor is the difference of two squares -- now we're done:

$$\boxed{(x - 7)(x + 3)(x - 3)}$$

EXAMPLE 7: **Factor: $ab + cd + ad + bc$**

Solution: Group the first two terms and the last two terms (after all, this technique worked quite well in the previous two examples):

$(ab + cd) + (ad + bc)$

We're stuck; there's no way to factor either pair of terms (the GCF = 1 in each case), so let's swap the two middle terms of the original problem and again group in pairs:

$(ab + ad) + (cd + bc)$

Pull out the GCF from each set of parentheses:

$a(b + d) + c(d + b)$

Do we have a common factor in these two terms? Well, does $b + d = d + b$? Since addition is commutative, of course they are equal. So the GCF is $b + d$, and when we pull it out in front, we're done:

$$\boxed{(b + d)(a + c)}$$

EXAMPLE 8: **Factor: $2ax - bx - 2ay + by$**

Solution: Group in pairs, as usual:

$$(2ax - bx) + (-2ay + by)$$

Pull out the GCF in each grouping:

$$x(2a - b) + y(-2a + b)$$

Problem: There's no common factor; however, the factors $2a - b$ and $-2a + b$ are opposites of each other, and that gives us a clue. Let's go back to our first step and factor out $-y$ rather than y:

$$x(2a - b) - y(2a - b)$$ (distribute to make sure we're right)

Now we see a good GCF, so we pull it out in front, and we're done:

$$\boxed{(2a - b)(x - y)}$$

Homework

3. Factor each expression:

a. $xw + xz + wy + yz$
b. $a^2 + ac + ab + bc$
c. $x^3 - 4x^2 + 3x - 12$
d. $n^3 - n^2 - 5n + 5$
e. $x^3 + x^2 - 9x - 9$
f. $ac - bd + bc - ad$
g. $xw + yz - xz - wy$
h. $2ac - 2ad + bc - bd$
i. $6xw - yz + 3xz - 2wy$
j. $hj - j^2 - hk + jk$
k. $ax + ay - bx - by$
l. $x^3 - 4x^2 + 3x - 12$
m. $xw + 2wy - xz - 2yz$
n. $a^3 - a^2 - 5a + 5$
o. $4tw - 2tx + 2w^2 - wx$
p. $6x^3 + 2x^2 - 9x - 3$
q. Not factorable
r. $6a^3 - 15a^2 + 10a - 25$

❑ MORE GROUPING AND SUBSTITUTION PROBLEMS

EXAMPLE 9: **Factor: $(w + z)^2 - a^2$**

Solution: After some practice, you might not need a substitution for this kind of problem, but we'll use one for this problem. Let $n = w + z$. The starting problem then becomes

$$n^2 - a^2$$

This is just a standard difference of squares:

$$(n + a)(n - a)$$

Now substitute in the other direction:

$$\boxed{(w + z + a)(w + z - a)}$$

EXAMPLE 10: **Factor: $x^2 + 6x + 9 - y^2$**

Solution: Grouping in pairs has worked quite well so far, so let's try it again:

$$(x^2 + 6x) + (9 - y^2)$$

We see that the first pair of terms has a nice GCF of x, and the second is the difference of squares:

$$x(x + 6) + (3 + y)(3 - y)$$

Good try, but there's no common factor in these two terms. In fact, no grouping into pairs will result in a common factor -- a dead end. Let's go back to the original problem and regroup so that the first three terms are together:

$$(x^2 + 6x + 9) - y^2$$

The first set of three terms is a perfect square trinomial, and factors into the square of a binomial:

$$(x + 3)^2 - y^2$$

leaving us with another difference of squares (just like the previous example), which factors to

$$\boxed{(x + 3 + y)(x + 3 - y)}$$

Homework

4. Factor each expression:

a. $(x + y)^2 - z^2$

b. $(a - b)^2 - c^2$

c. $x^2 + 4x + 4 - y^2$

d. $n^2 - 6n + 9 - Q^2$

e. $(u + w)^2 - T^2$

f. $y^2 + 10y + 25 - x^2$

g. $a^2 + 2ab + b^2 - c^2$

h. $w^2 - 2wy + y^2 - 49$

i. $4x^2 + 4x + 1 - t^2$

j. $9x^2 - 12x + 4 - y^2$

Practice Problems

5. Factor each expression:

a. $10ax^4 - 160a$

b. $Z^2(P - Q) - 144(P - Q)$

c. $50x^3 - 75x^2 - 2x + 3$

d. $12ac - 10bd + 8bc - 15ad$

e. $a^2 - 2ab + b^2 - c^2$

f. $x^2 + 2xy + y^2 - 144$

g. $x^4 - 34x^2 + 225$

h. $x^4 - 8x^2 - 9$

i. $x^3 - 7x^2 + 9x - 63$ j. $n^3 + 3n^2 - 16n - 48$

k. $(a + b)^2 - 5(a + b) + 6$ l. $(x - y)^2 + 7(x - y) + 6$

m. $(a - b)^2 + 6(a - b) - 16$ n. $hm - hn + km - kn$

Solutions

1. a. $(x^2 + 1)(x + 1)(x - 1)$ b. $(x^2 + 2)(x^2 - 3)$
c. $(n + 1)(n - 1)(n + 3)(n - 3)$ d. $(a^2 + 9)(a + 3)(a - 3)$
e. $(2w + 1)(2w - 1)(3w + 2)(3w - 2)$ f. $(x + 1)(x - 1)(3x + 5)(3x - 5)$
g. $(c^2 + 4)(c + 2)(c - 2)$ h. $(x^2 + 1)(x + 3)(x - 3)$
i. $(x^2 + 2)(x^2 - 5)$ j. $(g^2 + 16)(g +4)(g - 4)$
k. $(2u + 3)(2u - 3)(3u + 1)(3u - 1)$ l. Not factorable

2. a. $(x + y)(x + y + 7)$ b. $(a - b)(a - b - c)$
c. $(c + d)(x^2 + 5)$ d. $(a - b)(n + 3)(n - 3)$
e. $(a + 4)(x + 3)(x + 2)$ f. $y(m + n)(y + 7)$
g. $(a + b)(2x + 5)(x - 1)$ h. $(w + z)(2x + 3)(2x - 3)$
i. $(u - w)(u - w - 9)$ j. $n(a + b)(n - 9)$
k. $(t + r)(y + 10)(y - 10)$ l. $a(3x + 1)(x - 7)$

3. a. $(x + y)(w + z)$ b. $(a + b)(a + c)$ c. $(x^2 + 3)(x - 4)$
d. $(n^2 - 5)(n - 1)$ e. $(x + 3)(x - 3)(x + 1)$ f. $(a + b)(c - d)$
g. $(x - y)(w - z)$ h. $(2a + b)(c - d)$ i. $(3x - y)(2w + z)$
j. $(h - j)(j - k)$ k. $(a - b)(x + y)$ l. $(x^2 + 3)(x - 4)$
m. $(x + 2y)(w - z)$ n. $(a^2 - 5)(a - 1)$ o. $(2t + w)(2w - x)$
p. $(2x^2 - 3)(3x + 1)$ q. Not factorable r. $(3a^2 + 5)(2a - 5)$

4. a. $(x + y + z)(x + y - z)$ b. $(a - b + c)(a - b - c)$
c. $(x + 2 + y)(x + 2 - y)$ d. $(n - 3 + Q)(n - 3 - Q)$

e. $(u + w + T)(u + w - T)$

f. $(y + 5 + x)(y + 5 - x)$

g. $(a + b + c)(a + b - c)$

h. $(w - y + 7)(w - y - 7)$

i. $(2x + 1 + t)\ (2x + 1 - t)$

j. $(3x - 2 + y)(3x - 2 - y)$

5. a. $10a(x^2 + 4)(x + 2)(x - 2)$

b. $(P - Q)(Z + 12)(Z - 12)$

c. $(2x - 3)(5x + 1)(5x - 1)$

d. $(3a + 2b)(4c - 5d)$

e. $(a - b + c)(a - b - c)$

f. $(x + y + 12)(x + y - 12)$

g. $(x + 5)(x - 5)(x + 3)(x - 3)$

h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 9)(x - 7)$

j. $(n + 4)(n - 4)(n + 3)$

k. $(a + b - 3)(a + b - 2)$

l. $(x - y + 6)(x - y + 1)$

m. $(a - b + 8)(a - b - 2)$

n. $(m - n)(h + k)$

"A college degree is not a sign that one is a finished product, but an indication a person is prepared for life."

Reverend Edward A. Malloy, *Monk's Reflections*