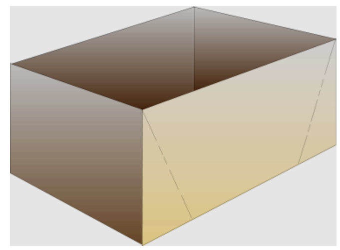
CH 20 – CUBIC FUNCTIONS

You're given a square piece of cardboard, 8 centimeters on a side, and told to cut squares out of each corner and fold up the resulting flaps, all in order to create an open box (no top) with the *largest* possible volume. What size corners should you cut from the cardboard? Solving this problem involves a cubic function.

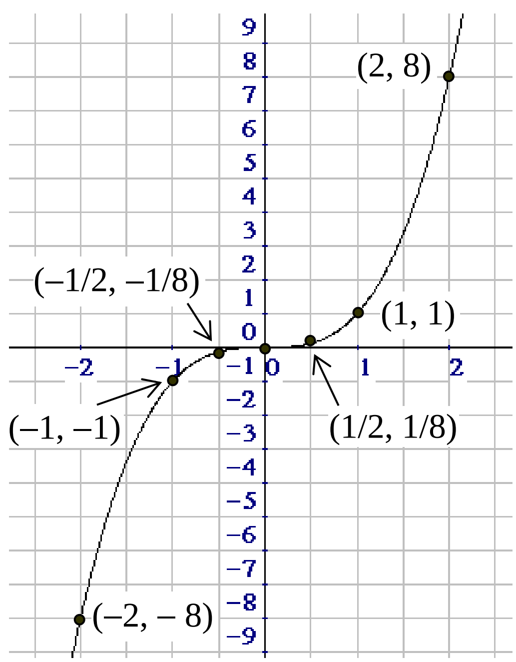


❑ INTRODUCTORY EXAMPLE

EXAMPLE 1: **Graph: $y = x^3$**

Solution: This is the classic cubic function -- it's as simple as a cubic can be. The **domain** is $\mathbb{R}$ (since any number can be cubed without a problem). Confirm the values in the following *x*-*y* table.

x	y
-3	-27
-2	-8
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
3	27



Note that when $x = 0$, $y = 0$; and when $y = 0$, $x = 0$. Therefore, the only **intercept** is the origin. Also notice that the **range** of the function is $\mathbb{R}$.

Also, the graph of the function appears to have **origin symmetry**. To prove this, replace x with $-x$ and y with $-y$, giving

$$-y = (-x)^3 \quad \Rightarrow \quad -y = -x^3 \quad \Rightarrow \quad y = x^3, \text{ the original equation.}$$

In addition, the following **limits** should be clear:

As $x \to \infty$, $y \to \infty$ As $x \to -\infty$, $y \to -\infty$

Homework

1. a. In Example 1, we claimed that the domain is $\mathbb{R}$, so we should be able to cube any real number. Cube each of the following numbers: 10, π, $\sqrt[3]{7}$, $-\sqrt[6]{11}$.

   b. It's also the case that y can be any real number. Find an x-value that will yield the given y-value: 64, $\frac{1}{27}$, -125, π, $-e$. [e is a number we'll about later in the course.]

2. Graph $y = -x^3$. Remember, $-x^3$ means cube x first, and then attach the minus sign.

3. Graph $f(x) = x^3 + 2$ and $g(x) = x^3 - 3$, and then explain what effect the 2 and the -3 have on the graph of $y = x^3$.

4. Graph $f(x) = (x-1)^3$ and $g(x) = (x+4)^3$, and then explain what effect the -1 and the 4 have on the graph of $y = x^3$.

5. For the function $y = x^3$, verify the limit "as $x \to \infty$, $y \to \infty$" by finding the x-value needed to guarantee that y can be made as large as one billion.

6. What is the slope of the curve $y = x^3$ at the origin?

7. This problem will prepare us for an issue in the next section. Let's solve the quadratic equation $x^2 + 7x - 18 = 0$ by factoring.

$$x^2 + 7x - 18 = 0$$
$$\Rightarrow \quad (x + 9)(x - 2) = 0$$
$$\Rightarrow \quad x + 9 = 0 \text{ or } x - 2 = 0$$
$$\Rightarrow \quad x = -9 \text{ or } 2$$

Notice that the solution $x = -9$ came from the factor $x + 9$, while the solution $x = 2$ came from the factor $x - 2$. So here's the question: What quadratic equation has the solutions $x = 7$ and $x = -5$? If 7 and -5 are solutions, then $x - 7$ and $x + 5$ must be factors: $(x - 7)(x + 5) = 0$, or $x^2 - 2x - 35 = 0$.

Find a quadratic equation whose solutions are given:

a. $x = 5, 10$ b. $x = -4, 9$ c. $x = -2, -11$

d. $x = 12$ e. $x = \pm 7$ f. $x = 0, 5$

❑ *INTERCEPTS AND GRAPHING*

EXAMPLE 2: **Graph:** $\boldsymbol{y = x^3 - 9x}$

Solution: This is another cubic function whose **domain** is $\mathbb{R}$.

Let's analyze **intercepts**. If we let $x = 0$, then $y = 0$, and so the graph has a y-intercept at the origin. Setting y to 0 gives

$$0 = x^3 - 9x = x(x^2 - 9) = x(x + 3)(x - 3),$$

whose solutions are $x = 0$, $x = -3$, and $x = 3$. Thus the x-intercepts are the points (0, 0), (–3, 0), and (3, 0).

Let's check for **symmetry** -- we'll try y-axis symmetry first. Replace x with $-x$ and we get $y = (-x)^3 - 9(-x) = -x^3 + 9x$. This is not the same formula as the original, so we do not have y-axis symmetry. But you can check that if we replace x with $-x$ and y

with $-y$, then we do get the same equation, and we can conclude that the graph has origin symmetry.

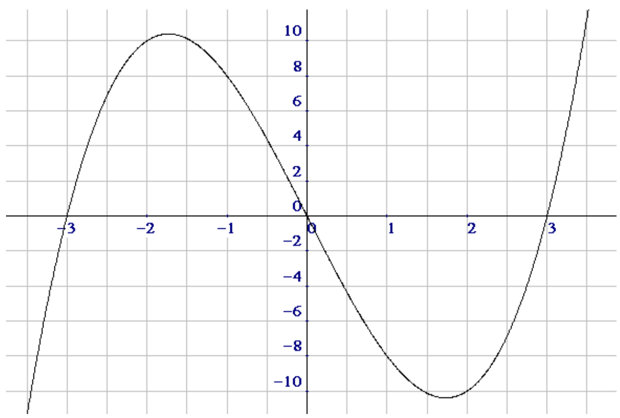
A few more calculations will give us the points

$$(-4, -28) \quad (-2, 10) \quad (-1, 8) \quad (1, -8) \quad (2, -10) \quad (4, 28).$$

Now let x be really big and really small, and you should be convinced of the following **limits**:

As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$

These points, along with the intercepts, the origin symmetry, and the limits, lead us to the following graph:



EXAMPLE 3: **Find all the x-intercepts of**

$$y = x^3 - x^2 - 14x + 24.$$

Hint: One of them is (2, 0).

Solution: Why the hint? To find the x-intercepts of any graph, we set $y = 0$. This yields the equation

$$x^3 - x^2 - 14x + 24 = 0$$

We can't use the Quadratic Formula since we have a cubic equation, not a quadratic one. That leaves factoring as the only viable technique, but factoring a cubic is difficult.

It's time to take advantage of the hint: (2, 0) is one of the x-intercepts. This means, of course, that the point (2, 0) is on the graph of the cubic function, which implies that the coordinates $x = 2$ and $y = 0$ must satisfy the equation of the cubic function. Let's verify this:

$$\mathbf{0} = \mathbf{2}^3 - \mathbf{2}^2 - 14(\mathbf{2}) + 24$$

$$\Rightarrow \quad 0 = 8 - 4 - 28 + 24$$

$$\Rightarrow \quad 0 = 0 \quad \checkmark$$

Now back to the cubic equation above:

$$x^3 - x^2 - 14x + 24 = 0$$

Here's the clue we need to solve this equation: Since $x = 2$ is one of the solutions of this equation (verified above), then $x - 2$ must be one of the factors of the left side of the equation. [See the previous homework problem.]

To find the other factors, we'll divide $x^3 - x^2 - 14x + 24$ by $x - 2$, the details of which are left to you.

$$\begin{array}{r|l} & x^2 + x - 12 \\ \hline x-2 & x^3 - x^2 - 14x + 24 \end{array}$$

Since the quotient is $x^2 + x - 12$ and the divisor is $x - 2$, we can write

$$x^3 - x^2 - 14x + 24 = (x^2 + x - 12)(x - 2)$$

Factoring the quadratic gives us the complete factorization. Here's the process from the beginning:

$$x^3 - x^2 - 14x + 24 = 0 \qquad \text{(the equation we're trying to solve)}$$

$$\Rightarrow \quad (x^2 + x - 12)(x - 2) = 0 \qquad \text{(from the long division)}$$

$$\Rightarrow \quad (x + 4)(x - 3)(x - 2) = 0 \qquad \text{(factor the quadratic)}$$

$$\Rightarrow \quad x = -4 \text{ or } x = 3 \text{ or } x = 2 \qquad \text{(set each factor to 0)}$$

And we thus have all three x-intercepts:

(–4, 0)	(3, 0)	(2, 0)

Can you see that the y-intercept is (0, 24)?

EXAMPLE 4: **Consider the cubic function $y = x^3 + 2x^2 - x - 2$. Use the fact that one of the x-intercepts is (–1, 0) to find all the intercepts, and then graph the function.**

Solution: First, the **domain** of the function is $\mathbb{R}$. Second, as in the previous example, we realize that an x-intercept of (–1, 0) means that $x + 1$ is one of the factors of the cubic. Dividing $x + 1$ into $x^3 + 2x^2 - x - 2$ gives a quotient of $x^2 + x - 2$, which itself factors into $(x + 2)(x - 1)$. In short, the complete factorization of the cubic function $y = x^3 + 2x^2 - x - 2$ is

$$y = (x + 1)(x + 2)(x - 1).$$

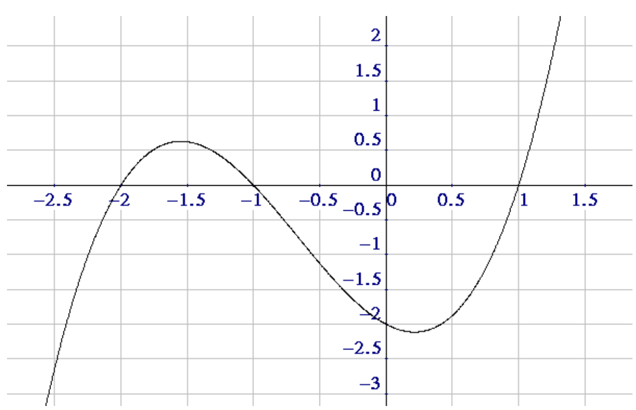
Setting y to 0 gives the following three ***x*-intercepts**:

(–1, 0) (–2, 0) (1, 0)

Setting x to 0 gives a ***y*-intercept** of (0, –2). You can verify (via calculator) that the following points are on the graph, too:

(–2.25, –1.015625) (–1.5, 0.625) (0.25, –2.109375)

If you plot the three x-intercepts, the y-intercept, and the three points above, you'll get a graph like the following:



The graph does not appear to have any **symmetries**. Also, we note the following two **limits**: As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$.

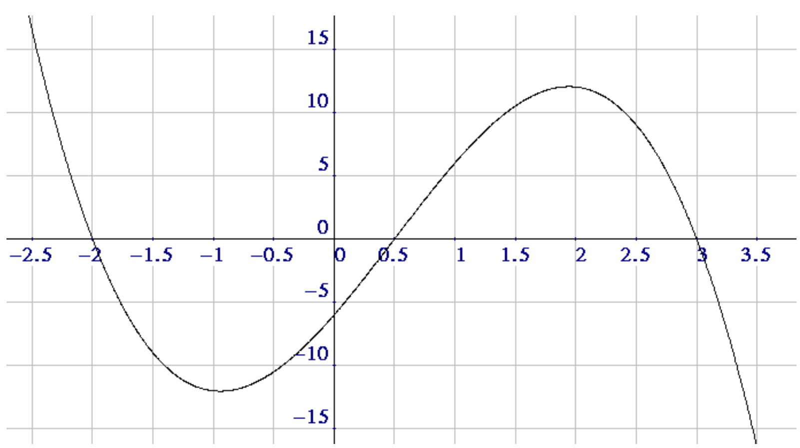
Postscript: It's impossible to accurately determine the *maximum point* (the top of the hill) in the second quadrant, or the *minimum point* (the bottom of the valley) in the fourth quadrant. Calculus is the subject where we find the tools needed to determine these two *extreme points* precisely.

EXAMPLE 5: **Find the intercepts and then sketch the graph of $y = -2x^3 + 3x^2 + 11x - 6$. Hint: One of the x-intercepts is (3, 0).**

Solution: An x-intercept of (3, 0) implies that $x - 3$ is one of the factors of the cubic. Dividing the cubic by $x - 3$ yields a quotient of $-2x^2 - 3x + 2$, which factors into $-(2x - 1)(x + 2)$. When this product is set to 0, we find that two additional x-intercepts are $(\frac{1}{2}, 0)$ and $(-2, 0)$. When we throw in the given x-intercept, and then calculate the y-intercept, we get four intercepts:

$(-2, 0)$ $(\frac{1}{2}, 0)$ $(3, 0)$ $(0, -6)$

With these four intercepts and a few other points which I'll leave for you to plot, we get the following graph:



The limits for this graph are different from the previous examples: As $x \to \infty$, $y \to -\infty$, and as $x \to -\infty$, $y \to \infty$. Do you know what it is about the cubic equation which produces these particular limits?

Homework

8. Graph $y = x^3 - 4x$.

9. Find all the x-intercepts of $y = x^3 - 4x^2 - 7x + 10$. Hint: One of them is (1, 0).

10. In Example 4, perform the long division to verify the factorization of the cubic. Now verify the calculations of the three additional points on the graph. Finally, without referring to the graph, prove that the graph has no symmetries.

11. Graph $y = x^3 - 3x^2 + 2x$. Label all the intercepts clearly. Estimate the maximum and minimum points on the graph.

12. Graph $y = -x^3 + 3x - 2$. Hint: One of the x-intercepts is (–2, 0). Estimate the maximum and minimum points on the graph. As $x \to \infty$, $y \to$ _____. As $x \to -\infty$, $y \to$ _____.

13. Graph $y = x^3 - 2x^2 - 5x + 6$. Hint: (–2, 0) is an x-intercept.

14. Graph $y = -x^3 - 4x^2 - 4x$.
As $x \to \infty$, $y \to$ _____. As $x \to -\infty$, $y \to$ _____.

❑ An Application of Cubic Functions

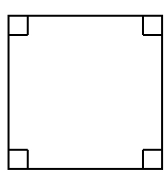
Remember the box question in the Introductory Example? Now we're ready to answer that query. You may recall from Gertrude's pigpen problem that a given amount of fence could produce different areas, depending on what dimensions we chose for the rectangle. The same concept applies to the box problem. We'll begin with two specific scenarios which should convince you that we could get different volumes with the same square piece of cardboard.

EXAMPLE 6: **Start with a cardboard square 8 cm by 8 cm. First cut out 1-cm squares from the corners and fold up the flaps to create an open box, and compute its volume. Then, starting with the original 8-cm square again, cut out 2-cm squares from the corners and compute the volume of the resulting box. Prove that the two boxes have different volumes.**

Solution: The volume of a box with dimensions l, w, and h is given by

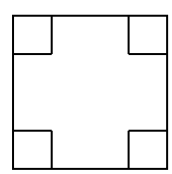
$$V = lwh$$

1-cm corners removed leaves a box 6 cm on each side and 1 cm high



$V = 6 \text{ cm} \times 6 \text{ cm} \times 1 \text{ cm}$
$= \mathbf{36 \text{ cm}^3}$

2-cm corners removed leaves a box 4 cm on each side and 2 cm high

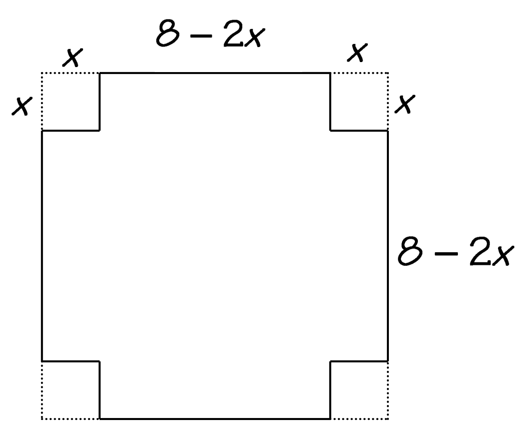


$V = 4 \text{ cm} \times 4 \text{ cm} \times 2 \text{ cm}$
$= \mathbf{32 \text{ cm}^3}$

Therefore, the boxes have different volumes, even though each box was created from identical 8-cm squares of cardboard. The next example will ask us to find the size of the cut-out corner that will produce the *maximum* volume. After all, given an 8-cm square of cardboard, we might as well get the most volume that we possibly can from the box.

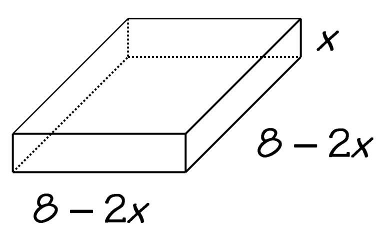
EXAMPLE 7: **An 8-cm square piece of cardboard is to be made into an open box by cutting squares from the corners and folding up the flaps. What size squares should be cut to achieve a box of maximum volume? What will the maximum volume be?**

Solution: Here's the plan of attack. We'll sketch the piece of cardboard with the corners removed; we'll assume that each square corner removed has dimensions of x cm by x cm. Then we'll write an expression that represents the length and width of the box which results from folding up the flaps. Since the height of the box is x cm, we'll be able to create a volume formula from the length, the width, and the height. Last, we'll graph the resulting function and estimate the value of x which produces the maximum volume.



The following are the dimensions of the box when the flaps are folded up:

length $= 8 - 2x$
width $= 8 - 2x$
height $= x$



And so the volume has the formula:

$$\begin{aligned} V &= lwh \\ &= (8-2x)(8-2x)x \\ &= (64-32x+4x^2)x \\ &= 4x^3-32x^2+64x \end{aligned}$$

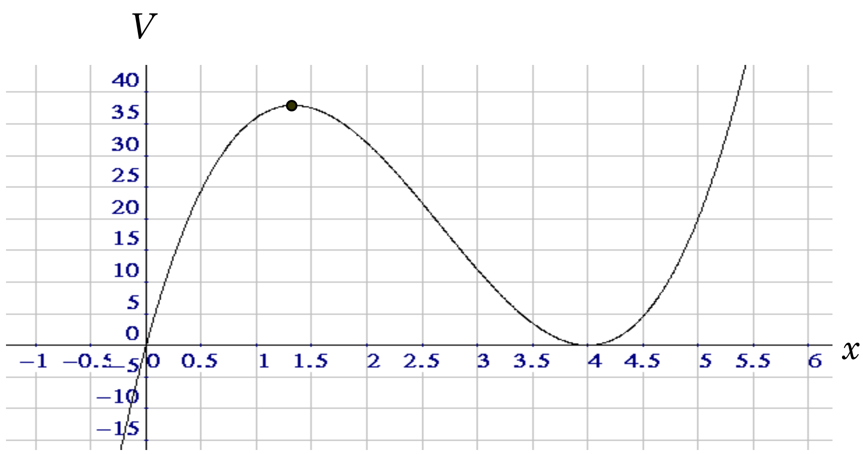
Therefore, the function we're trying to maximize is given by

$$V = 4x^3 - 32x^2 + 64x.$$

Factoring produces

$$V = 4x(x-4)^2,$$

which yields two x-intercepts, (0, 0) and (4, 0). Clearly we could plot lots of other points to get a decent graph, but let's just cut to the chase with the following graph:



Before we get to the goal of this problem, let's notice what the graph is telling us. It says that when $x = 0$, $V = 0$ (i.e., the graph passes through the origin). This makes sense because if we don't cut out any corners, we can't fold up the flaps, we get no box, and so obviously the volume is 0. By the same token, if we cut out flaps that are 4 cm long, there's no base left to the box, since it's only 8 cm to begin with. The graph indicates this situation at the point (4, 0).

Now to find the size of the squares to cut out and the maximum volume: Look at the top of the mountain -- this is the point which represents maximum volume. Going straight down to the x-axis, we estimate the x-coordinate to be about 1.3 cm. Now move from the top of the mountain to the left, and the volume appears to be about 38 cm^3. In summary,

> Cutting a 1.3 cm square from each corner will produce a box with volume 38 cm^3.

Homework

15. Each of the following numbers is the length of the side of a piece of square cardboard. The cardboard square is to be made into an open box by cutting squares from the corners and folding up the flaps. What size square should be cut from each corner to achieve a box of maximum volume? Note that the solutions are approximations only. As long as your answers are in the ballpark (and you really understand what you're doing!), consider yourself correct.

    a. 2   b. 3   c. 4   d. 6

Practice Problems

16. Let $y = 2x^3 - 4x^2 + 9$.
    - a. Is it a function? Why?
    - b. Why is it cubic?
    - c. What's its domain?
    - d. Find the y-intercept.
    - e. What would you do to find the x-intercepts?

17. How does the graph of $g(x) = (x - 2)^3 + 7$ compare to the graph of $f(x) = x^3$?

18. Prove that the graph of $y = 2x^3 + 7x$ has origin symmetry.

19. Let $f(x) = 3x^3 + 4x - 7$ and $g(x) = -\pi x^3 + x^2 + 100$.

    a. As $x \to \infty$, $f(x) \to$ _____ 　　b. As $x \to -\infty$, $f(x) \to$ _____

    c. As $x \to \infty$, $g(x) \to$ _____ 　　d. As $x \to -\infty$, $g(x) \to$ _____

20. If one of the x-intercepts of the cubic graph $y = x^3 + 3x^2 - 61x - 63$ is (7, 0), find the other two intercepts.

21. Graph $y = -x^3 + 2x$. Be sure to discuss symmetry, intercepts, and domain. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

22. Graph $y = x^3 + 3x^2 - x - 3$. Hint: (1, 0) is one of the intercepts. Discuss symmetry, intercepts, and domain. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

23. A 5-cm square piece of aluminum is to be made into an open box by cutting squares from the corners and folding up the flaps. What size squares should be cut to achieve a box of maximum volume?

24. True/False:

    a. Both the domain and range of the function $y = x^3$ are $\mathbb{R}$.

    b. The slope of the curve $y = x^3$ at the origin is about 1.

    c. The curve $y = x^3$ never touches the 4th quadrant.

    d. For the function $y = x^3$, as $x \to -\infty$, $y \to 0$.

    e. The graphs of $f(x) = (-x)^3$ and $g(x) = -x^3$ are identical.

    f. The graph of $y = x^3 - 16x$ has three x-intercepts.

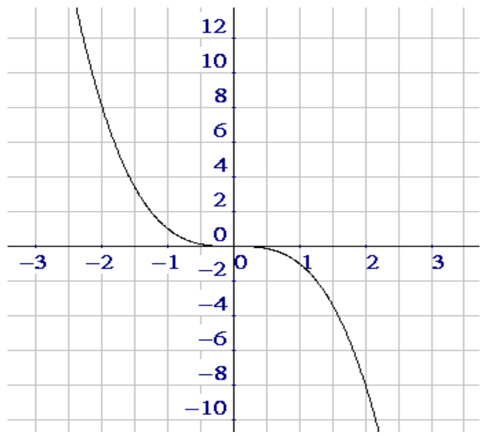
    g. The graph of $y = x^3 - 25x$ has its minimum point in Quadrant II.

    h. If you know one of the x-intercepts of a cubic function, long division will help you find the others.

Solutions

1. a. $1000,\ \pi^3,\ 7,\ -\sqrt{11}$ b. $4,\ \frac{1}{3},\ -5,\ \sqrt[3]{\pi},\ \sqrt[3]{-e}$

2.



3. The graph of f is x^3 shifted up 2 units.
The graph of g is x^3 shifted down 3 units.

4. The graph of f is x^3 shifted right 1 unit.
The graph of g is x^3 shifted left 4 units.

5. Make $x \ge 1000$ and y will be $\ge 1{,}000{,}000{,}000$.

6. I'd like to know your opinion.

7. a. $(x-5)(x-10) = 0 \Rightarrow x^2 - 15x + 50 = 0$

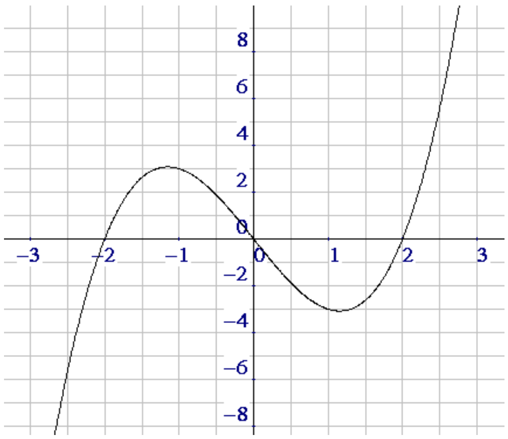
b. $x^2 - 5x - 36 = 0$

c. $x^2 + 13x + 22 = 0$

d. Only one solution is given, but we're required to provide a quadratic equation; so we must use the solution $x = 12$ twice:
$(x-12)(x-12) = 0$, or $x^2 - 24x + 144 = 0$.

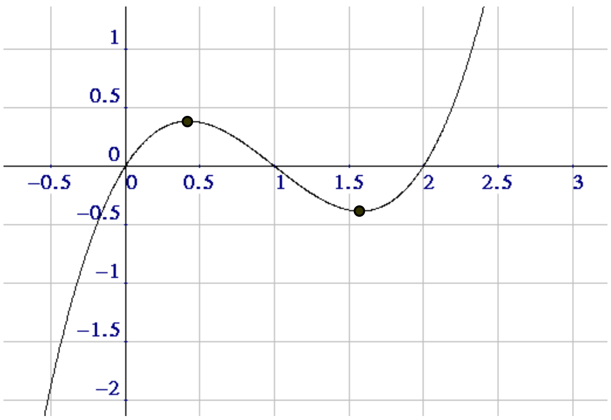
e. $x^2 - 49 = 0$

f. $x(x-5) = 0 \Rightarrow x^2 - 5x = 0$

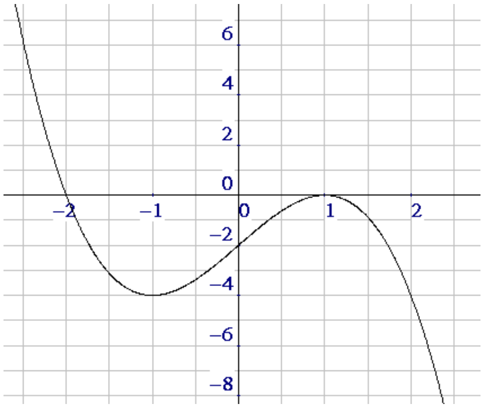
8. 

9. Divide $x - 1$ into $x^3 - 4x^2 - 7x + 10$; the quotient is $x^2 - 3x - 10$. The complete factorization is $(x - 1)(x - 5)(x + 2)$, and therefore the x-intercepts are (1, 0), (5, 0) and (–2, 0).

10. For the symmetries, use the litmus tests; e.g., to test for y-axis symmetry, substitute $-x$ for x and show that you do not get the same equation.

11. 

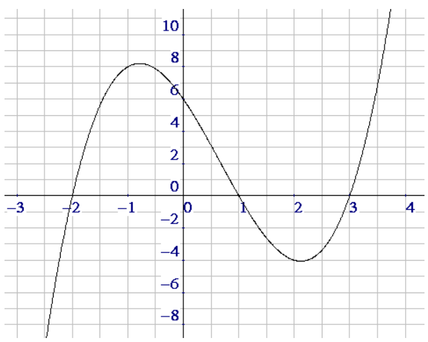
A maximum point appears roughly at (.4, .4), and a minimum point near (1.6, –.4).

12. 

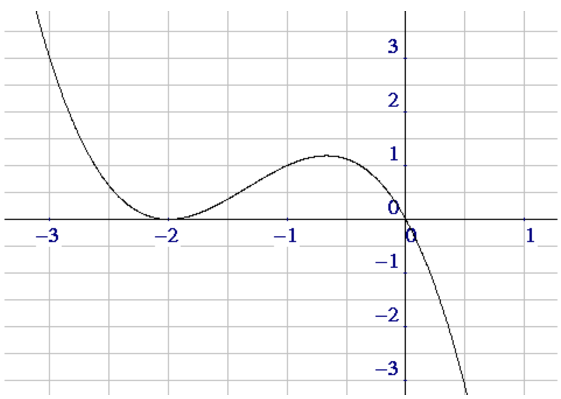
max at (1, 0)

min at (–1, –4)

Limits: –∞; ∞

13. 

Make sure your four intercepts match those on the graph.

14. 

Limits: –∞; ∞

15. a. 0.3   b. 0.5   c. 0.7   d. 1

16. a. Yes; Given any input (an x), there's only one output (a y).
b. Because the highest exponent is 3.
c. $\mathbb{R}$, since there's no real number that could possibly cause any problems.
d. Setting $x = 0$ gives $y = 9$. Therefore, the y-intercept is (0, 9).
e. Set $y = 0$, then solve the resulting equation for x.

17. Compared to f, the graph of g is 2 units to the right and 7 units up.

18. We replace x with $-x$ and y with $-y$ at the same time:

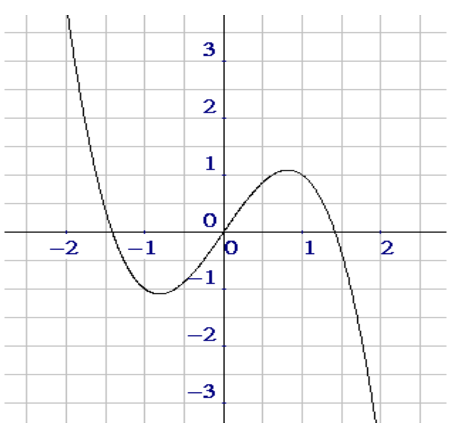
$$-y = 2(-x)^3 + 7(-x)$$
$$\Rightarrow \quad -y = 2(-x^3) + (-7x)$$
$$\Rightarrow \quad -y = -2x^3 - 7x$$
$$\Rightarrow \quad y = 2x^3 + 7x, \text{ the same as the original equation.}$$

Therefore, the graph has origin symmetry.

19. a. ∞ b. $-\infty$ c. $-\infty$ d. ∞

20. (–1, 0) and (–9, 0)

21.



Origin symmetry

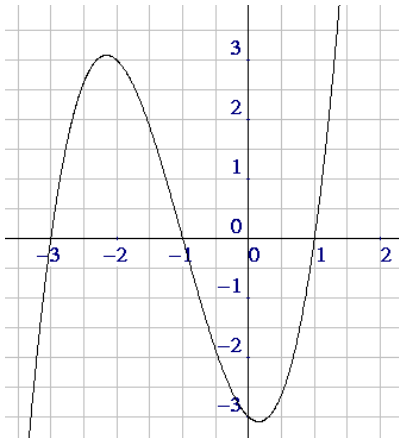
x-int: (0, 0) and $(\pm\sqrt{2}, 0)$

y-int: (0, 0)

Domain = $\mathbb{R}$

Limits: $-\infty$; ∞

22.



No symmetry

x-int: (1, 0), (−1, 0), and (−3, 0)

y-int: (0, −3)

Domain = $\mathbb{R}$

Limits: ∞; $-\infty$

23. approximately 0.8

24. a. T b. F c. T d. F e. T f. T g. F h. T

Give a man a fish

and you feed him for a day.

Teach a man to fish

and you feed him for a lifetime.

Chinese Proverb