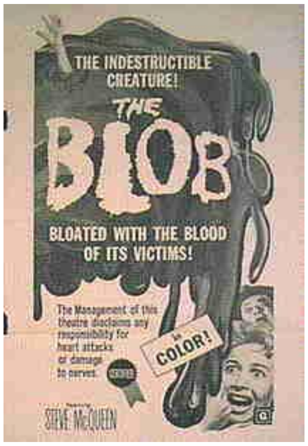
CH 22 – EXPONENTIAL FUNCTIONS

What worried Steve McQueen was not that The Blob was growing by a constant amount every hour, but rather that it was doubling in size every hour. If our hero had known his math, he could have warned the town that The Blob was not growing linearly, but exponentially.



☐ Function Review

This is a good place to review some of the functions we've seen so far in this class:

Linear -- These functions are of the form

$$y = mx + b$$

They are straight lines with slope m and y-intercept (0, b). A special case of a linear function is the constant function $y = b$, which is a horizontal line (with slope 0) passing through the point (0, b).

Another form for a straight line is

$$y - y_1 = m(x - x_1)$$

where m is the slope and (x_1, y_1) is a point on the line.

Quadratic -- These are the parabolas:

$$y = \frac{1}{4p}(x-h)^2 + k$$

p is called the focal length. If $p > 0$, then the parabola opens up, and if $p < 0$, then it opens down. The vertex of the parabola is (h, k). A quadratic function can also be written in the form

$$y = ax^2 + bx + c$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Cubic -- These functions are of the form

$$y = ax^3 + bx^2 + cx + d$$

where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$.

Polynomial -- A polynomial function is a catch-all for the linear, quadratic, and cubic functions just discussed, as well as 4th degree, 5th degree, etc. functions. A typical example is $y = 7x^5 - \pi x^3 + x^2 - \sqrt{2}$. The standard form of a polynomial can be written

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0 \in \mathbb{R}$ and $a_n \neq 0$. The key to any polynomial is that all the exponents on the x come from the set of whole numbers { 0, 1, 2, 3, . . . }.

Rational -- These are the ratios of polynomials: $\frac{P}{Q}$. For example, $y = \dfrac{3x-3}{x^2+2x+7}$ is a rational function. It's the kind of function that may have vertical and horizontal asymptotes.

Miscellaneous -- These are functions like $f(x) = \sqrt{7x-9}$ and $g(x) = 3\left|8x-2\right|$.

Homework

1. Explain why $x = 4$ is not mentioned in the linear function discussion.

2. The graph of $y^2 = x$ is a parabola, but it's not mentioned in the quadratic function section. Why not?

3. Give an example of a cubic function containing only odd powers of x.

4. Explain why $y = \pi x^7$ is a polynomial, but $y = 7x^\pi$ is not.

5. In the definition of the cubic function, there's a stipulation that $a \neq 0$. Why?

☐ Exponential Functions

EXAMPLE 1: **Analyze The Blob function.**

Solution: To give you an idea of the concept of an exponential function, let's look at the two scenarios regarding the rate of growth of The Blob. An example of a linear rate of growth might be the formula $B = 4t$, where t is the number of hours and B is the size of The Blob. An exponential formula could be $B = 2^t$. If we construct a table, showing the time and the amount of The Blob for each formula, we can see the true effect of exponential growth.

t	1	2	3	4	5	6	7	8	9	10
$4t$	4	8	12	16	20	24	28	32	36	40
2^t	2	4	8	16	32	64	128	256	512	1024

For the first three hours, there's more blob in the linear formula than in the exponential formula. At $t = 4$, the blob amounts are equal. But after that, it's not even a contest -- the exponential formula shows that The Blob will probably eat the town, the state, and eventually the entire Earth!

EXAMPLE 2: **Explain the connection between subsets and exponential functions.**

Solution: For another example of an exponential function, we need only go back to the last section of Chapter 2 on Sets. We learned that if a set has n elements, then that set has 2^n subsets. For example, a set with 5 elements has $2^5 = 32$ subsets. We can write an official exponential function like this: If we let n represent the number of elements in a set, and let S represent the number of subsets of that set, then S is an exponential function of n, and we can write

$$S(n) = 2^n.$$

For example, to find the number of subsets of a set containing 10 elements, we calculate

$$S(10) = 2^{10} = 1024.$$

EXAMPLE 3: **Find some ordered pairs for the exponential function $f(x) = 4^x$.**

Solution:

$$f(1) = 4^1 = 4 \qquad \Rightarrow \qquad (1,4)$$

$$f(-2) = 4^{-2} = \frac{1}{4^2} = \frac{1}{16} \qquad \Rightarrow \qquad (-2, \tfrac{1}{16})$$

$$f(-1) = 4^{-1} = \frac{1}{4} \qquad \Rightarrow \qquad (-1, \tfrac{1}{4})$$

$$\begin{aligned} f(0) &= 4^0 = 1 && \Rightarrow && (0, 1) \\ f(\tfrac{1}{2}) &= 4^{1/2} = \sqrt{4} = 2 && \Rightarrow && (\tfrac{1}{2}, 2) \\ f(2) &= 4^2 = 16 && \Rightarrow && (2, 16) \end{aligned}$$

We now try to determine exactly what the formula for an exponential function looks like, and how it differs from that of a polynomial. Look at the exponential functions given in the previous three examples:

$$B = 2^t \qquad S(n) = 2^n \qquad f(x) = 4^x$$

Notice that in each function, the base is a constant and the exponent is a variable. Thus, an ***exponential function*** is a function of the form

$$f(x) = \underset{\substack{\uparrow \\ \boxed{\text{constant}}}}{b}{}^{x} \longleftarrow \boxed{\text{variable}}$$

where b is some appropriate real number (a constant). This is in sharp contrast to the polynomial, which is the other way around. Thus,

$y = 10^x$ is an *exponential* function,

$y = x^{10}$ is a *polynomial* function, and

$y = x^x$ is neither exponential nor polynomial.

The question of which real numbers b serve nicely as the base of an exponential function will be discussed later.

Homework

6. a. Fill in the following chart, similar to Example 1:

t	1	2	3	4	5	6	7	8	9	10
$9t$										
3^t										

b. Is the $9t$ row of the chart a linear or exponential function?

c. Is the 3^t row of the chart a linear or exponential function?

d. For how many hours does the linear growth produce more blob than the exponential growth?

e. At what hour do both growths give the same amount of blob?

f. At 10 hours, what is the ratio of the exponential amount of blob to the linear amount of blob?

7. Let's find some more ordered pairs in the function $f(x) = 4^x$ from Example 3.

a. (3, ___) b. (–3, ___) c. $(-\frac{1}{2},$ ___) d. $(\frac{3}{2},$ ___)

8. Take a guess what the domain and range of $f(x) = 4^x$ are.

9. Melanie tells her father that she will give up her allowance for the rest of her life if he will agree to the following plan: He gives her 2¢ on the first day of the month, 4¢ on the second day of the month, 8¢ on the third day of the month, 16¢ on the fourth day of the month, and so on till the 30th of the month. Dad (who was a philosophy major) thinks this is a great idea for him and accepts Melanie's proposal. Calculate the amount of money that Melanie will receive on the last day of the month.

☐ Graphing Exponential Functions

Let's use the function $y = 4^x$ discussed in the previous section to make our first exponential graph.

EXAMPLE 4: **Graph: $y = 4^x$**

Solution: Here are some of the ordered pairs for this function that we found in Example 3 and Homework #7:

$(-1, \frac{1}{4})$ $(-\frac{1}{2}, \frac{1}{2})$ $(0, 1)$ $(\frac{1}{2}, 2)$ $(1, 4)$ $(2, 16)$

Notice that the point (0, 1) is the ***y*-intercept**.

Next we analyze a pair of **limits**. First we'll let $x \to \infty$. As it does, the functional value 4^x approaches ∞ much faster than x does. For example, as x takes the values 6, 8, 10, the y-values go 4096, 65536, 1048576. The functional values are growing like crazy. The following limit should now be clear:

$$\text{As } x \to \infty,\ y \to \infty$$

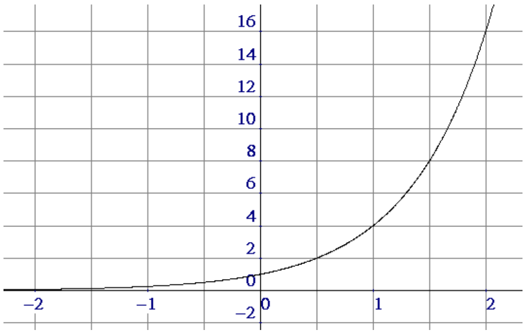
Second, we analyze what the y-values do when $x \to -\infty$. Consider the three ordered pairs:

(–5, .000977) (–8, .000015) (–10, .000000954)

It appears that as x grows smaller, the y-values are positive numbers shrinking toward zero. That is,

$$\text{As } x \to -\infty,\ y \to 0$$

The ordered pairs we listed and the limits we calculated lead us to the following graph:



The **domain** is ℝ, and the range is all positive numbers, $(0, \infty)$. Also, there is no vertical **asymptote**, but the line $y = 0$ (the x-axis) is a horizontal asymptote.

Last, it appears that there is no ***x*-intercept** on our graph (which is confirmed by the fact that the range of the function is $(0, \infty)$). Even more importantly, this confirms that the equation $4^x = 0$ has no solution.

EXAMPLE 5: **Graph:** $y = \left(\frac{1}{2}\right)^x$

Solution: Let's get right to some ordered pairs.

If $x = -3$, then $y = \left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8$, which gives us the ordered pair $(-3, 8)$. It's now your job to verify each of the following ordered pairs in our function:

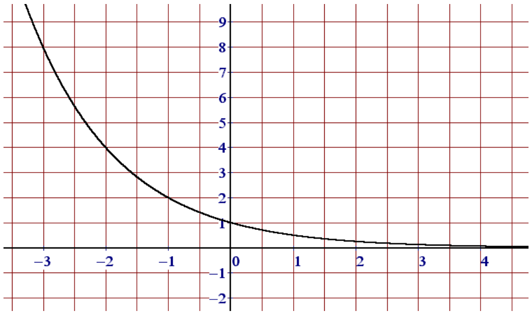
$(0, 1)$ $(1, \frac{1}{2})$ $(2, \frac{1}{4})$ $(3, \frac{1}{8})$ $(4, \frac{1}{16})$

$(-1, 2)$ $(-2, 4)$ $(-3, 8)$ $(-4, 16)$

The y-intercept is $(0, 1)$. There is no x-intercept, since the equation $\left(\frac{1}{2}\right)^x = 0$ has no solution. The ordered pairs listed above give credence to the following limits:

As $x \to \infty, y \to 0$ and As $x \to -\infty, y \to \infty$

The ordered pairs and the limits lead us to the following graph:



We can see that the domain of this function is ℝ, while the range is $(0, \infty)$. Notice also that we have a horizontal asymptote at $y = 0$, but there is no vertical asymptote.

EXAMPLE 6: **Graph:** $f(x) = 2^{-x}$

Solution: Two observations and we'll be done in a jiffy. First, we know that $f(x)$ can be written simply as y. Second, check out the following calculation:

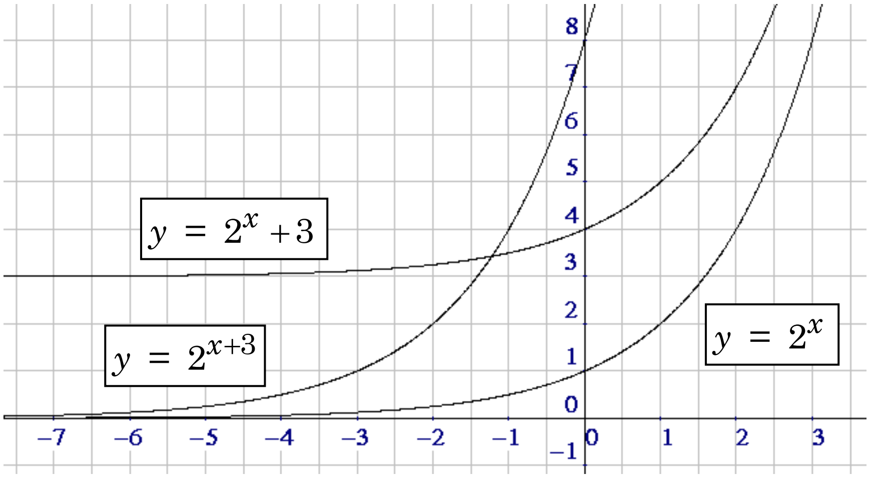
$$2^{-x} = \frac{1}{2^x} = \frac{1^x}{2^x} = \left(\frac{1}{2}\right)^x$$

In other words, the original formula can be written $y = \left(\frac{1}{2}\right)^x$, which we just finished graphing. So the solution to this problem is identical to that of the previous example.

EXAMPLE 7: **Graph:** $y = 2^x$ **and** $y = 2^x + 3$ **and** $y = 2^{x+3}$.

Solution: Let's make one table showing x and all the y-values at once and a single grid containing all the graphs at once.

x	2^x	$2^x + 3$	2^{x+3}
–6	1/64	3 1/64	1/8
–5	1/32	3 1/32	1/4
–4	1/16	3 1/16	1/2
–3	1/8	3 1/8	1
–2	1/4	3 1/4	2
–1	1/2	3 1/2	4
0	1	4	8
1	2	5	16
2	4	7	32
3	8	11	64



You should note that the graph of $2^x + 3$ is just the graph of 2^x but shifted 3 units up. Also, we see that the graph of 2^{x+3} is the result of taking the graph of 2^x and shifting it 3 units to the left.

Homework

10. Referring to Example 4, explain the last paragraph.

11. Graph: $f(x) = 3^x$

12. Graph: $y = 3^x - 2$

13. Graph: $y = 3^{x+2}$

14. Graph: $g(x) = \left(\frac{1}{3}\right)^x$

15. Graph: $h(x) = 3^{-x}$

☐ The Legal Bases of an Exponential Function

In the previous section we graphed exponential functions with bases 4, $\frac{1}{2}$, 2, 3, and $\frac{1}{3}$. Now it's time to figure out exactly which bases we'll allow in the exponential function

$$f(x) = b^x.$$

Whatever values of b we allow to be the base of an exponential function, we'd like the domain of the function (the legal x-values) to be $\mathbb{R}$, the set of real numbers. And we don't want the exponential function to degenerate into some simple function that we've seen before in algebra.

$b < 0$ What about negative bases? Consider $f(x) = (-4)^x$. If we choose $x = \frac{1}{2}$, the functional value is $(-4)^{1/2} = \sqrt{-4} \notin \mathbb{R}$ (i.e., not an element of the real numbers). We thus disallow any base b that is negative.

$b = 0$ Now consider $f(x) = 0^x$. But 0^x is fraught with problems. For example, if $x = 0$, we get 0^0. Can we assign a value to 0^0? On the one hand, 0 to any power should be 0. On the other hand, anything to the 0 power is supposed to be 1. So 0^0 is meaningless. Even worse, consider 0^{-2}. Since a negative exponent indicates reciprocal, we get $\frac{1}{0^2} = \frac{1}{0}$, which is undefined. All in all, a base of 0 really stinks.

$0 < b < 1$ These bases are just fine. We used bases of $\frac{1}{2}$ and $\frac{1}{3}$ in the previous section. Even the number $1/\pi$ would be a legal base, though I've never seen it used.

$b = 1$ This gives us the function $f(x) = 1^x$, which is the function $f(x) = 1$, a constant function (the horizontal line $y = 1$). Exponential functions aren't supposed to be flat, so b can't be 1.

$b > 1$ Any base bigger than 1 is appropriate. In fact, in computer science a base of 2 is very popular. In basic science the best base is 10 (for things like acids, earthquakes, and the volume of sound). And in calculus and the more advanced sciences, we use a number called "e", to be developed in the next chapter.

Homework

16. Describe precisely the legal bases for an exponential function.

17. Explain why –9 is not a good base for an exponential function.

18. Which of the following real numbers are legal bases for an exponential function?

$$-1 \quad -.01 \quad 0 \quad \frac{2}{3} \quad 0.987 \quad 1 \quad \pi \quad 200$$

Practice Problems

19. T/F: $y = x^3$ is an exponential function.
20. Describe the real numbers which can be used as the base of an exponential function.
21. Explain why 1 is not a good base for an exponential function.
22. Give a function which is both an exponential function and an polynomial function.
23. Graph $y = 5^x$ and state its range.
24. Graph $y = 3^{-x} - 2$ and state its horizontal asymptote and range.
25. Let $f(x) = 2^x$. Now let g be the graph which is the result of taking the graph of f and shifting it 7 units to the left and 4 units up. Find a formula for g.
26. T/F: $y = 3^x$ is an exponential function.
27. T/F: In the exponential function $f(x) = b^x$, b can be any positive real number.

28. A set contains k elements. How many subsets does the set have? Is the answer classified as polynomial or exponential?

29. What is the domain of the function $y = 10^{7x+1} - 10$?

30. Find all the asymptotes of the function $f(x) = 99^x$.

31. Explain why the graphs of $g(x) = \left(\frac{1}{3}\right)^x$ and $h(x) = 3^{-x}$ are the same.

32. How does the graph of $f(x) = 5^{x-2} + 4$ compare with that of $y = 5^x$?

33. T/F: All exponential functions are increasing functions.

34. Explain why 0 is not a good base for an exponential function.

35. True/False:

    a. $y = x^3$ is an exponential function.

    b. $y = \pi^x$ is an exponential function.

    c. A set has m elements. Then the set has m^2 subsets.

    d. The domain of the function $y = 4^x$ is $[0, \infty)$.

    e. The range of the function $y = 4^x$ is $[0, \infty)$.

    f. The function $y = x^x$ is neither polynomial nor exponential.

    g. The function $y = 2^x - 1$ has an x-intercept.

    h. For the function above, as $x \to -\infty$, $y \to 0$.

    i. Consider the function $g(x) = \left(\frac{1}{3}\right)^x$. As $x \to \infty$, $y \to \infty$.

    j. Compared to the graph of $y = 4^x$, the graph of $y = 4^{x-5}$ is four units lower.

    k. Any real number $b > 0$ is a legal base for an exponential function.

    l. Any real number $b \geq 0$, but not equal to 1, is a legal base for an exponential function.

    m. The number $\pi + \sqrt{2}$ is a legal base for an exponential function.

    n. All exponential functions are decreasing functions.

Solutions

1. $x = 4$ is not a function. Given an input of 4, there are <u>infinitely</u> many outputs (any real number). Also, the graph of $x = 4$ is a vertical line, which clearly violates the Vertical Line Test.

2. It's not a function (since it's a sideways parabola).

3. Example: $y = 9x^3 + \sqrt{2}\,x$

4. 7 is a whole number, but π isn't.

5. If a were 0, it would be quadratic.

6. a.

t	1	2	3	4	5	6	7	8	9	10
$9t$	9	18	27	36	45	54	63	72	81	90
3^t	3	9	27	81	243	729	2187	6561	19683	59049

b. linear c. exponential d. first two hours e. $t = 3$
f. 59,049 / 90 $\approx$ 656

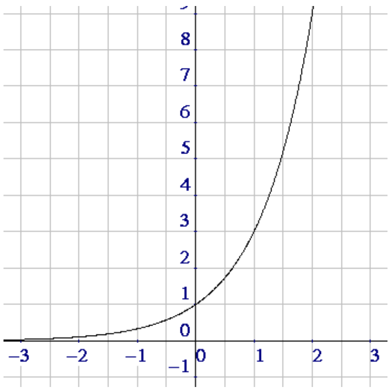
7. a. 64 b. $\frac{1}{64}$ c. $\frac{1}{2}$ d. 8

8. x can be all kinds of numbers, so the domain is probably $\mathbb{R}$. For the range, look at the outputs we calculated: 1/16, 1/4, 1, 2, 16, 64, 1/64, 1/2, 8. What do you think the range is?

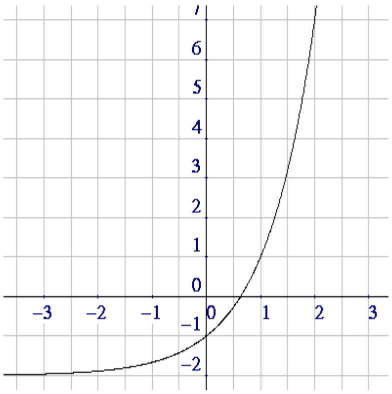
9. Almost 11 million dollars.

10. If $4^x = 0$, we're saying that the graph of $y = 4^x$ has an x-intercept, which it doesn't. Therefore, $4^x = 0$ has no solution.

11.



12.



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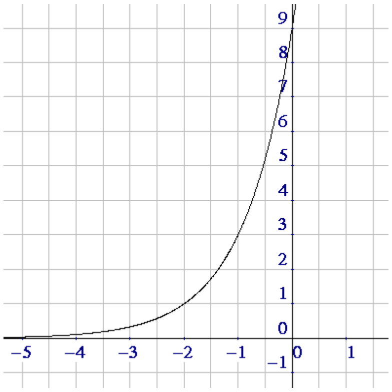
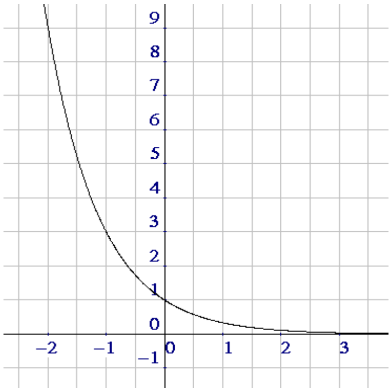


14.



15. Same graph as #14, since $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

16. The base must be positive but not equal to 1. That is, the function $f(x) = b^x$ is an exponential function if $b > 0$, but $b \neq 1$. In other words, the base b must be in the set $(0, \infty) - \{1\}$.

17. If we consider the exponential function $y = (-9)^x$, then we could not use $x = \frac{1}{2}$, since $y = (-9)^{\frac{1}{2}} = \sqrt{-9} \notin \mathbb{R}$.

18. 2/3, 0.987, π, 200

19. False

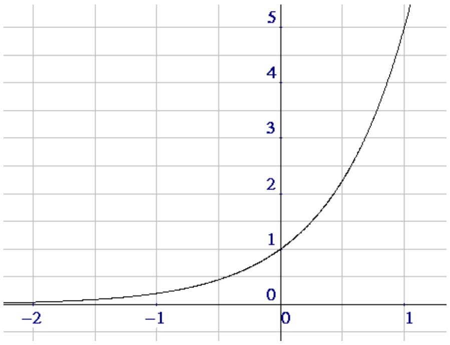
20. $\{x \in \mathbb{R} \mid x > 0, x \neq 1\}$ –OR– Any positive real number $\neq 1$

–OR– $(0, 1) \cup (1, \infty)$ –OR– $(0, \infty) - \{1\}$

21. If b were 1, then $f(x) = b^x$ would become $f(x) = 1^x = 1$, which is a simple constant function whose graph is a horizontal line -- rather useless to describe exponential growth and decay.

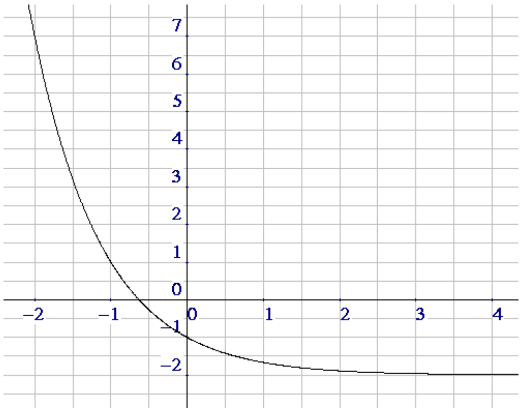
22. Ain't no such animal

23.



Range: $(0, \infty)$

24.



Horizontal asymptote: $y = -2$

Range: $(-2, \infty)$

25. $g(x) = 2^{x+7} + 4$

26. T **27**. F (any positive real number $\neq 1$)

28. 2^k; exponential **29**. $\mathbb{R}$ **30**. horiz: $y = 0$; vert: none

31. Because $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

32. The graph of f is the graph of y shifted 2 units to the right and 4 units up.

33. F

34. Because if $f(x) = 0^x$, then $f(x) = 0$, which is just a horizontal line.

35. a. F b. T c. F d. F e. F f. T g. T h. F
i. F j. F k. F l. F m. T n. F

The most beautiful experience we can have
is the mysterious.
It is the fundamental emotion
which stands at the cradle
of true art and true science.

Albert Einstein