CH 22 – SYSTEMS OF LINEAR EQUATIONS

❑ INTRODUCTION

Planning production is the key to any manufacturing business. We have various constraints, which are restrictions that limit our options. For example, we might need to produce 20 widgets today, but how many red widgets and how many green widgets can we afford to make? Or perhaps we should produce the red and green widgets in a ratio that's in line with our customers' demands. The next chapter will solve these kinds of business problems, but solving such a problem will lead to two equations with two variables. And so this is the subject of the current chapter.

❑ TWO EQUATIONS IN TWO VARIABLES

Here's an example of a ***system of two equations in two variables***:

$$\begin{aligned} x + y &= 10 \\ 2x - 3y &= 5 \end{aligned}$$

The *two equations* are easy to see, as are the *two variables*. The term *system* refers to the fact that these two equations are tied together -- our final solution must be a pair of numbers, one of x and one for y, which satisfy both equations.

For example, in the system above, the values $x = 8$ and $y = 2$ will satisfy the first equation [8 + 2 = 10], but will not satisfy the second equation [2(8) – 3(2) = 16 – 6 = 10, not 5]. Therefore, $x = 8$ and $y = 2$ is not a solution to the *system* of equations.

But $x = 7$ and $y = 3$ is a solution of the system of equations. Here's why:

$$7 + 3 = 10 \checkmark \quad \text{and} \quad 2(7) - 3(3) = 14 - 9 = 5 \checkmark$$

Homework

1. Consider the system of equations: $\begin{aligned} a + b &= 9 \\ a - b &= 7 \end{aligned}$

   a. Show that $a = 5$ and $b = 4$ is a solution of the first equation, but is not a solution of the system.

   b. Show that $a = 20$ and $b = 13$ is a solution of the second equation, but is not a solution of the system.

   c. Show that $a = 8$ and $b = 1$ is a solution of the system.

2. Try to solve the system $\begin{aligned} u + w &= 12 \\ u - w &= 0 \end{aligned}$ by guessing.

   Now try the system $\begin{aligned} 3x - 17y &= 200 \\ -5x - 12y &= 29 \end{aligned}$ by guessing (just kidding!).

   I hope this problem motivates you to study the next section carefully.

❑ *THE ADDITION METHOD*

To solve the applications in future chapters, we need to be able to solve two equations in two variables. A method that works very well in many cases is called the ***Addition Method***. We multiply one or both equations by appropriate numbers (whatever that means), **add** the resulting equations to eliminate a variable, and then solve for the variable that survived. The *elimination method* is another term used to describe this procedure.

<u>EXAMPLE 1:</u> **Solve the system:** $\mathbf{5x - 2y = 20}$

$\mathbf{3x + 7y = -29}$

<u>Solution:</u> In the Addition Method we may eliminate either variable. But there's a certain "orderliness" that comes in handy in future math courses if we always eliminate the first variable, so in this case we will eliminate the x. As mentioned above, we multiply one or both equations by some numbers, add the resulting equations to kill off one of the variables, and then solve for the variable that still lives. How do we find these numbers? Rather than some mystifying explanation, just watch -- you'll catch on.

$$5x - 2y = 20 \xrightarrow{\text{times 3}} 15x - 6y = 60$$

$$3x + 7y = -29 \xrightarrow{\text{times -5}} -15x - 35y = 145$$

<u>Add</u> the equations: $0x - 41y = 205$

The x's are gone!

Divide by -41: $\dfrac{-41y}{-41} = \dfrac{205}{-41}$

$\underline{y = -5}$

Now that we have the value of y, we can substitute its value of -5 into either of the two original equations to find the value of x. Using the first equation:

$$5x - 2(\mathbf{-5}) = 20$$
$$\Rightarrow \quad 5x + 10 = 20$$
$$\Rightarrow \quad 5x = 10$$
$$\Rightarrow \quad \underline{x = 2}$$

Therefore, the final solution to the system of equations is

$$\boxed{x = 2 \ \& \ y = -5}$$

EXAMPLE 2: **Solve the system:** $\mathbf{12a + 7b = 9}$
$\mathbf{-18a - 5b = 3}$

Solution: Notice how the numbers we choose to multiply each equation by will accomplish our goal: They make the a's disappear when the equations are added.

$$\begin{aligned} 12a+7b &= 9 \xrightarrow{\text{times } 3} & 36a+21b &= 27 \\ -18a-5b &= 3 \xrightarrow{\text{times } 2} & -36a-10b &= 6 \end{aligned}$$

Add the equations: $0a + 11b = 33$

Divide by 11: $\underline{b = 3}$

Placing this value of b into the first equation gives us

$$12a + 7(\mathbf{3}) = 9 \Rightarrow 12a + 21 = 9 \Rightarrow 12a = -12 \Rightarrow \underline{a = -1}$$

We now have our complete solution to the system of equations:

$$\boxed{a = -1 \ \& \ b = 3}$$

Let's use this example to learn how to check our solution. The main theme is this: The values of a and b must work in <u>both</u> of the <u>original</u> equations in order to constitute a valid solution.

1st equation:	2nd equation:
$12a + 7b = 9$	$-18a - 5b = 3$
$12(\mathbf{-1}) + 7(\mathbf{3}) \stackrel{?}{=} 9$	$-18(\mathbf{-1}) - 5(\mathbf{3}) \stackrel{?}{=} 3$
$-12 + 21 \stackrel{?}{=} 9$	$18 - 15 \stackrel{?}{=} 3$
$9 = 9$ ✓	$3 = 3$ ✓

Our final conclusion is that the values $a = -1$ and $b = 3$ work perfectly. The solution can also be written as the ordered pair $(-1, 3)$.

Homework

3. Solve each system using the Addition Method, and be sure you practice checking your solution (your pair of numbers) in <u>both</u> of the <u>original</u> equations:

a. $\begin{aligned} 2x + y &= 5 \\ -2x + 7y &= 19 \end{aligned}$

b. $\begin{aligned} 5a - 3b &= 5 \\ 10a + 4b &= -40 \end{aligned}$

c. $\begin{aligned} -2u - 3v &= -16 \\ -7u + 8v &= -56 \end{aligned}$

d. $\begin{aligned} 7x + 12y &= -24 \\ 6x - 7y &= 14 \end{aligned}$

e. $\begin{aligned} 3m - 2n &= 34 \\ -6m + n &= -62 \end{aligned}$

f. $\begin{aligned} -3s - 3t &= -24 \\ 10s + 8t &= 64 \end{aligned}$

g. $\begin{aligned} 2c - 3d &= 13 \\ 5c + 6d &= -8 \end{aligned}$

h. $\begin{aligned} -5w - 4x &= -20 \\ 20w + 3x &= 15 \end{aligned}$

i. $\begin{aligned} -5x - 4n &= -8 \\ 11x + 6n &= -2 \end{aligned}$

j. $\begin{aligned} 2w - 4a &= 6 \\ -3w + 9a &= -12 \end{aligned}$

k. $\begin{aligned} 2n - 3y &= -2 \\ 8n - 11y &= -2 \end{aligned}$

l. $\begin{aligned} 4c + 9y &= 4 \\ -5c - 11y &= 12 \end{aligned}$

m. $\begin{aligned} 5g - 2h &= -6 \\ 4g + 2h &= 3 \end{aligned}$

n. $\begin{aligned} -4w + 3h &= -1 \\ -3w + 4h &= 5 \end{aligned}$

o. $\begin{aligned} -3w + 4m &= 6 \\ -3w - m &= 1 \end{aligned}$

p. $\begin{aligned} 3a + 3q &= 1 \\ -5a + 5q &= 6 \end{aligned}$

❑ THE SUBSTITUTION METHOD

Another method that's really useful in solving systems of equations is the Substitution Method, where a variable is solved for in one equation, and then its value is substituted into another equation. Let's get right to an example.

<u>EXAMPLE 3:</u> **Solve the system:**

$$\begin{aligned} \mathbf{3x + y} &= \mathbf{12} \\ \mathbf{-5x + 2y} &= \mathbf{13} \end{aligned}$$

<u>Solution:</u> The first step is to choose an equation and a variable within that equation to solve for. For this system, the best variable to solve for is the y in the first equation. Why y? Because its coefficient is 1, which means we avoid fractions when solving for y.

Solving the first equation for y gives us

$y = 12 - 3x$ (by subtracting $3x$ from each side)

Now we substitute the expression $12 - 3x$ into the second equation where the y is.

$-5x + 2y = 13$ (the second equation)

$\Rightarrow$ $-5x + 2(\mathbf{12 - 3x}) = 13$ (since $y = 12 - 3x$)

$\Rightarrow$ $-5x + 24 - 6x = 13$ (distribute)

$\Rightarrow$ $-11x + 24 = 13$ (combine the x's)

$\Rightarrow$ $-11x = -11$ (subtract 24 from each side)

$\Rightarrow$ $\underline{x = 1}$ (divide each side by -11)

Since $y = 12 - 3x$, and we just found out that $x = 1$, we can solve for y:

$$\begin{aligned} y &= 12 - 3(\mathbf{1}) \\ y &= 12 - 3 \\ \underline{y} &\underline{= 9} \end{aligned}$$

$$\boxed{x = 1 \;\&\; y = 9}$$

Notes: First, if you'd like to check our solution, substitute the values of x and y into both of the original equations. Second, what if none of the variables has a coefficient of 1? Well, you'll just have to pick a variable and then deal with whatever fractions may arise. In the real world, however, it's probably best to solve the system using the Addition Method.

Homework

4. Solve each system using the Substitution Method, and be sure you practice checking your solution (your pair of numbers) in both of the original equations:

a. $\begin{aligned} 2x + y &= 5 \\ -2x + 7y &= 19 \end{aligned}$

b. $\begin{aligned} 3m - 2n &= 34 \\ -6m + n &= -62 \end{aligned}$

c. $\begin{aligned} -3w + 4m &= 6 \\ -3w - m &= 1 \end{aligned}$

Review Problems

5. Consider the system of equations $\begin{aligned} x - y &= 12 \\ x + y &= 8 \end{aligned}$

   a. Show that $x = 15$ and $y = 3$ is a solution of the first equation, but is not a solution of the system.

   b. Show that $x = 7$ and $y = 1$ is a solution of the second equation, but is not a solution of the system.

   c. Show that $x = 10$ and $y = -2$ is a solution of the system.

6. Solve the system of equations by Addition: $\begin{aligned} 7x - 3y &= 8 \\ -9x + 5y &= -8 \end{aligned}$

7. Solve the system of equations by Substitution:
$$\begin{aligned} 7a + 4b &= -1 \\ a - 3b &= -18 \end{aligned}$$

Solutions

1. a. $5 + 4 = 9$ ✓ But $5 - 4 = 1 \neq 7$.

b. $20 - 13 = 7$ ✓ But $20 + 13 = 33 \neq 9$.

c. $8 + 1 = 9$ ✓ And $8 - 1 = 7$ ✓

2. Both variables = 6, since 6 + 6 = 12, and 6 – 6 = 0.

3. a. $x = 1$, $y = 3$

Complete Check:

$2x + y = 5$	$-2x + 7y = 19$
$2(\mathbf{1}) + \mathbf{3} = 5$	$-2(\mathbf{1}) + 7(\mathbf{3}) = 19$
$2 + 3 = 5$	$-2 + 21 = 19$
$5 = 5$ ✓	$19 = 19$ ✓

b. $a = -2$, $b = -5$ c. $u = 8$, $v = 0$ d. $x = 0$, $y = -2$

e. $m = 10$, $n = -2$ f. $s = 0$, $t = 8$ g. $c = 2$, $d = -3$

h. $w = 0$, $x = 5$ i. $x = -4$, $n = 7$ j. $w = 1$, $a = -1$

k. $n = 8$, $y = 6$ l. $c = -152$, $y = 68$ m. $g = -\frac{1}{3}, h = \frac{13}{6}$

n. $w = \frac{19}{7}, h = \frac{23}{7}$ o. $w = -\frac{2}{3}, m = 1$ p. $a = -\frac{13}{30}, q = \frac{23}{30}$

4. a. $x = 1$, $y = 3$ b. $m = 10$, $n = -2$ c. $w = -\frac{2}{3}, m = 1$

5. a. $15 - 3 = 12$ ✓ But $15 + 3 = 18 \neq 8$

b. $7 + 1 = 8$ ✓ But $7 - 1 = 6 \neq 12$

c. $10 - (-2) = 12$ ✓ And $10 + (-2) = 8$ ✓✓

6. $x = 2$, $y = 2$ **7.** $a = -3$, $b = 5$

"To educate a man in mind and not in morals is to educate a menace to society."

Theodore Roosevelt
(1858–1919)