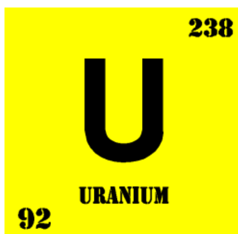
CH 24 – EXPONENTIAL EQUATIONS PART I

Watching your investments grow, tracking populations, the decaying of a radioactive substance – these are the kinds of problems which lead to exponential equations.



Uranium, atomic number 92, decays exponentially – the less that remains, the less that decays.

❑ Introductory Examples

An ***exponential equation*** is an equation with the variable in the exponent, not something we're used to seeing.

EXAMPLE 1: **Solve for x: $3^{7x} = 3^{14}$**

Solution: Each side of the equation is an exponential expression. Notice that the bases are the same (the 3's), so the only way the two sides of the equation can be equal is if the exponents are equal. In other words, $7x$ must equal 14,

$$7x = 14$$

from which we determine that

$$\boxed{x = 2}$$

<u>EXAMPLE 2:</u> **Solve for y: $5^{4y} = 25$**

<u>Solution:</u> We're not as lucky here as we were in Example 1 -- the bases are not the same. But maybe we can make them the same. Suppose we think of 25 as 5^2. Then each side of the equation will have the same base, and we can set the exponents equal to each other and find the value of y. Let's try all of this:

$5^{4y} = 25$ (the original equation)

$\Rightarrow$ $5^{4y} = 5^2$ (rewrite 25 with a base of 5)

$\Rightarrow$ $4y = 2$ (set the exponents equal to each other, since the bases are the same)

$\Rightarrow$ $\boxed{y = \frac{1}{2}}$

<u>EXAMPLE 3:</u> **Solve for z: $27^{-4z} = \frac{1}{9}$**

<u>Solution:</u> This one's terrible! The bases aren't the same, and it contains a fraction. But look at the 27 in the equation -- it's equal to 3^3. And check out the 9 in the denominator -- it can be written as 3^2. So maybe we can write everything in terms of the base **3**:

$27^{-4z} = \frac{1}{9}$ (the original equation)

$\Rightarrow$ $(3^3)^{-4z} = \frac{1}{3^2}$ (express 27 and 9 as powers of 3)

$\Rightarrow$ $3^{-12z} = 3^{-2}$ (law of exponents & negative exponent)

$\Rightarrow$ $-12z = -2$ (bases are the same - set exponents equal)

$\Rightarrow$ $\boxed{z = \frac{1}{6}}$ (solve for z)

<u>EXAMPLE 4:</u> **Solve each equation for *x*:**

A. $e^{5x} = e^{20}$

$$\Rightarrow\ 5x = 20 \ \Rightarrow\ \boldsymbol{x = 4}$$

B. $e^{2x+1} = (e^3)^{x-5}$

$$\Rightarrow\ e^{2x+1} = e^{3x-15} \ \Rightarrow\ 2x+1 = 3x-15$$

$$\Rightarrow\ -x = -16 \ \Rightarrow\ \boldsymbol{x = 16}$$

C. $\dfrac{1}{e} = e^{2x-1}$

$$\Rightarrow\ e^{-1} = e^{2x-1} \ \Rightarrow\ -1 = 2x-1 \ \Rightarrow\ \boldsymbol{x = 0}$$

D. $\dfrac{1}{\sqrt[3]{e}} = \left(\dfrac{1}{e^2}\right)^x$

$$\Rightarrow\ \frac{1}{e^{1/3}} = \frac{1^x}{(e^2)^x} \ \Rightarrow\ \frac{1}{e^{1/3}} = \frac{1}{e^{2x}}$$

$$\Rightarrow\ \frac{1}{3} = 2x \ \Rightarrow\ \boldsymbol{x = \frac{1}{6}}$$

<u>EXAMPLE 5:</u> **Solve for *x*: $\boldsymbol{3^x = 5}$**

<u>Solution:</u> Now we're really up a stump! How can we make the bases the same? We can't, so our method for solving exponential equations dies at this point. In fact, the solution to this equation will not be a rational number as in the previous four examples, so the best we can do is approximate it. Here's one way to solve the problem, using a calculator.

This is simply a guess-and-check method. Clearly, the solution x of the equation $3^x = 5$ is bigger than 1, since $3^1 = 3$; but x must be smaller than 2, since $3^2 = 9$. So the solution is between 1 and 2:

$$1 < x < 2$$

Now try 1.5. $3^{1.5} = 5.196$, a little too big. Let $x = 1.4$; then $3^{1.4} = 4.656$, a little too small. Thus, we've narrowed the solution to a number between 1.4 and 1.5:

$$1.4 < x < 1.5$$

Since 1.45 is halfway between 1.4 and 1.5, let's give it a try. $3^{1.45} = 4.918$, which is really close to 5. Consider 1.46: $3^{1.46} = 4.973$. Now nudge x up to 1.47: $3^{1.47} = 5.028$, a little too big. So the value of x is between 1.46 and 1.47:

$$1.46 < x < 1.47$$

This is getting boring, but I hope you're getting the point. Let's just take the average of 1.46 and 1.47, and figure we're close enough.

Our best guess is therefore

$$\boxed{x = 1.465}$$

In fact, $3^{1.465} = 5.0001$, so we have a really accurate value of x. The following chart is a summary of the calculations we made in this problem:

x	1	1.4	1.45	1.46	1.465	1.47	1.5	2
3^x	3	4.656	4.918	4.973	5.0001	5.028	5.196	9

In the next chapter we will study what are called "log" functions. Then we'll have a third (and much better) way of solving any kind of exponential equation, even if the base is e.

Homework

Solve each equation for exact solutions:

1. $2^{7x+1} = 2^{3x}$
2. $6^{8y-1} = 36$
3. $8^{-6z} = \frac{1}{2}$
4. $\frac{1}{25^x} = 125^{3x-1}$
5. $7^{3x} = 7^{5x}$
6. $81^{4-3x} = \sqrt{729}$
7. $e^{2x+1} = e$
8. $\frac{1}{e^2} = e^{1-7x}$
9. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e}$

Use your calculator to approximate the solution of each equation:

10. $2^x = 6$
11. $10^y = 50$
12. $e^z = 12$

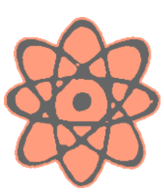
13. Solve by inspection (this means stare at it and guess): $8^x = 5^x$

❑ The Growth and Decay Formula

Many things grow and shrink (decay) at a rate that's based on how much of the stuff there is at the moment.

For instance, the number of births in a population is based on the number of people living at the time (since the more people there are, the more new people there will be).

Another example is compound interest: As money accumulates (due to earned interest) in the account, the more interest the account will earn; that is, the interest will earn interest.



In science, we can observe the radioactive decay of an element like uranium. As the uranium begins to disintegrate, less of it will decay, because as time goes on, there's less of it left to decay.

Here's a formula to help you predict how much of something there will be in the future. It works for population growth, continuous compounding of interest, and the decay of radioactive substances:

Let

A = ending amount
A_0 = starting amount (read: "A sub zero" or "A naught")
e = the constant from Chapter 23
k = growth or decay rate
t = time

Then

$$A = A_0 e^{kt}$$

<u>EXAMPLE 6:</u> **Assuming an initial population of 1506, and a growth rate of 25% per year compounded continuously, predict the population in 10 years.**

<u>Solution:</u> Let's start by writing the formula that will solve this problem:

$$A = A_0 e^{kt}$$

The initial population is 1506; so $A_0 = 1506$.

The growth rate is 25%; thus $k = .25$.

We're talking about a period of 10 years; therefore $t = 10$.

Plug all these values into our formula:

$$A = 1506e^{(.25)(10)}$$

$$\Rightarrow \quad A = 1506e^{2.5}$$

$$\Rightarrow \quad A = 1506(12.1825)$$

$$\Rightarrow \quad \boxed{A = 18{,}347}$$

EXAMPLE 7: **Assuming a starting investment of $2401, and an annual interest rate of 13% compounded continuously, predict the value of the investment in 11 years.**

Solution: The formula we used for population growth works equally well for compound interest.

$$A = A_0e^{kt}$$

$$\Rightarrow \quad A = 2401e^{(.13)(11)}$$

$$\Rightarrow \quad A = 2401e^{1.43}$$

$$\Rightarrow \quad A = 2401(4.1787)$$

$$\Rightarrow \quad \boxed{A = \$10{,}033}$$

EXAMPLE 8: **Between 1950 and 1965 the population was growing at a rate of 6% per year. If the population in 1965 was 32,400, what was the population back in 1950?**

Solution: We'll continue to use the growth formula, but notice that in contrast to the two previous examples, this problem asks for the initial amount, not the ending amount. No problem -- we know algebra.

$$A = A_0 e^{kt}$$

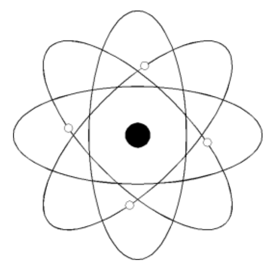
$\Rightarrow$ $A_0 = \dfrac{A}{e^{kt}}$ (solve the equation for A_0)

$\Rightarrow$ $A_0 = \dfrac{32,400}{e^{(.06)(15)}}$ (1950 to 1965 implies 15 years)

$\Rightarrow$ $A_0 = \dfrac{32,400}{2.4596}$ ($e^{(.06)(15)} = e^{.9} = 2.4596$)

$\Rightarrow$ $\boxed{A_0 = 13,173}$

<u>EXAMPLE 9</u>: **Starting with 624 grams of uranium, and assuming an annual decay rate of 5%, compute the number of grams of uranium remaining after 9 years.**



<u>Solution</u>: We use the same growth/decay formula except for one thing: Since the amount of uranium is shrinking (decaying) rather than growing, we will use a decay rate of –5% in our formula:

$$A = A_0 e^{kt}$$

$\Rightarrow$ $A = 624e^{(-.05)(9)}$

$\Rightarrow$ $A = 624e^{-.45}$

$\Rightarrow$ $A = 624(.6376)$

$\Rightarrow$ $\boxed{A = 397.9 \text{ grams}}$

Note: The growth and decay formula, $A = A_0\,e^{kt}$, has four variables in it (Which symbol in the formula is not a variable?). Conceivably, we should be able to solve for any one of them. But the examples we've done involved solving only for A or A_0. Solving for *k* or *t* would produce an exponential equation that can be solved only by the calculator guess-and-check method (see Example 5). Thus, we'll postpone solving for *k* or *t* until Chapter 28, when we'll have the proper tools to solve the growth and decay formula efficiently.

Homework

14. Explain why we can't solve a growth/decay formula when the rate or time is unknown.

15. Assuming an initial population of 23,900 and a growth rate of 7% per year compounded continuously, predict the population in 12 years.

16. Assuming a starting investment of $5575 and an annual interest rate of 12.5% compounded continuously, predict the value of the investment in 10 years.

17. Starting with 450 grams of plutonium and assuming an annual continuous decay rate of 12%, compute the number of grams of plutonium remaining after 15 years.

18. Mr. and Mrs. Abernathy want to have $50,000 in an annuity when their newborn daughter begins college in 18 years. If the annual interest rate of the annuity is 9%, how much must they invest in order to achieve their goal?

19. Assuming an initial population of 142,500 and a growth rate of 12.3% per year compounded continuously, predict the population in 10 years.

20. Assuming a starting investment of \$25,000 and an annual interest rate of 9.25% compounded continuously, predict the value of the investment in 15 years.

21. Starting with 5600 grams of uranium and assuming an annual decay rate of 0.7%, compute the number of grams of uranium remaining after 85 years.

22. Sarah wants to have \$75,000 in an IRA (individual retirement account) by the time she retires in 12 years. If the annual interest rate of the IRA is 9%, how much must Sarah invest in order to achieve her goal?

Practice Problems

23. Consider the equations $x^4 = 10$ and $4^x = 10$. Which is exponential?

24. Solve for x: $10^x = 17^x$ [to be solved by inspection]

25. Find the exact solution: $27^{x+1} = \frac{1}{9}$

26. Find the exact solution: $\frac{1}{125} = 25^{4-2x}$

27. Find the exact solution: $\frac{1}{e^3} = \left(e^{-5}\right)^{2x+1}$

28. Find an approximate solution: $6^x = 20$

29. Assuming an initial population of 200, and a growth rate of 12% per year compounded continuously, predict the population in 50 years.

30. Ms. Abraham invested $15,000 for her 2-year-old daughter's college education at an annual interest rate of 8% compounded continuously. How much will the investment be worth in 16 years?

31. Starting with 32 grams of thallium, how much thallium remains after 4 years, assuming a decay rate of 15% per year?

32. The population was 30,500 in 1900. What was the population in 1800 if the growth rate was 4% per year?

33. A radioactive enzyme decays at an annual rate of 0.5%. If 200 mg of the enzyme are present in the bloodstream today, how much of the enzyme remains after 18 months have passed?

34. What was the population back in 1950 if the population is 5,750 in 2002, assuming an annual growth rate of 7%?

35. True/False:

a. $3x^4 = 48$ is an exponential equation.

b. The solution of $\dfrac{1}{e^{-3x+4}} = \dfrac{e^{2x}}{e}$ is $x = 3$.

c. To the nearest 10th, the solution of $7^x = 20$ is $x = 1.6$.

d. $x = 0$ is a solution of the equation $e^x = 10^x$.

e. The formula $A = A_0e^{kt}$ assumes continuous growth or decay.

f. Assume an initial population of 2000, a growth rate of 15% per year compounded continuously, and a period of 10 years. At that point the population will be about 8963.

g. Starting with 200 g of plutonium and assuming an annual decay rate of 2%, the number of grams of plutonium remaining after 50 years is 12 g.

h. Mr. and Mrs. Garcia wish to have $150,000 in a college investment account when their newborn son begins college in 18 years. Assuming an annual interest rate of 12%, they need to fund their account with approximately $17,299.

Solutions

1. $-\frac{1}{4}$

2. $6^{8y-1} = 6^2 \Rightarrow 8y-1 = 2 \Rightarrow y = \frac{3}{8}$

3. $8^{-6z} = \frac{1}{2} \Rightarrow (2^3)^{-6z} = 2^{-1} \Rightarrow 2^{-18z} = 2^{-1}$

$\Rightarrow -18z = -1 \Rightarrow z = \frac{1}{18}$

4. $\frac{3}{11}$ **5.** 0 **6.** $\frac{13}{12}$ **7.** 0 **8.** $\frac{3}{7}$

9. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e} \Rightarrow \left(e^{-1/2}\right)^x = e^{1/4} \Rightarrow e^{-\frac{1}{2}x} = e^{\frac{1}{4}}$

$\Rightarrow -\frac{1}{2}x = \frac{1}{4} \Rightarrow x = -\frac{1}{2}$

10. 2.58 **11.** 1.7 **12.** 2.48

13. Hint: The solution is the only number with no reciprocal.

14. Because *k* and *t* are in the exponent of the formula, and we have no way of solving for these variables (yet).

15. 55,361 **16.** $19,459 **17.** 74.4 g **18.** $9895

19. 487,525 **20.** $100,121 **21.** 3089 g **22.** $25,470

23. The second one is exponential, since the variable is in the exponent. The first equation is a polynomial equation.

24. $x = 0$, since each side would then become 1.

25. $-\frac{5}{3}$ **26**. $\frac{11}{4}$ **27**. $-\frac{1}{5}$ **28**. 1.67195

29. 80,686 **30**. \$53,950 **31**. 17.56 g **32**. 559

33. 198.5 mg **34**. 151

35. a. F b. T c. F d. T e. T f. T g. F h. T

Not only is mathematics a rich and often beautiful product of the human mind - a major part of our culture - but there is almost no area of life that is not affected by it in deep and profound ways. Quite simply, mathematics is the "invisible universe."

Keith Devlin, Mathematician