CH 28 – EXPONENTIAL EQUATIONS, PART II

When we studied growth and decay back in Chapter 24, we found that we had no way to determine the growth rate or the amount of time. The previous chapter on the Laws of Logs will give us the necessary tools.

The exponential decay of radioactive elements allows us to determine the age of dinosaur bones.

❑ Exponential Equations

Remember the hassles we encountered in trying to solve the exponential equation $3^x = 5$? We had to use a calculator to guess an answer, and then adjust and guess again, and so on, until we had a couple of digits of accuracy. We're now ready to solve this problem more efficiently.

EXAMPLE 1: **Solve for x: $3^x = 5$**

Solution: The variable is in the exponent. This is a dilemma -- how do we get the unknown out of the exponent so that we can solve for it? Do you recall the third law of logs

$$\log_b a^x = x \log_b a \ ?$$

It allows us to move the exponent to the front (making it a coefficient), but only if we're taking the log of an expression. So the procedure here will be to take a log (we'll choose *ln*, since it's on your calculator and ln is used in calculus), bring down the exponent, and then solve for it.

	$3^x = 5$	(the original equation)
$\Rightarrow$	$\ln 3^x = \ln 5$	(take the *ln* of both sides)
$\Rightarrow$	$x \ln 3 = \ln 5$	(the third law of logs)
$\Rightarrow$	$x = \dfrac{\ln 5}{\ln 3}$	(simple algebra -- solve for x)
$\Rightarrow$	$x = \dfrac{1.609437912}{1.098612289}$	(use your calculator)
$\Rightarrow$	$\boxed{x = 1.464973521}$	

When we did this problem by guessing with a calculator, a lot of work produced a guess of 1.465. Clearly, our new method is superior. Note that the true answer, $x = \dfrac{\ln 5}{\ln 3}$, is an irrational number, while the answer in the box is a rational approximation of the true answer.

EXAMPLE 2: **Solve for n: $2^{3n+2} = 7$**

Solution:

	$2^{3n+2} = 7$	(the given equation)
$\Rightarrow$	$\ln 2^{3n+2} = \ln 7$	(take the *ln* of each side)
$\Rightarrow$	$(3n+2)\ln 2 = \ln 7$	(third law of logs)

[Notice the parentheses around the $3n + 2$]

$\Rightarrow \quad (3\ln 2)\,n + 2\ln 2 = \ln 7$ (distribute)

$\Rightarrow \quad (3\ln 2)n = \ln 7 - 2\ln 2$ (subtract 2 ln2)

$\Rightarrow \quad n = \dfrac{\ln 7 - 2\ln 2}{3\ln 2}$ (divide each side by 3 ln 2 to get the exact answer)

$\Rightarrow \quad \boxed{n = 0.269118307}$ (use your calculator to get a rational approximation)

EXAMPLE 3: **Solve for a: $5^{2a-3} = 6^{a+1}$**

Solution:

$5^{2a-3} = 6^{a+1}$ (the original equation)

$\Rightarrow \quad \ln 5^{2a-3} = \ln 6^{a+1}$ (take the *ln* of each side)

$\Rightarrow \quad (2a-3)\ln 5 = (a+1)\ln 6$ (third law of logs)

$\Rightarrow \quad 2a\ln 5 - 3\ln 5 = a\ln 6 + \ln 6$ (distribute)

$\Rightarrow \quad 2a\ln 5 - a\ln 6 = \ln 6 + 3\ln 5$ (variables to the left and constants to the right)

$\Rightarrow \quad a(2\ln 5 - \ln 6) = \ln 6 + 3\ln 5$ (factor out the variable)

$\Rightarrow \quad \boxed{a = \dfrac{\ln 6 + 3\ln 5}{2\ln 5 - \ln 6}}$ (divide to isolate the a)

This is the exact answer. A rational approximation would be $a = 4.638776075$.

Homework

Solve each equation and round your answers to the nearest ten thousandths place:

1. $2^x = 72$
2. $5^{-y} = 3$
3. $3^{4n-1} = 5$
4. $3^{z+1} = 8^{3-z}$
5. $3^{5c} = 7^{10c}$
6. $e^{3x+4} = 25$
7. $3^{-n} = 43$
8. $7^{4-3x} = 2$
9. $e^x = 2^{x-6}$

❑ The Growth and Decay Formula Revisited

The growth and decay formula

$$A = A_0 e^{kt}$$

worked just fine back in Chapter 24 when we were searching for either the starting amount A_0 or the final amount A. But when the unknown was in the exponent, we were stuck. Now we're not stuck.

EXAMPLE 4: **Assuming an initial population of 7500, a final population of 12,000, and a time period of 7 years, find the annual growth rate.**

Solution: We will write the growth formula, substitute the given values, and then solve for the unknown k:

$$A = A_0 e^{kt} \quad \text{(the growth formula)}$$

$$\Rightarrow \quad 12{,}000 = 7500 e^{k\cdot 7} \quad \text{(substitute the given values)}$$

$$\Rightarrow \quad e^{7k} = \frac{12{,}000}{7500} \quad \text{(isolate the } e^{7k}\text{)}$$

$$\Rightarrow \quad e^{7k} = 1.6 \quad \text{(calculator)}$$

$$\Rightarrow \quad \ln e^{7k} = \ln 1.6 \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad 7k\ln e = \ln 1.6 \qquad \text{(the third law of logs)}$$

$$\Rightarrow \quad 7k = \ln 1.6 \qquad (\ln e = 1)$$

$$\Rightarrow \quad k = \frac{\ln 1.6}{7} \qquad \text{(solve for } k\text{)}$$

$$\Rightarrow \quad k = 0.067 \qquad \text{(calculator gives a decimal)}$$

And therefore the annual growth rate is

6.7%

EXAMPLE 5: **How long will it take for an investment of \$10,000 to reach a total of \$32,000 if the interest rate is 7.3% per year compounded continuously?**

Solution: The phrase "compounded continuously" justifies the use of our growth formula.

$$A = A_0e^{kt} \qquad \text{(the growth formula)}$$

$$\Rightarrow \quad 32{,}000 = 10{,}000e^{0.073t} \qquad \text{(remember: } 7.3\% = .073\text{)}$$

$$\Rightarrow \quad e^{0.073t} = \frac{32{,}000}{10{,}000} = 3.2 \qquad \text{(divide each side by 10,000)}$$

$$\Rightarrow \quad \ln e^{0.073t} = \ln 3.2 \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad 0.073t = \ln 3.2 \qquad \text{(third law of logs \& } \ln e = 1\text{)}$$

$$\Rightarrow \quad t = \frac{\ln 3.2}{0.073} = 15.933 \qquad \text{(solve for } t\text{)}$$

Thus, the amount of time it will take to reach the goal is

15.933 years

EXAMPLE 6: **A population is growing at 13% per year, compounded continuously. How long will it take for the population to triple?**

Solution: It would appear that the answer should depend on the starting and ending populations. But an interesting aspect of the exponential growth/decay formula is that it doesn't matter. If it takes 10 years for the population to triple from 12 to 36, then it takes the same 10 years for the population to triple from 5,000 to 15,000. The following calculations should prove this.

$$A = A_0 e^{kt}$$

$$36 = 12e^{0.13t} \qquad\qquad 15{,}000 = 5000e^{0.13t}$$

In each case we divide by the initial population:

$$\frac{36}{12} = e^{0.13t} \qquad\qquad \frac{15{,}000}{5000} = e^{0.13t}$$

$$3 = e^{0.13t} \qquad\qquad 3 = e^{0.13t}$$

Under either scenario, we arrive at exactly the same equation:

$$e^{0.13t} = 3$$

$$\Rightarrow \quad \ln e^{0.13t} = \ln 3$$

$$\Rightarrow \quad 0.13t = \ln 3$$

$$\Rightarrow \quad t = \frac{\ln 3}{0.13} = 8.45$$

Therefore, the amount of time needed for the population to triple, regardless of the actual populations, is

$$\boxed{8.45 \text{ years}}$$

Half-life The ***half-life*** of a radioactive substance is the <u>time required for **half** of the material to disintegrate</u> into energy and other particles, which therefore means it's the time it takes for the final amount of the substance to be half of the starting amount. For instance, start with 100 grams of plutonium, and assume the half-life is 7 years. After 7 years, 50 g remain. After another 7 years, 25 g remain. Another 7 years and $12\frac{1}{2}$ g remain, etc., etc. The concept of half-life also pertains to the time it takes for the amount of medication remaining in the body to be half of its starting amount.

EXAMPLE 7: **Find the half-life of a radioactive substance which decays at an annual rate of 9%.**

Solution: Recall the preceding example on tripling the population. It didn't matter whether we went from 12 to 36 or from 5,000 to 15,000 -- the tripling time was the same either way. In the same manner, the half-life of a substance does not depend on the starting and ending amounts, as long as the ending amount is half of the starting amount. We can prove this contention perfectly by using variables instead of numbers for the starting and ending amounts.

Let's assume a starting amount of A_0 grams. When half of it decays, the remaining amount is half of A_0, or $\frac{1}{2}A_0$. In other words, $A = \frac{1}{2}A_0$.

$$A = A_0 e^{kt} \qquad \text{(the growth/decay formula)}$$

$$\Rightarrow \quad \tfrac{1}{2}A_0 = A_0 e^{-0.09t} \qquad \text{(note: } k \text{ is negative)}$$

$$\Rightarrow \quad \tfrac{1}{2} = e^{-0.09t} \qquad \text{(divide both sides by } A_0\text{)}$$

$$\Rightarrow \quad \ln\tfrac{1}{2} = \ln e^{-0.09t} \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad -0.09t = \ln\tfrac{1}{2}$$

$$\Rightarrow \quad t = \frac{\ln\frac{1}{2}}{-0.09} = 7.7$$

And thus the half-life of the substance is

$$\boxed{7.7 \text{ years}}$$

Homework

10. In the growth formula $A = A_0e^{kt}$, solve for
    a. A_0 b. k c. t

11. Find the interest rate if an investment of \$7200 reached a total of \$18,000 in 5 years.

12. If the population is growing 9% per year, how long will it take for a population of 25,600 to reach a population of 100,000?

13. How long will it take for a population to double in size if the growth rate is 18% per year?

14. Find the half-life of a radioactive substance whose annual decay rate is 23%.

15. At an annual rate of 8%, how many years will it take for you to quadruple your money?

16. The decay rate of a radioactive substance is k (where $k < 0$). Prove that the half-life of the substance is $t = \dfrac{-\ln 2}{k}$.

Practice Problems

17. Solve for a: $5^a = 3$

18. Solve for x: $T = T_0 e^{ax}$

19. Solve for x: $7^{3x-1} = 2^{6-5x}$

20. Solve for n: $2^{2n+3} = 3^{4-5n}$

21. Solve for b: $e^{3b} = 2^{b+1}$

22. Solve for x: $10^{7x-14} = 1$

23. Find the annual growth rate if a population increased from 2000 to 7,500 in a period of 9 years.

24. $25,000 in invested in a money market account paying 9.5% per year compounded continuously. How many years will it take for that investment to reach a total of $75,000?

25. How long would it take a population to quintuple in size if the growth rate is 12.7% per year?

26. Find the half-life of a substance whose annual decay rate is 6.3%.

27. True/False:

    a. The solution of the exponential equation $6^x = 4$ is about 0.774.

    b. The exact solution of the equation $2^{3x+2} - 7 = 0$ is $\frac{\ln 1.75}{\ln 8}$.

    c. If it took 12 years for the population to increase from 500 to 2,250, then the annual growth rate was 20%.

    d. If the amount in the 7%/year continuous compounding savings account went from $7,500 to $14,000, then the time span was about 8.916 years.

e. A population is growing at 23% per year compounded continuously. It should take about 1.76 years for the population to grow 50%.

f. The half-life of a substance which decays at an annual rate of 1.3% is about 35 years.

g. If $A = A_0e^{kt}$, then $t = \frac{1}{k}\ln\frac{A}{A_0}$.

Solutions

1. 6.1699 **2.** –0.6826

3. $3^{4n-1} = 5 \Rightarrow \ln(3^{4n-1}) = \ln 5 \Rightarrow (4n-1)\ln 3 = \ln 5$

$$\Rightarrow n = \frac{\frac{\ln 5}{\ln 3}+1}{4} \approx .6162$$

4. 1.6173

5. $3^{5c} = 7^{10c} \Rightarrow \ln(3^{5c}) = \ln(7^{10c}) \Rightarrow 5c\ln 3 = 10c\ln 7$

$\Rightarrow 5c\ln 3 - 10c\ln 7 = 0 \Rightarrow c(5\ln 3 - 10\ln 7) = 0$

$$\Rightarrow c = \frac{0}{5\ln 3 - 10\ln 7} \Rightarrow c = 0$$

6. –0.2604 **7.** –3.4236 **8.** 1.2146 **9.** –13.5533

10. a. $A_0 = \frac{A}{e^{kt}}$

b. $A = A_0e^{kt} \Rightarrow \frac{A}{A_0} = e^{kt} \Rightarrow \ln\frac{A}{A_0} = \ln e^{kt}$

$$\Rightarrow \ln A - \ln A_0 = kt \Rightarrow k = \frac{\ln A - \ln A_0}{t}$$

c. $t = \dfrac{\ln A - \ln A_0}{k}$

11. 18% **12**. 15 yrs **13**. 3.9 yrs **14**. 3 yrs **15**. 17.3 yrs

16. $A = A_0e^{kt}$. We assume that the final amount is half of the starting amount: $A = \frac{1}{2}A_0$. Also assume that $k < 0$, since it's a decay problem and not a growth problem.

$$\tfrac{1}{2}A_0 = A_0e^{kt} \Rightarrow \ln\tfrac{1}{2} = \ln e^{kt} \Rightarrow kt = \ln 1 - \ln 2$$

$$\Rightarrow t = \frac{0 - \ln 2}{k} \Rightarrow t = \frac{-\ln 2}{k} \qquad \text{Q.E.D.}$$

17. $a = \dfrac{\ln 3}{\ln 5} \approx 0.6826$

18. $x = \dfrac{\ln T - \ln T_0}{a}$

19. $x = \dfrac{6\ln 2 + \ln 7}{3\ln 7 + 5\ln 2} \approx 0.6562$

20. $n = \dfrac{4\ln 3 - 3\ln 2}{2\ln 2 + 5\ln 3} \approx 0.3365$

21. $b = \dfrac{\ln 2}{3 - \ln 2} \approx 0.3005$

22. $x = 2$

23. $k = 14.7\%$ **24**. 11.56 yrs **25**. 12.67 yrs **26**. 11 yrs

27. a. T b. T c. F d. T e. T f. F g. T

It's better to light a candle

than to curse the darkness.

Chinese proverb