CH 25 – CALCULATING LOGS

The **decibel scale** for measuring the loudness of sound, the **Richter scale** for the intensity of an earthquake, and the **pH scale** for acids and bases – all of these concepts are described using what's called a *logarithm*.

❑ Introduction

When we encountered an exponential function, say $g(x) = 10^x$, we would think up an x-value, 3 for example, and then calculate $g(3) = 10^3$, and our y-value would be 1000. That is, the input was 3 and the output was 1000.

Now we play Jeopardy: Using the same function as above, if I tell you that the output was 1000, could you give me the input; that is, could you tell me what exponent I need to make it work? In other words, can you solve the exponential equation

$$10^x = 1000 \text{ ?}$$

If you can, you have found the "**log** (base 10) of 1000," which we write as

$$\log_{10} 1000$$

Since $10^3 = 1000$, we write

$$\log_{10} 1000 = 3$$

In summary, the expression $\log_{10} 1000$ is asking "10 raised to <u>what power</u> makes 1000?" *A log is an exponent* -- the exponent needed on the given base to equal the specified number. *Log* is an abbreviation of the word logarithm.

One more way of describing a log: If you can fill in the box in the equation

$$5^{\square} = 125,$$

then you have found the "log, base 5, of 125" (which is 3).

This is really abstract, isn't it? Let's go right to some homework.

Homework

1. To find $\log_5 25$, which is read "log base 5 of 25," ask yourself "5 raised to what power equals 25?" $5^{\square} = 25$
2. To find $\log_2 8$, which is read "log base 2 of 8," ask yourself "2 raised to what power equals 8?" $2^{\square} = 8$
3. To find $\log_9 9$, which is read "log base 9 of 9,"ask yourself "9 raised to what power equals 9?" $9^{\square} = 9$
4. To find $\log_{17} 1$, which is read "log base 17 of 1,"ask yourself "17 raised to what power equals 1?" $17^{\square} = 1$
5. To find $\log_{100} 10$, which is read "log base 100 of 10,"ask yourself "100 raised to what power equals 10?" $100^{\square} = 10$
6. To find $\log_6 \frac{1}{6}$, which is read "log base 6 of 1/6,"ask yourself "6 raised to what power equals $\frac{1}{6}$?" $6^{\square} = \frac{1}{6}$

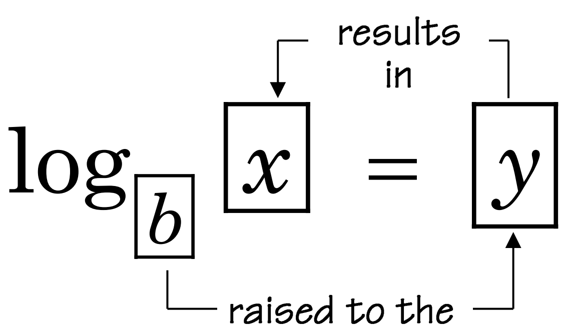
❑ Definition of Log

$$\log_b x = y \quad \text{means} \quad b^y = x$$

The notation "$\log_b x$" is read either
"log base *b* of *x*" or "log of *x*, base *b*"

NOTE: To make logs useful, we assume that the value of *b*, the base, is a positive real number not equal to 1. This is exactly what we did when we analyzed the possible bases of the exponential functions.

Here's another way to visualize the meaning of the log function:



<u>EXAMPLE 1:</u>

A. $\log_{10} 10{,}000 = \mathbf{4}$ because $10^4 = 10{,}000$

B. $\log_e e^2 = \mathbf{2}$ because $e^2 = e^2$

C. $\log_{17} 17 = \mathbf{1}$ because $17^1 = 17$

D. $\log_e 1 = \mathbf{0}$ because $e^0 = 1$

E. $\log_{25} 5 = \mathbf{\frac{1}{2}}$ because $25^{1/2} = \sqrt{25} = 5$

F. $\log_{64} 4 = \mathbf{\frac{1}{3}}$ because $64^{1/3} = \sqrt[3]{64} = 4$

G. $\log_8 4 = \mathbf{\frac{2}{3}}$ because $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

H. $\log_{13} \frac{1}{13} = \mathbf{-1}$ because $13^{-1} = \frac{1}{13}$

I. $\log_6 \frac{1}{36} = \mathbf{-2}$ because $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$

J. $\log_{49} \frac{1}{7} = -\mathbf{\frac{1}{2}}$ because $49^{-1/2} = \frac{1}{49^{1/2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

K. $\log_{1/2} \frac{1}{8} = \mathbf{3}$ because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

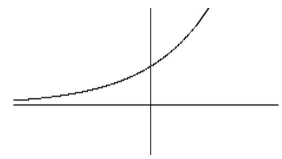
<u>EXAMPLE 2:</u>

A. $\log_b b = \mathbf{1}$ since $b^1 = b$

B. $\log_b 1 = \mathbf{0}$ since $b^0 = 1$

C. $\log_b b^n = \boldsymbol{n}$ since $b^n = b^n$

D. $\log_{10} 0$ is **undefined** because there's <u>no</u> exponent that you could raise 10 to and get an answer of 0. That is, there's no solution to the equation $10^x = 0$. Recall the graph of $y = 10^x$ -- it never has a y value of 0 because it's always above the x-axis.



E. $\log_5(-25)$ is **undefined** because there's <u>no</u> exponent that you could raise 5 to and get an answer of -25. This is due to the fact that there's no solution to the equation $5^x = -25$.

NOTE: What these 16 logs, especially the last two, should tell you is that the log function is defined only on positive numbers; that is, you can take the log of positive numbers only. Equivalently, you can never take the log of 0 or any negative number. In short, the <u>domain</u> of the function $y = log_b x$ is $(0, \infty)$.

Homework

Find the value of each log:

7.	a. $\log_{10} 100$	b. $\log_5 125$	c. $\log_8 64$	d. $\log_2 32$
8.	a. $\log_e e^5$	b. $\log_b b^2$	c. $\log_{\sqrt{2}} \sqrt{2}$	d. $\log_L L$
9.	a. $\log_{10} 1$	b. $\log_e 1$	c. $\log_{\sqrt[5]{99}} 1$	d. $\log_b 1$
10.	a. $\log_{36} 6$	b. $\log_{49} 7$	c. $\log_{144} 12$	d. $\log_b \sqrt{b}$
11.	a. $\log_5 \frac{1}{5}$	b. $\log_e \frac{1}{e}$	c. $\log_{1/e} 1$	d. $\log_n \frac{1}{n}$
12.	a. $\log_Q Q^n$	b. $\log_x 1$	c. $\log_{2.3} 2.3$	d. $\log_9 81$
13.	a. $\log_8 2$	b. $\log_{64} 4$	c. $\log_{125} 5$	d. $\log_a \sqrt[3]{a}$
14.	a. $\log_7 0$	b. $\log_8(-8)$	c. $\log_e 0$	d. $\log_2(-1)$

❑ Calculating Logs

The homework problems above were designed to be solved by inspection. Some logs aren't easy to do that way. So now we present a longer, but systematic way of evaluating logs by solving a certain exponential equation.

EXAMPLE 3: **Calculate: $\log_{27} 9$**

Solution: Let's give our log expression a name -- call it y. Now we can write an equation:

$$y = \log_{27} 9$$

The definition of log shows us how we can convert our log equation into an exponential equation:

$$27^y = 9$$

And now we solve for y. Chapter 24 showed us how:

$$27^y = 9 \Rightarrow (3^3)^y = 3^2 \Rightarrow 3^{3y} = 3^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

But y was the name we gave to the original log problem. So we can conclude that

$$\boxed{\log_{27} 9 = \frac{2}{3}}$$

To check our result, we can raise 27 to the $\frac{2}{3}$ power and make sure it comes out 9:

$$27^{2/3} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9 \quad \checkmark$$

Homework

Find the value of each log:

15. $\log_{64} 16$
16. $\log_{25} \frac{1}{5}$
17. $\log_{16} 8$
18. $\log_{27} \frac{1}{9}$
19. $\log_{8} 16$
20. $\log_{32} \frac{1}{2}$
21. $\log_{1/3} \frac{1}{9}$
22. $\log_{1/4} 64$
23. $\log_{9} 1$
24. $\log_{\pi} \pi^5$
25. $\log_{\pi} \sqrt{\pi}$
26. $\log_{\sqrt{2}} 32$

❑ Two Important Bases

The real numbers 10 and *e* are the two most important bases for logs. A third important base, especially in computers, is 2.

Notation:

Since 10 and *e* are the most-used bases for logs, we have special math notation for logs with these bases (computers may do it differently):

$\log_{10} x$ is called the "**common log**" and is abbreviated $\boldsymbol{\log x}$.

$\log_{e} x$ is called the "**natural log**" and is abbreviated **ln *x***. This can be read "log base-*e* of *x*" or "el en of *x*."

<u>EXAMPLES:</u>

$\log 100 = 2$ (the understood base is 10)

$\log 47.3 = 1.6749$ (use your calculator)

$\ln e^5 = 5$ (the understood base is *e*)

$\ln 47.3 = 3.8565$ (use your calculator)

Homework

27. Find the value of each log -- no calculator:

a. $\log 1000$ b. $\log 0.01$ c. $\log \sqrt{10,000}$

d. $\log 10^{100}$ e. $\log 1$ f. $\log 0$

g. $\log (-10)$ h. $\log 10$ i. $\log \frac{1}{10}$

j. $\log 10000$ k. $\log (-1)$ l. $\log 0.001$

The *common* log has a base of 10.

28. Find the value of each log -- no calculator:

a. $\ln e^{99}$ b. $\ln e$ c. $\ln \sqrt[3]{e}$

d. $\ln \frac{1}{e}$ e. $\ln 1$ f. $\ln 0$

g. $\ln (-1)$ h. $\ln \frac{1}{e^5}$ i. $\ln \sqrt[5]{e^4}$

j. $(\ln e)^e$ k. $\ln (e^e)$ l. $3^{\ln e}$

The *natural* log has a base of *e*.

29. Find the value of each expression -- use your calculator:

a. $\log 83$ b. $\ln 83$ c. $\log 1,000,000$

d. $\ln 1$ e. $\log 1$ f. $\log 0$

g. $\ln (-e)$ h. $\log (-1)$ i. $\log 10^{2.7}$

j. $\ln e^{-5.9}$ k. $10^{\log 7}$ l. $e^{\ln 13}$

30. Use your calculator to prove that each statement is <u>false</u>:

a. $\log(12+50) = \log 12 + \log 50$

b. $\ln(8 \cdot 20) = (\ln 8)(\ln 20)$

c. $\log \frac{90}{35} = \frac{\log 90}{\log 35}$

d. $\ln(48-24) = \ln 48 - \ln 24$

e. $\log 7^5 = (\log 7)^5$

f. $\ln(1 \cdot 0) = \ln 1 + \ln 0$

❑ The pH Scale for Acids and Bases

One of the uses of logs is in the definition of the pH scale for acids and bases. The official definition of the pH of a substance is the negative of the common logarithm of the hydrogen ion concentration of the substance. Acids (like lemonade) have a pH smaller than 7, while bases (like Drano) have a pH higher than 7. The pH of water is a neutral 7.

We'll use the official chemistry notation for the hydrogen ion concentration, $[H^+]$, which has the units of moles/liter. It is not necessary to understand any of the chemistry -- indeed, the math takes care of everything. Devised by a biochemist while working on the brewing of beer, the pH of a substance is defined to be the negative common logarithm of the hydrogen ion concentration:

$$\text{pH} = -\log\,[H^+]$$

Calculator Hint: On a TI-30, to enter a number like 1.6×10^{-13}, first press 1.6, then press the "EE" button, and then press 13 +/–. Your display should then look like $1.6\ ^{-13}$ (the base of 10 is understood).

EXAMPLE 4: **Some orange juice has a hydrogen ion concentration of 2.9×10^{-4} moles/liter. Find the pH of the orange juice.**

Solution: According to the definition,

$$pH = -\log[H^+] = -\log(2.9 \times 10^{-4}) = -(-3.5376) \approx 3.54$$

According to the text next to the lemonade stand above, this should mean that orange juice is an acid, as indeed it is (the sour taste is one clue). Thus, the pH of the orange juice is

$$\boxed{3.54}$$

Homework

31. The hydrogen ion concentration of household ammonia is 1.26×10^{-12} moles/liter. Find the pH of the ammonia. Is it an acid or a base?

32. Pure water has a hydrogen ion concentration of 1.0×10^{-7} moles/liter. Prove that water has a neutral pH of 7.

33. Find the pH of each substance given its molarity:

a. 1.3×10^{-2} moles/L
b. 2.8×10^{-6} moles/L
c. 0.3×10^{-10} moles/L
d. 9.2×10^{-12} moles/L
e. 5.9×10^{-7} moles/L
f. 8.0×10^{-1} moles/L

Practice Problems

34. a. What is the common log of 100?
    b. What is the natural log of e^4?

35. a. Translate the expression $3^2 = 9$ to log form.
    b. Translate the expression $\log_5 125 = 3$ to exponent form.

36. a. $\log_8 \frac{1}{4} =$ b. $\log_b b =$ c. $\log_b 1 =$

37. a. $\ln \sqrt[3]{e^2} =$ b. $\log_b b^n =$ c. $\log(-1000) =$

38. Find the value of each log:

a. $\ln e^3$ b. $\ln e^{1/10}$ c. $\log 1{,}000{,}000$

d. $\ln e$ e. $\ln 1$ f. $\ln 0$

g. $\log 10$ h. $\log 1$ i. $\log 0$

j. $\log_{1/3} \frac{1}{81}$ k. $\ln \frac{1}{e}$ l. $\log_b \sqrt[7]{b}$

m. $\ln(-e)$ n. $\log_9 \frac{1}{27}$ o. $\log_\pi \sqrt[10]{\pi^7}$

39. Use your calculator to give some evidence that the statement $\log(23 \times 42) = \log 23 + \log 42$ is true.

40. Use your calculator to prove that the statement $\frac{\ln 10}{\ln 2} = \ln 5$ is false.

41. A very strong acid has a hydrogen ion concentration of $[H^+] = 1.3 \times 10^{-2}$ moles/L. Use the formula $pH = -\log [H^+]$ to calculate the pH of the acid.

42. True/False:

a. $\log 1000 + \ln e + \log_4 4 - \log_9 1 = 5$

b. $\log_{25} 5 = -\frac{1}{2}$

c. $\log_{1/3} \frac{1}{27} = 3$

d. $\log_b b^n + \log_b 1 - \log_b b = n - 1$

e. Both log 0 and ln (–1) are undefined.

f. The domain of the log function is $(0, \infty)$.

g. The domain of the ln function is $[0, \infty)$.

h. The natural log of e^4 is 4.

i. The common log of 10 is 0.

j. $10^{\log 8} + e^{\ln 12} = 20$ [You may use your calculator.]

k. $\ln(10 \times 15) = \ln 10 + \ln 15$ [You may use your calculator.]

l. $\log(100 - 10) = \log 100 - \log 10$ [You may use your calculator.]

m. $\ln 5^6 = 6 \ln 5$ [You may use your calculator.]

n. The negative common log of the hydrogen ion concentration is called Hp. [Hint: logs are used to make paper, and paper is used in printers, and Hp makes printers.]

o. A pH of 13 represents a strong acid.

p. A pH of 7 represents neutral.

q. A hydrogen ion concentration of 2.3×10^{-12} moles/L has a pH of about 11.6.

r. The exponential expression $b^a = c$ is equivalent to $\log_b a = c$.

Solutions

1. $5^{\boxed{?}} = 25$; since $5^2 = 25, \log_5 25 = 2$.

2. $2^{\boxed{?}} = 8$; since $2^3 = 8, \log_2 8 = 3$.

3. 1 **4**. 0

5. $100^{\boxed{?}} = 10$; since $100^{1/2} = \sqrt{100} = 10, \log_{100} 10 = \frac{1}{2}$. **6**. -1

7. a. 2 b. 3 c. 2 d. 5 **8**. a. 5 b. 2 c. 1 d. 1

9. a. 0 b. 0 c. 0 d. 0 **10**. a. $\frac{1}{2}$ b. $\frac{1}{2}$ c. $\frac{1}{2}$ d. $\frac{1}{2}$

11. a. -1 b. -1 c. 0 d. -1 **12**. a. n b. 0 c. 1 d. 2

13. a. $\frac{1}{3}$ b. $\frac{1}{3}$ c. $\frac{1}{3}$ d. $\frac{1}{3}$

14. a. Undefined b. Undefined c. Undefined d. Undefined

15. Let $y = \log_{64} 16 \Rightarrow 64^y = 16 \Rightarrow (4^3)^y = 4^2$

$\Rightarrow 4^{3y} = 4^2 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$

16. $-\frac{1}{2}$ **17**. $\frac{3}{4}$ **18**. $-\frac{2}{3}$ **19**. $\frac{4}{3}$ **20**. $-\frac{1}{5}$ **21**. 2

22. -3 **23**. 0 **24**. 5 **25**. $\frac{1}{2}$ **26**. 10

27. a. 3 b. –2 c. 2 d. 100 e. 0
f. Undefined g. Undefined h. 1 i. –1 j. 4
k. Undefined l. –3

28. a. 99 b. 1 c. $\frac{1}{3}$ d. –1 e. 0
f. Undefined g. Undefined h. –5 i. $\frac{4}{5}$ j. 1
k. e l. 3

29. a. 1.9191 b. 4.4188 c. 6 d. 0 e. 0
f. Undefined g. Undefined h. Undefined i. 2.7 j. –5.9
k. 7 l. 13

30. a. $\log(12+50) = \log 62 = 1.7924,$
but $\log 12 + \log 50 = 1.0792 + 1.6990 = 2.7782$

The rest of the problems are similar.

31. pH = 11.9; it's a base (an alkali), since its pH is greater than 7.

32. $\text{pH} = -\log[\text{H}^+] = -\log(1.0\times 10^{-7}) = -(-7) = 7$

33. a. 1.89 b. 5.55 c. 10.52 d. 11.04 e. 6.23 f. 0.10

34. a. 2 b. 4

35. a. $\log_3 9 = 2$ b. $5^3 = 125$

36. a. $-\frac{2}{3}$ b. 1 c. 0

37. a. $\frac{2}{3}$ b. n c. Undefined

38. a. 3 b. 1/10 c. 6 d. 1 e. 0 f. Undefined
g. 1 h. 0 i. Undefined j. 4 k. –1
l. 1/7 m. Undefined n. – 3/2 o. 7/10

39. $\log(23 \times 42) = \log 966 = 2.984977126$
$\log 23 + \log 42 = 1.361727836 + 1.62324929 = 2.984977126$
Note: This is not a perfect proof due to possible rounding errors by the calculator. But it's darned good evidence that the statement is true.

40. $\frac{\ln 10}{\ln 2} = \frac{2.302585093}{0.693147181} = 3.321928095$; $\ln 5 = 1.609437912$, not even close.

41. pH = 1.886

42.

a. T	b. F	c. T	d. T	e. T	f. T	g. F	h. T	i. F
j. T	k. T	l. F	m. T	n. F	o. F	p. T	q. T	r. F

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

Bertrand Russell