CH 29 – LOG EQUATIONS

Now that we have the laws of logs at our disposal, we can solve equations that have logs in them. We will also revisit the applications from previous chapters on the Richter scale, the pH scale, and the decibel scale.

Richter Scale

❑ Introduction

pH Scale

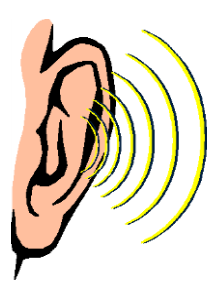
Some log equations can be solved by inspection. For instance, the solution of the equation $\log x = 2$ is $x = 100$, since $\log 100 = 2$. But some others aren't so easy. The key to solving log equations will be the conversion from a log expression to an exponential expression; this conversion is none other than the definition of log:

$$\log_b x = y \quad \text{means} \quad b^y = x$$

In addition, let's restate the 1st and 2nd laws of logs -- these will also be used in this chapter.

First Law of Logs: $\log_b(xy) = \log_b x + \log_b y$

Second Law of Logs: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$



Decibel Scale

❑ Level One Equations

EXAMPLE 1: **Solve for x: $\log_8 x - 2 = 0$**

Solution:

	$\log_8 x - 2 = 0$	(the original equation)
$\Rightarrow$	$\log_8 x = 2$	(begin to isolate the x)
$\Rightarrow$	$8^2 = x$	(change to exponential form)
$\Rightarrow$	$\boxed{x = 64}$	(calculate the value of x)

EXAMPLE 2: **Solve for y: $\log(3y + 5) = 1$**

Solution: Note: This is the common log, base 10.

	$\log(3y + 5) = 1$	(the original equation)
$\Rightarrow$	$10^1 = 3y + 5$	(change to exponential form)
$\Rightarrow$	$3y + 5 = 10$	(rearrange and simplify)
$\Rightarrow$	$3y = 5$	(subtract 5 from each side)
$\Rightarrow$	$\boxed{y = \frac{5}{3}}$	(divide each side by 3)

EXAMPLE 3: **Solve for x: $\ln(7 - 4x) = \frac{1}{2}$**

Solution: *ln* is just another log, and its base is understood to be e.

	$\ln(7 - 4x) = \frac{1}{2}$	(the original equation)
$\Rightarrow$	$e^{1/2} = 7 - 4x$	(convert to exponent form)

$$\Rightarrow \quad -4x+7 = \sqrt{e} \qquad \text{(rearrange equation)}$$

$$\Rightarrow \quad -4x = \sqrt{e}-7 \qquad \text{(subtract 7 from each side)}$$

$$\Rightarrow \quad x = \frac{\sqrt{e}-7}{-4} \qquad \text{(divide each side by } -4\text{)}$$

$$\Rightarrow \quad x = \frac{7-\sqrt{e}}{4} \qquad \text{(multiply top and bottom by } -1\text{)}$$

So the exact solution is

$$\boxed{x = \frac{7-\sqrt{e}}{4}}$$

We can use a calculator to approximate the solution as 1.338.

Homework

1. Solve each log equation:

a. $\log_6 x = 2$

b. $\log_5 x = 3$

c. $\log_2 x = \frac{1}{2}$

d. $\log_3 x = -4$

e. $\log_7(x+2) = 1$

f. $\log(2x-1) = 2$

g. $\ln x = 3$

h. $\ln(x-1) = 1$

i. $\ln(3x-5) = 0$

j. $\ln(2x+5) = \frac{1}{4}$

❑ Level Two Equations

EXAMPLE 4: **Solve for x: $\log x + \log 2 = 3$**

Solution: Be honest -- you don't like logs, and this equation has two of them in it! But we have a way out, using the first law of logs: $\log(ab) = \log a + \log b$

Let's begin:

	$\log x + \log 2 = 3$	(the original equation)
$\Rightarrow$	$\log(2x) = 3$	(first law of logs)
$\Rightarrow$	$10^3 = 2x$	(exponent form -- base is 10)
$\Rightarrow$	$2x = 1000$	(rearrange and simplify)
$\Rightarrow$	$\boxed{x = 500}$	(divide each side by 2)

EXAMPLE 5: **Solve for n: $\log(n - 3) = \log 5 + 4$**

Solution: For this problem, the second law of logs, $\log\frac{a}{b} = \log a - \log b$, comes into play:

	$\log(n - 3) = \log 5 + 4$	(the given equation)
$\Rightarrow$	$\log(n - 3) - \log 5 = 4$	(bring the logs together)
$\Rightarrow$	$\log\left(\frac{n-3}{5}\right) = 4$	(second law of logs)
$\Rightarrow$	$10^4 = \frac{n-3}{5}$	(convert to exponent form)
$\Rightarrow$	$n - 3 = 5(10^4) = 50{,}000$	(multiply each side by 5)
$\Rightarrow$	$\boxed{n = 50{,}003}$	(solve for n)

As a quick check, note that if we plug 50,003 into the original equation, we are taking the log of a positive number -- a good sign, since the log of a negative number or zero is undefined. Using a calculator, we can compute the following:

$\log(n - 3) = \log 5 + 4$

$\log(\mathbf{50{,}003} - 3) = \log 5 + 4$

$\log(50{,}000) = \log 5 + 4$

$4.69897 = 0.69897 + 4$

$4.69897 = 4.69897$ ✓

EXAMPLE 6: **Solve for z: $\log z + \log(z + 15) = 2$**

Solution: Let's just do it:

	$\log z + \log(z + 15) = 2$	(the given equation)
$\Rightarrow$	$\log[z(z + 15)] = 2$	(the first law of logs)
$\Rightarrow$	$\log(z^2 + 15z) = 2$	(distribute)
$\Rightarrow$	$10^2 = z^2 + 15z$	(change to exponent form)
$\Rightarrow$	$z^2 + 15z - 100 = 0$	(put in quadratic form)
$\Rightarrow$	$(z + 20)(z - 5) = 0$	(factor)
$\Rightarrow$	$z = -20$ OR $z = 5$	(set each factor to 0)

That yields two potential solutions, but only checking will tell which, if either, of the two numbers will work in the original equation:

$z = -20$: Immediate meltdown:

$\log z + \log(z + 15) = 2$

$\log(-20) + \ldots$ we can stop here;

this log is undefined; thus, –20 fails

z = 5: It might work. Using a calculator:

$\log z + \log (z + 15) = 2$

$\log 5 + \log (5 + 15) = 2$

$\log 5 + \log 20 = 2$

$.698970004 + 1.301029996 = 2$

$2 = 2$ ✓

Thus, the final solution to the equation is $\boxed{z = 5}$

EXAMPLE 7: **Solve for *x*:** $\mathbf{\log_2(x+5) = 3+\log_2(x-2)}$

Solution: Notice that the base of the log is 2, but the equation is really no more difficult than if it had a base of 10.

$\log_2(x+5) = 3+\log_2(x-2)$ (the original equation)

$\Rightarrow \log_2(x+5)-\log_2(x-2) = 3$ (bring the logs together)

$\Rightarrow \log_2\left(\frac{x+5}{x-2}\right) = 3$ (the second law of logs)

$\Rightarrow 2^3 = \frac{x+5}{x-2}$ (change to exponent form)

$\Rightarrow 8x - 16 = x + 5$ (clear the fraction)

$\Rightarrow 7x = 21$ (subtract x and add 16)

$\Rightarrow \boxed{x = 3}$

To check our candidate, $x = 3$, we see that we won't be able to use our calculator, since log base 2 is not on the machine:

$\log_2(x+5) \stackrel{?}{=} 3+\log_2(x-2)$

$\log_2(\boxed{3}+5) \stackrel{?}{=} 3+\log_2(\boxed{3}-2)$

$\log_2 8 \stackrel{?}{=} 3+\log_2 1$

$3 \stackrel{?}{=} 3+0$

$3 = 3$ ✓

<u>EXAMPLE 8</u>: **Solve for x: $\ln x^2 + \ln x + \ln 7 = 3$**

<u>Solution</u>: For this log equation, notice that the log has the understood base of e.

$$\ln x^2 + \ln x + \ln 7 = 3$$

$\Rightarrow \quad \ln (x^2 \cdot x \cdot 7) = 3$ (sum of logs = log of product)

$\Rightarrow \quad \ln (7x^3) = 3$ (simple algebra)

$\Rightarrow \quad e^3 = 7x^3$ (change to exponent form)

$\Rightarrow \quad x^3 = \dfrac{e^3}{7}$ (divide each side by 7)

$\Rightarrow \quad x = \sqrt[3]{\dfrac{e^3}{7}}$ (take the cube root of each side)

$\Rightarrow \quad x = \dfrac{\sqrt[3]{e^3}}{\sqrt[3]{7}}$ (split the radical)

$\Rightarrow \quad \boxed{x = \dfrac{e}{\sqrt[3]{7}}}$ (simplify the top)

An approximate answer, given by the calculator, is 1.421.

<u>EXAMPLE 9</u>: **Solve for a: $\ln a = \ln (a + 5) - 4$**

<u>Solution</u>: $\ln a = \ln (a + 5) - 4$ (the original equation)

$\Rightarrow \quad \ln a - \ln (a + 5) = -4$ (subtract $\ln (a + 5)$)

$\Rightarrow \quad \ln\left(\dfrac{a}{a+5}\right) = -4$ (second law of logs)

$\Rightarrow \quad \dfrac{a}{a+5} = e^{-4}$ (convert to exponent form)

$\Rightarrow \quad \dfrac{a}{a+5} = \dfrac{1}{e^4}$ (convert exponent to fraction)

$$\Rightarrow \quad ae^4 = a + 5 \qquad \text{(cross-multiply)}$$

$$\Rightarrow \quad ae^4 - a = 5 \qquad \text{(subtract } a \text{ from each side)}$$

$$\Rightarrow \quad a(e^4 - 1) = 5 \qquad \text{(factor out the variable)}$$

$$\Rightarrow \quad \boxed{a = \frac{5}{e^4 - 1}} \qquad \text{(divided each side by } e^4 - 1\text{)}$$

Homework

Solve each log equation:

2. $\log y + \log 3 = 4$
3. $\ln z = 1 + \ln 5$
4. $\log_3(w+1) - \log_3 w = 3$
5. $\log u + \log (u + 246) = 3$
6. $\log (2x + 1) = 1 - \log (3x - 4)$
7. $\log (2x + 1) - \log 5 = \log x$
8. $\ln (x + 2) - \ln (x - 3) = 2$
9. $\ln x - \ln (x + 5) = 4$
10. $\log_2 x^2 + \log_2 x + \log_2 3 = 5$
11. Solve each log equation:
    a. $\ln x + \ln 5 = 2$
    b. $\log a = 2 + \log 3$
    c. $\log_2(y+1) - \log_2 y = 4$
    d. $\log(u+24) + \log(u+3) = 2$
    e. $\log_5(8x+1) = 2 - \log_5(x-2)$
    f. $\ln(2x) - \ln(x-1) = 4$
    g. $\log_3 n^2 + \log_3 n + \log_3 7 = 2$
    h. $\log(5-x) + \log(x+15) = 2$
12. Check your solution to problem 11h.

❑ Log Applications Revisited

Now that we can solve equations containing logs, we will go back to our formulas for pH, decibels, and Richter numbers and solve more problems. Here are the three formulas from the previous chapters:

$\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles/liter

$D = 10\log\left(\dfrac{I}{10^{-12}}\right)$ where I is the intensity in watts/m^2

$M = \frac{2}{3}\log\left(\dfrac{E}{2.5\times10^4}\right)$ where E is the energy in joules

<u>EXAMPLE 10</u>: **If the pH of an acid is 2.3, find the hydrogen ion concentration.**

<u>Solution</u>: To simplify the symbols, let's use just the letter H to represent the concentration.

$$\text{pH} = -\log H$$

$$\Rightarrow \quad 2.3 = -\log H$$

$$\Rightarrow \quad \log H = -2.3$$

$$\Rightarrow \quad H = 10^{-2.3}$$, and so the concentration is

$$\boxed{0.005 \text{ moles/liter}}$$

<u>EXAMPLE 11</u>: **A certain sound has a decibel value of 85. Find the intensity of the sound.**

<u>Solution</u>: $D = 10\log\left(\dfrac{I}{10^{-12}}\right)$

$$\Rightarrow \quad 85 = 10\log\left(\frac{I}{10^{-12}}\right)$$

$$\Rightarrow \quad 8.5 = \log\left(\frac{I}{10^{-12}}\right)$$

$$\Rightarrow \quad \frac{I}{10^{-12}} = 10^{8.5}$$

$$\Rightarrow \quad I = 10^{-12} \times 10^{8.5}$$

$$\Rightarrow \quad I = 10^{-3.5}$$

$$\Rightarrow \quad \boxed{I = .00032 \text{ W/m}^2}$$

<u>EXAMPLE 12:</u> **An earthquake had a Richter magnitude of 6. Find the energy release in joules.**

<u>Solution:</u>

$$M = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$$

$$\Rightarrow \quad 6 = \frac{2}{3}\log\left(\frac{E}{2.5\times10^4}\right)$$

$$\Rightarrow \quad 9 = \log\left(\frac{E}{2.5\times10^4}\right)$$

$$\Rightarrow \quad \frac{E}{2.5\times10^4} = 10^9$$

$$\Rightarrow \quad E = 10^9 \times 2.5 \times 10^4$$

$$\Rightarrow \quad \boxed{E = 2.5 \times 10^{13} \text{ J}}$$

Homework

13. The pH of a strong base is 12.3. Find the hydrogen ion concentration.

14. Find the intensity of a sound wave whose decibel value is 50.

15. Calculate the energy released in an earthquake of magnitude 4.5 on the Richter scale.

16. The pH of a weak acid is 5.8. Find the hydrogen ion concentration.

17. Find the intensity of a sound wave whose decibel value is 100.

18. Calculate the energy released in an earthquake of magnitude 7.8 on the Richter scale.

Practice Problems

19. Solve for x: $\log_3(x+4) - \log_3 x = 4$

20. Solve for x: $\ln x^3 + \ln x^2 - \ln x - 2 = 0$

21. Solve for x: $\log(x-3) = \log x - 3$

22. Solve for x: $\log_2(x+14) = 5 - \log_2 x$

23. a. Find the hydrogen ion concentration of a base with pH = 12.

b. Find the intensity of a sound whose decibel level is 40.

c. Find the energy released in an earthquake of Richter 3.

24. True/False:

a. The equation $\log_2 x = 0$ has a solution.

b. The solution of $\ln(7-4x) = \frac{1}{2}$ is $x = \frac{1}{14-8x}$.

c. The solution of $\log x + \log 5 = 4$ is $x = 2000$.

d. The solution of $\log_3(x+1) - \log_3(x-1) = 3$ is $x = 14/13$.

e. The equation $\log n = 2 - \log(n+15)$ has two solutions.

f. The solution of $\ln x^3 + \ln x^2 + \ln x + \ln 2 = 10$ is $x = \sqrt[6]{\frac{e^{10}}{2}}$.

g. If an acid has a pH of 2.5, then its hydrogen ion concentration is 0.02 moles/liter.

h. The intensity of a sound whose decibel level is 125 is 3.1623 W/m^2.

i. If the Richter magnitude of an earthquake is 9, then the energy release is 7.9057×10^{17} joules.

Solutions

1. a. 36 b. 125 c. $\sqrt{2}$ d. $\frac{1}{81}$ e. 5

f. $\frac{101}{2}$ g. e^3 h. $e+1$ i. 2 j. $\frac{\sqrt[4]{e}-5}{2}$

2. $\frac{10{,}000}{3}$ **3.** $5e$, which is ≈ 13.5914 **4.** $\frac{1}{26}$

5. -250 is an extraneous solution; the solution is $\{4\}$.

6. You should reach the quadratic equation $6x^2 - 5x - 14 = 0$, whose two solutions are 2 and $-\frac{7}{6}$. Only 2 works.

7. $\frac{1}{3}$

8. $\ln(x+2) - \ln(x-3) = 2 \Rightarrow \ln\frac{x+2}{x-3} = 2 \Rightarrow \frac{x+2}{x-3} = e^2$

$\Rightarrow x+2 = e^2x - 3e^2 \Rightarrow x - e^2x = -3e^2 - 2$

$\Rightarrow x(1-e^2) = -3e^2 - 2 \Rightarrow x = \frac{-3e^2-2}{1-e^2}$, or $\frac{3e^2+2}{e^2-1}$

9. $x = \frac{5e^4}{1-e^4}$ is the solution obtained by following the rules. The problem is that this value of x is negative. Placing this value of x into the original equation yields an error; the final solution is therefore $\varnothing$.

10. You should get the equation $3x^3 = 32$.

Solving for x gives $\sqrt[3]{\frac{32}{3}}$ or $2\sqrt[3]{\frac{4}{3}}$ or $\frac{2\sqrt[3]{36}}{3}$.

11. a. $\frac{e^2}{5}$ b. 300 c. $\frac{1}{15}$ d. 1

e. 3 f. $\frac{e^4}{e^4-2}$ g. $\sqrt[3]{\frac{9}{7}}$ h. –5

12. We suppose that $x = -5$ is a solution of $\log(5-x) + \log(x+15) = 2$.

Placing –5 in for x in the original equation gives

$$\log(5-(-\mathbf{5})) + \log(-\mathbf{5}+15) = \log(5+5) + \log(-5+15)$$
$$= \log 10 + \log 10 = 1+1 = 2$$

It works.

13. 5.01×10^{-13} moles/liter

14. 10^{-7} W/m^2

15. 1.41×10^{11} joules

16. 0.000001585 moles/liter

17. 0.01 W/m^2

18. 1.253×10^{16} J

19. $x = \frac{1}{20}$

20. $x = \sqrt{e}$

21. $x = \frac{1000}{333}$

22. $x = 2$

23. a. 1×10^{-12} moles/liter b. 1×10^{-8} W/m^2 c. 7.91×10^{8} joules

24. a. T b. F c. T d. T e. F f. T g. F h. T i. T

Education is what survives when what has been learned has been forgotten.

BF Skinner