CH 0 – PROLOGUE

❑ THE REAL NUMBERS

You have seen many different kinds of numbers in your previous math courses. Consider these examples:

$$7 \qquad 0 \qquad -9 \qquad 2.835 \qquad -\frac{15}{4} \qquad \frac{2}{3} \qquad \frac{364}{495} \qquad \pi \qquad -\sqrt{2}$$

As varied as all these numbers may seem, they actually have one critical <u>common</u> characteristic: They can all be written as ***decimal numbers***, some repeating and some non-repeating:

$7 = 7.0$	Repeating decimal (zeros forever)
$0 = 0.0$	Repeating decimal (zeros forever)
$-9 = -9.0$	Repeating decimal (zeros forever)
2.835	Repeating decimal (zeros forever)
$-\frac{15}{4} = -3.75$	Repeating decimal (zeros forever)
$\frac{2}{3} = 0.6666666\ldots$	Repeating decimal (6's forever)
$\frac{364}{495} = 0.7353535\ldots$	Repeating decimal (35's forever)
$\pi = 3.14159265\ldots$	Non-repeating decimal
$-\sqrt{2} = -1.41421356\ldots$	Non-repeating decimal

> The number **π** is the ratio of the circumference of any circle to its diameter.
>
> **$\sqrt{2}$** is the length of the hypotenuse of a right triangle where each leg has length 1.

Note that some of the decimals listed above repeat a block of digits forever (the ***rational*** numbers), while some don't repeat (the ***irrational*** numbers). Nevertheless, they are all decimals.

> The repeating decimals are called ***rational numbers***, and the non-repeating decimals are called ***irrational numbers.***

But what about a number like $\sqrt{-9}$? This number must be a number whose square is –9. Now, what number do we know which, when squared, would come out –9? Does 3 work? No, since $3^2 = 9$. Does –3 work? No, since $(-3)^2 = 9$, also. We conclude that there is no decimal in the world that can represent the number $\sqrt{-9}$, or for that matter, the square root of any negative number.

To distinguish between the numbers that are decimals and numbers like $\sqrt{-9}$, which can never be written as a decimal, the term ***real number*** was given to the decimals, and the term ***imaginary number*** was given to numbers like $\sqrt{-9}$.

> The ***Real Numbers*** is the combination of the *Rational Numbers* and the *Irrational Numbers*.

Hundreds of years ago, mathematicians thought it was obvious which numbers were real and which were imaginary. But this demonstrates a rather arrogant attitude. After all, to a beginning algebra student, a real number like $\sqrt{2}$ (which is an infinite, non-repeating decimal) may not seem "real" at all. Moreover, imaginary numbers, like $\sqrt{-1}$, seem very real to people (such as electronics engineers) who use them every day. The bottom line is, the terms *real* and *imaginary* are completely arbitrary — one person's reality is another's imagination. But we're stuck with the terms, so we might as well learn them.

In summary, we call any number that can be written as a decimal a ***real number***, whether or not it repeats. The set of real numbers is often denoted by writing $\mathbb{R}$.

Homework

1. Classify each number as **real** or **imaginary**. If it's real, further classify as **rational** or **irrational**:

a. 123 b. -42 c. 0

d. 2.3 e. $\sqrt{-8}$ f. $\sqrt{144}$

g. $-\sqrt{81}$ h. $\sqrt{10}$ i. -23.78

j. $-\pi$ k. $\sqrt{3}$ l. $\sqrt{-121}$

m. 0.239057 n. $2.787878...$ o. 3.092748526

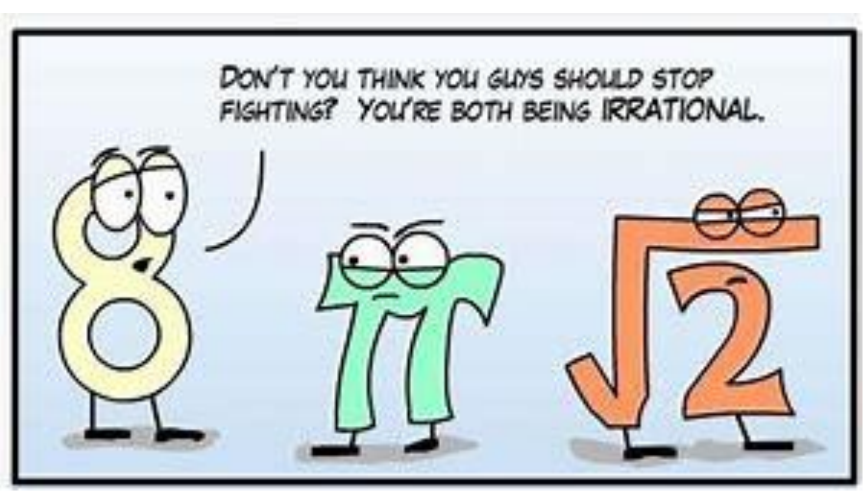
p. $3.1428669...$ q. $\sqrt{-(-8765)}$ r. $\sqrt{-0.25}$

s. $-\sqrt{-25}$ t. $\sqrt{-(-71)}$ u. 10^6

2. Put the following 13 real numbers in ascending order (smallest to biggest):

$$2\pi, \quad \sqrt{5}, \quad \sqrt{0}, \quad \frac{11}{3}, \quad 3.0808..., \quad -\pi,$$

$$\frac{1}{101}, \quad -\sqrt{3}, \quad 2^3, \quad 3^2, \quad -1, \quad \sqrt{25}, \quad \sqrt{1}$$



❑ THREE OF THE THINGS WE DO TO REAL NUMBERS

Opposite

The ***opposite*** of a real number is found by changing the sign of the number. For example, the opposite of 7 is -7, the opposite of $-\pi$ is π, and the opposite of 0 is 0 (since 0 doesn't really have a sign). The opposite of n is $-n$, and the opposite of $-n$ is n. Also notice that <u>the sum of a number and its opposite is always 0</u>; for example, $17 + (-17) = 0$.

When considering numbers on the real number line, two numbers are opposites of each other if they're the same distance from 0, but on opposite sides of 0. [Note that although <u>0 is the opposite of 0</u>, it's kind of hard to justify the claim that they're on "opposite" sides of 0.]

Homework

3. What is the **opposite** of each number?

   a. 17 b. 0 c. -3.5 d. 8π e. $-\sqrt{2}$

4. a. T/F: Every number has an opposite.
   b. The opposite of 0 is ____.
   c. The opposite of a negative number is always _____.
   d. The opposite of a positive number is always _____.
   e. The sum of a real number and its opposite is always _____.

5. Using the formula $y = -x$, find the y-value for the given x-value:

   a. $x = 9$ b. $x = -3$ c. $x = 0$ d. $x = \pi$ e. $x = -\sqrt{2}$

Reciprocal

The ***reciprocal*** of a real number is found by dividing the number into 1. Equivalently, the reciprocal of x is $\frac{1}{x}$, and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. Every real number has a reciprocal except 0; the reciprocal of 0 would be $\frac{1}{0}$, which is undefined, as explained later in the Prologue.

Notice that the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative. In addition, **the product of any real number with its reciprocal is always 1**; for example, $\frac{2}{7} \cdot \frac{7}{2} = 1$.

Homework

6. Find the **reciprocal** of each real number:

   a. 5 b. $\frac{2}{9}$ c. $-\frac{7}{3}$ d. 1 e. 0 f. $\frac{1}{\pi}$ g. $-\sqrt{3}$

7. a. T/F: Every number has a reciprocal.
   b. The reciprocal of 0 is ____.
   c. The reciprocal of a negative number is always _____.
   d. The reciprocal of a positive number is always _____.
   e. The product of a real number and its reciprocal is always ____.

8. Using the formula $y = \frac{1}{x}$, answer each question:

   a. If $x = 14$, then $y =$ ____.
   b. If $x = \frac{2}{3}$, then $y =$ ____.
   c. If $x = -99$, then $y =$ ____.
   d. If $x = -\frac{5}{4}$, then $y =$ ____.
   e. If $x = 0$, then $y =$ ____.

Absolute Value

The ***absolute value*** of a real number is its distance to 0 on the number line.

For example, the absolute value of 9 is 9, because the number 9 is 9 units from 0 on the number line. The absolute value of –5 is 5, because the number –5 is 5 units from 0 on the number line. As for the real number 0, its absolute value is 0, since the number 0 is 0 units away from 0.

The notation for absolute value is two vertical bars around the number. So, for example, the absolute value of –12 is written $|-12|$, and equals 12. Here are three more examples:

$|35| = 35$ $\qquad |-2.7| = 2.7$ $\qquad |0| = 0$

> In computer programming, $|-12|$ might be written
> **`abs(–12)`**

Homework

9. Evaluate each expression:

   a. $|5\pi|$ b. $|-23.9|$ c. $|0|$ d. $|7-9|$ e. $|-\sqrt{7}|$

10. a. T/F: Every number has an absolute value.
    b. The absolute value of 0 is ____.
    c. The absolute value of a negative number is always _____.
    d. The absolute value of a positive number is always _____.

11. Which two of the following operations can be applied to all real numbers?

    Opposite Reciprocal Absolute Value

12. Find the **absolute value** of each number:

    a. 72 b. –99 c. 0 d. π e. $-\pi$ f. $-\sqrt{2}$

13. Evaluate each expression:

a. $|17-7|$ b. $|3-25|$ c. $|2(3)-6(1)|$ d. $|2\pi+3\pi|$

14. Using the formula $y = |x|$, answer each question:

a. If $x = 33$, then $y =$ ____.

b. If $x = 0$, then $y =$ ____.

c. If $x = -25$, then $y =$ ____.

d. If $y = 17$, then $x =$ ____ or ____.

e. If $y = 0$ then $x =$ ____.

f. If $y = -5$, then $x =$ ____.

❑ FRACTIONS

EXAMPLE 1:

A. **Express $\frac{7x}{2}$ as the product of two quantities:**

Solution: The easiest way to see this process is to just do it and then check that it's right. Here's what I claim:

$$\frac{7x}{2} = \frac{7}{2}x$$

And here's the reason:

$$\frac{7}{2}x = \frac{7}{2}\cdot\frac{x}{1} = \frac{7\cdot x}{2\cdot 1} = \frac{7x}{2}$$ ✓

B. **Express $\frac{7x+9}{-5}$ as the sum or difference of two fractions.**

Solution: What two fractions have a sum of $\frac{7x+9}{-5}$? The -5 tells us that we could use fractions with a denominator of -5. Since the numerator is $7x + 9$, we can make one numerator $7x$ and the other one 9. That is,

$$\frac{7x+9}{-5} = \frac{7x}{-5} + \frac{9}{-5} = -\frac{7}{5}x - \frac{9}{5}$$

C. **Combine the ideas of parts A and B to split up $\frac{8x-5}{7}$.**

Solution:

$$\frac{8x-5}{7} = \frac{8x}{7} - \frac{5}{7} = \mathbf{\frac{8}{7}x - \frac{5}{7}}$$

D. **Express $\frac{8x+16}{16}$ as the sum of two quantities.**

Solution:

$$\frac{8x+16}{16} = \frac{8x}{16} + \frac{16}{16} = \frac{x}{2} + 1 = \mathbf{\frac{1}{2}x + 1}$$

Homework

15. Express each fraction as the product of two quantities:

a. $\frac{9x}{4}$ b. $\frac{17x}{2}$ c. $\frac{3y}{16}$ d. $\frac{-33a}{17}$ e. $\frac{5n}{-23}$

f. $\frac{3x}{6}$ g. $\frac{-6x}{-2}$ h. $\frac{22m}{33}$ i. $\frac{-9x}{-15}$ j. $\frac{-39z}{52}$

16. Express each fraction as the sum or difference of two quantities, using parts C and D of Example 1 as a guide:

a. $\dfrac{3x+8}{4}$ b. $\dfrac{-9x+18}{6}$ c. $\dfrac{y-1}{8}$ d. $\dfrac{3n+15}{-5}$

e. $\dfrac{-3w-24}{2}$ f. $\dfrac{45x-75}{15}$ g. $\dfrac{33x+44}{55}$ h. $\dfrac{-31x-17}{-20}$

Operations with Fractions

$$-\frac{2}{3}-\frac{1}{2} = -\frac{4}{6}-\frac{3}{6} = -\mathbf{\frac{7}{6}}$$

$$\frac{4}{5}-\left(-\frac{2}{3}\right) = \frac{4}{5}+\frac{2}{3} = \frac{12}{15}+\frac{10}{15} = \mathbf{\frac{22}{15}}$$

$$\frac{2}{9}-7 = \frac{2}{9}-\frac{63}{9} = -\mathbf{\frac{61}{9}}$$

$$\left(\frac{2}{3}\right)\left(-\frac{5}{7}\right) = -\mathbf{\frac{10}{21}}$$

$$-\frac{4}{7}\div -2 = -\frac{4}{7}\times -\frac{1}{2} = \frac{4}{14} = \mathbf{\frac{2}{7}}$$

$$\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{5}\right) = \mathbf{\frac{1}{120}}$$

$$\frac{-8}{\frac{1}{3}} = -\frac{8}{1}\div\frac{1}{3} = -\frac{8}{1}\times\frac{3}{1} = -\mathbf{24}$$

$$\frac{-\frac{9}{4}}{-2} = -\frac{9}{4}\div -2 = -\frac{9}{4}\div -\frac{2}{1} = -\frac{9}{4}\times -\frac{1}{2} = \mathbf{\frac{9}{8}}$$

Note: A negative sign can "float." For instance,

$$\frac{-30}{6} = \frac{30}{-6} = -\frac{30}{6}$$

since all of these fractions have the value –5.

Powers and Square Roots of Fractions

An **exponent** still means what it always has, so these next examples should be clear.

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3}\cdot\frac{2}{3} = \mathbf{\frac{4}{9}} \qquad \left(-\frac{1}{4}\right)^3 = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) = -\mathbf{\frac{1}{64}}$$

$$\left(-\frac{9}{4}\right)^1 = -\mathbf{\frac{9}{4}} \qquad \left(\frac{1}{2}\right)^8 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \mathbf{\frac{1}{256}}$$

As for the **square root sign**, we still ask: What number (that's not negative) times itself gives the number in the radical sign?

$\sqrt{\frac{9}{25}} = \mathbf{\frac{3}{5}}$ This is true because $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$.

$\sqrt{\frac{1}{144}} = \mathbf{\frac{1}{12}}$ This is due to the fact that $\frac{1}{12}\times\frac{1}{12} = \frac{1}{144}$.

$\sqrt{-\frac{4}{49}}$ does <u>not</u> exist as a real number, because $-\frac{4}{49}$ is a negative number, and square roots of negative numbers are outside the real numbers. It's an imaginary number.

$\sqrt{\frac{-4}{-49}}$ <u>does</u> exist as a real number, because the fraction is actually a positive number: $\sqrt{\frac{-4}{-49}} = \sqrt{\frac{4}{49}} = \mathbf{\frac{2}{7}}$.

Homework

Perform the indicated operation:

17. a. $-\frac{1}{2}-\frac{4}{5}$ b. $-\frac{1}{3}-\left(-\frac{1}{3}\right)$ c. $\frac{2}{3}-\left(-\frac{5}{6}\right)$

d. $-\frac{4}{5}+\frac{2}{3}$ e. $9-\frac{4}{5}$ f. $-1-\frac{2}{3}$

g. $\frac{8}{3}-5$ h. $-\frac{2}{3}-(-1)$ i. $-\frac{1}{4}-\frac{2}{7}$

18. a. $\left(-\frac{1}{2}\right)\left(-\frac{5}{6}\right)$ b. $\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)$ c. $-\frac{5}{6}\cdot-\frac{6}{5}$

d. $-\frac{2}{3}\div-\frac{3}{2}$ e. $\frac{1}{2}\div-9$ f. $7\div-\frac{3}{4}$

g. $\dfrac{-\frac{2}{3}}{-\frac{1}{9}}$ h. $\dfrac{\frac{4}{5}}{-8}$ i. $\dfrac{-\frac{4}{5}}{\frac{5}{8}}$

19. True/False: $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$ [assuming $b \neq 0$]

20. a. $\left(-\frac{2}{3}\right)^2$ b. $\left(-\frac{1}{2}\right)^3$ c. $\left(-\frac{1}{3}\right)^4$

d. $\left(-\frac{14}{19}\right)^1$ e. $\left(-\frac{1}{2}\right)^5$ f. $\left(-\frac{2}{3}\right)^6$

g. $\left(\frac{99}{-99}\right)^{99}$

21. a. $\sqrt{\frac{81}{100}}$ b. $\sqrt{\frac{36}{64}}$ c. $\sqrt{\frac{1}{4}}$

d. $\sqrt{\frac{1}{9}}$ e. $\sqrt{\frac{121}{144}}$ f. $\sqrt{-\frac{25}{81}}$

g. $\sqrt{\frac{-256}{-289}}$ h. $\sqrt{-\frac{14}{17}-\left(-\frac{14}{17}\right)}$

❑ *ORDER OF OPERATIONS*

Order of Operations
Parentheses and Brackets [( )]
Exponents
Multiply & Divide (left to right)
Add & Subtract (left to right)

Note: Certainly $(-5)^2 = 25$, since both the 5 and the minus sign are being squared [i.e., $(-5)^2 = (-5)(-5) = 25$]. However, consider the expression

$$-5^2$$

Do we square the –5? The answer is NO; the exponent attaches to the 5 only. The justification is the Order of Operations, which states that exponents (near the top of the chart) are to be done before we deal with negative signs (which are at the bottom of the chart). So, although $(-5)^2 = 25$, we must agree that

$$-5^2 = -25$$

Homework

22. Evaluate (simplify) each expression:

a. $3 \cdot 10^2 - (8 - 4)^3 - 3 \times 2$

b. $(5 - 3)^2 + (10 - 7)^3$

c. $[3 + 2(5)] - 1 + 3 \cdot 10$

d. $2(10 - 5)^2 - 12 \div 3$

e. $[2(10 - 5)]^3 \div (10 \cdot 10^2)$

f. $(1 + 4)^2 - (4 + 1)^2$

g. $[(3^2 - 2^2)^3 - 80] \div (36 / 4)$

h. $3 \cdot 4^2 - (13 - 12)^3$

i. $10 + 8(8 - 1)^2 - 3 - 2 - 1$

j. $[8^2 - 2^3 + 3 \cdot 4 - 2(7)]^2$

k. $[20 - (5 - 2)^2]^2 - 2 \cdot 3 \cdot 4$

l. $[13 - (8 - 3) + (10 - 2)]^3$

23. Evaluate each expression for the given values:

a. $(x + y)^2$ for $x = 2$ and $y = 1$

b. $x^2 + y^2$ for $x = 10$ and $y = 5$

c. $x^2 + xy + y^2$ for $x = 3$ and $y = 6$

d. $(x + y)(x - y)$ for $x = 10$ and $y = 2$

e. $x^2 - y^2$ for $x = 12$ and $y = 10$

❑ TERMS

If the final operation in an expression is multiplication, then the expression consists of one term. The expression $A(B + C - D)$ consists of one term.

If the final operation consists of additions and/or subtractions, then the expression consists of at least two terms – the number of terms is found by counting the things being added or subtracted. For instance, the expression $ax - b + L(R + Q)$ consists of three terms: the ax, the b, and the $L(R + Q)$.

Homework

24. Determine the number of **terms** in each expression:

a. abc

b. $a + b - c$

c. $xyz - w$

d. $(x - 3)^2$

e. $x^2 + 25$

f. $ab + ac - xy$

g. $(a + b)(x - y)$

h. $rst - qrw$

i. $(rst)(qrw)$

j. $(rst)(qrw) - 1$

k. $[(rst)(qrw) - 1]^5$

l. $25 - (x + y - z)^2$

m. $a - b$

n. $x + y^2 + z$

o. $20 - mnpq$

p. $a + x - c + QT$

q. $(y + xm)^2A + B$

r. $(a - b)^3 - (c - d)^3 - w$

s. $abcd + x - y + wxyz$

t. $[a(b - c)ed - 7]^4$

u. $w(u - x)^3 - abc(def - mnpq)$

❑ *Division With Zeros*

This is the result of dividing by zero.

It's a mathematical fact of life that the only number that's never allowed to be in the denominator (bottom) of a fraction is zero. Sometimes this is phrased

"Never divide by zero."

Why the big deal?

Recall from elementary school that

$$\frac{56}{7} = 8 \quad \text{because } 8 \times 7 = 56.$$

Zero on the Top

How shall we interpret the division problem

$$\frac{0}{7} = ???$$

What number times 7 yields an answer of 0? Well, 0 works; that is,

$$\frac{0}{7} = 0 \quad \text{because } 0 \cdot 7 = 0.$$

Moreover, no other number besides 0 will work.

Zero on the Bottom

Now let's put a zero on the bottom and see what happens:

$$\frac{9}{0} = ???$$

Let's try an answer of 0; unfortunately $0 \cdot 0 = 0$, not 9.

How about we try an answer of 9? Then $9 \cdot 0$ is also 0, not 9.

Could the answer be π? No; $\pi \cdot 0 = 0$, not 9.

In fact, any number we surmise as the answer will have to multiply with 0 to make a product of 9. But this is impossible, since any number times 0 is always 0, never 9. In short, <u>no number in the whole world</u> will work in this problem.

Zero on the Top AND the Bottom

Now for an even stranger problem with division and zeros:

$$\frac{0}{0} = ???$$

We can try 0; in fact, since $0 \cdot 0 = 0$, a possible answer is 0.

Let's try an answer of 5; because $5 \cdot 0 = 0$, another possible answer is 5.

Could π possibly work? Since $\pi \cdot 0 = 0$, another possible answer is π.

Is there any end to this madness? Apparently not, since <u>any</u> number we conjure up will multiply with 0 to make a product of 0. In short, <u>every number in the whole world</u> will work in this problem.

Summary:

1) Zero on the top of a fraction is perfectly okay, as long as the bottom is NOT zero. The answer to this kind of division problem is always zero. For example, $\frac{0}{7} = 0$.

2) There is <u>no answer</u> to the division problem $\frac{9}{0}$. Clearly, we can never work a problem like this.

3) There are <u>infinitely many answers</u> to the division problem $\frac{0}{0}$. This may be a student's dream come true, but in mathematics we don't want a division problem with trillions of answers.

Each of the problems with a zero in the denominator leads to a major conundrum, so we summarize cases 2) and 3) by stating that

DIVISION BY ZERO IS UNDEFINED!

Thus,

$$\frac{0}{7} = 0$$

$$\frac{9}{0} \text{ is undefined}$$

$$\frac{0}{0} \text{ is undefined}$$

> "Black holes are where God divided by zero."
>
> *Steven Wright*

Homework

25. Evaluate each expression, and explain your conclusion:

a. $\frac{0}{15}$ b. $\frac{32}{0}$ c. $\frac{0}{0}$

26. Evaluate each expression:

a. $\frac{0}{17}$ b. $\frac{0}{-9}$ c. $\frac{6-6}{17+3}$ d. $\frac{3^2-8-1}{100}$

e. $\frac{98}{0}$ f. $\frac{-44}{0}$ g. $\frac{7+8}{2^3-8}$ h. $\frac{7^2-40}{-23+23}$

i. $\frac{0}{0}$ j. $\frac{-9+9}{10-10}$ k. $\frac{5^2-25}{0^2+0^3}$ l. $\frac{4\cdot5-2\cdot10}{3^3-9}$

27. $\frac{0}{\pi} = 0$ because

   a. 0 is the only number multiplied by π that will produce 0.
   b. no number times π equals 0.
   c. every number times π equals 0.

28. $\frac{0}{0}$ is undefined because

   a. no number times 0 equals 0.
   b. every number times 0 equals 0.
   c. any number divided by itself is 1.

29. $\frac{7}{0}$ is undefined because

   a. 0 is the only number multiplied by 0 that will produce 7.
   b. no number times 0 equals 7.
   c. every number times 0 equals 7.

30. a. The numerator of a fraction is 0. What can you conclude?
    b. The denominator of a fraction is 0. What can you conclude?

❑ *LINEAR EQUATIONS AND FORMULAS*

Solve for x: $\mathbf{2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)}$

Solution: The steps are

1) Distribute
2) Combine like terms
3) Solve the simplified equation

$2(3x - 7) - 5(1 - 3x) = -(-4x + 1) + (x + 7)$

$\Rightarrow$ $6x - 14 - 5 + 15x = 4x - 1 + x + 7$ (distribute)

$\Rightarrow$ $21x - 19 = 5x + 6$ (combine like terms)

$\Rightarrow$ $21x \mathbf{- 5x} - 19 = 5x \mathbf{- 5x} + 6$ (subtract $5x$ from each side)

$\Rightarrow$ $16x - 19 = 6$ (simplify)

$\Rightarrow$ $16x - 19 + \mathbf{19} = 6 + \mathbf{19}$ (add 19 to each side)

$\Rightarrow$ $16x = 25$ (simplify)

$\Rightarrow$ $\dfrac{16x}{\mathbf{16}} = \dfrac{25}{\mathbf{16}}$ (divide each side by 16)

$\Rightarrow$ $\boxed{x = \dfrac{25}{16}}$ (simplify)

Solve for x: $\dfrac{nx - w}{y + z} = e - f$

Solution: Notice the use of parentheses in the solution.

$\dfrac{nx - w}{y + z} = e - f$ (original formula)

$\Rightarrow$ $\dfrac{nx - w}{y + z}(y + z) = (e - f)(y + z)$ (multiply each side by $y + z$)

$\Rightarrow$ $nx - w = (e - f)(y + z)$ (simplify)

$\Rightarrow$ $nx = (e - f)(y + z) + w$ (add w to each side)

$\Rightarrow$ $\boxed{x = \dfrac{(e - f)(y + z) + w}{n}}$ (divide each side by n)

Homework

31. Solve each equation:

a. $-4(a - 6) + (-5a - 3) = 6(2a + 1) - (5a + 4)$

b. $2(-8e - 6) - 8(-3e - 2) = 3(-8e - 7) - 4(-2e + 9)$

c. $5(-9r - 5) + 3(8r + 3) = -2(8r - 3) - 3(7r + 7)$

d. $9(-7j - 6) - 7(-5j + 3) = 6(8j + 1) + 5(5j - 8)$

e. $-6(-9d + 6) + 3(-3d - 9) = -8(-d - 9) - 8(-3d - 4)$

32. Solve each formula for x:

a. $x - c = d$ b. $2x + b = R$ c. $abx = c$

d. $\frac{x}{u} = N$ e. $x(y + z) = a$ f. $\frac{x}{n} = c - d$

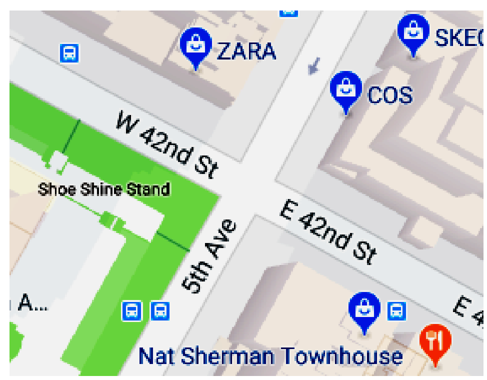
g. $\frac{x}{a+b} = m - n$ h. $\frac{x}{c-Q} = c + Q$ i. $\frac{x}{R} = a - b + c$

j. $x(b_1 + b_2) = A$ k. $\frac{x}{a} - e = m$ l. $\frac{x+a}{b} = y$

m. $\frac{ax - by}{c} = z$ n. $\frac{cx - a}{y+z} = h - g$ o. $\frac{ax+b}{c} - d = Q$

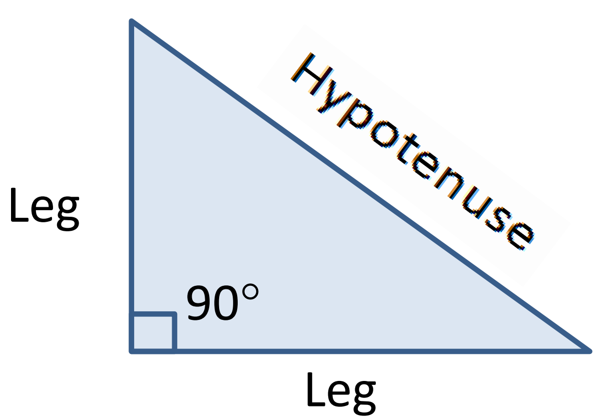
p. $\frac{9x + u - w}{Q + R} = m + n$ q. $9x - 7y + 13 = 0$

❑ *THE PYTHAGOREAN THEOREM*



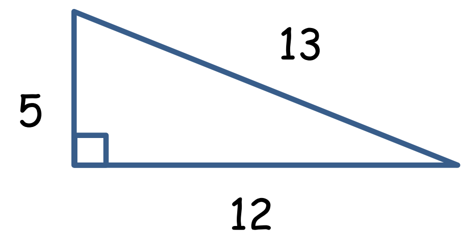
The Right Triangle

An angle of 90° is called a ***right angle***, and when two things meet at a right angle, we say they are ***perpendicular***. For example, the angle between the floor and the wall is 90°, and so the floor is perpendicular to the wall. And in Manhattan, 5th Avenue is perpendicular to 42nd Street.



If we have a triangle with a 90° angle in it, we call the triangle a ***right triangle***. The two sides which form the right angle (90°) are called the ***legs*** of the right triangle, and the side opposite the right angle is called the ***hypotenuse*** (accent on the 2nd syllable). It also turns out that the hypotenuse is always the longest side of a right triangle.

The Pythagorean Theorem



A Classic Right Triangle

Ancient civilizations discovered that a triangle with sides 5, 12, and 13 would actually be a right triangle — that is, a triangle with a 90° angle in it.
[By the way, is it obvious that the hypotenuse must be the side of length 13?]

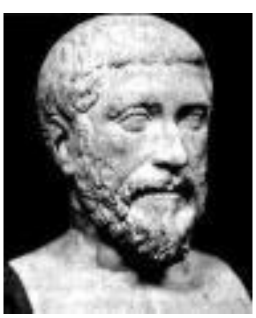
But what if just the two legs are known? Is there a way to calculate the length of the hypotenuse? The answer is yes, and the formula dates back to 600 BC, the time of Pythagoras and his faithful followers.

To discover this formula, let's rewrite the three sides of the above triangle:

$$leg = 5 \qquad leg = 12 \qquad hypotenuse = 13$$

Here's the secret: Use the idea of squaring. If we square the 5, the 12, and the 13, we get 25, 144, and 169; that is,

$$5^2 = 25 \qquad 12^2 = 144 \qquad 13^2 = 169$$



and we notice that the sum of 25 and 144 is 169:

$$25 + 144 = 169$$

In other words, a triangle with sides 5, 12, and 13 forms a right triangle precisely because

$$5^2 + 12^2 = 13^2$$

Now let's try to express this relationship in words — it appears that

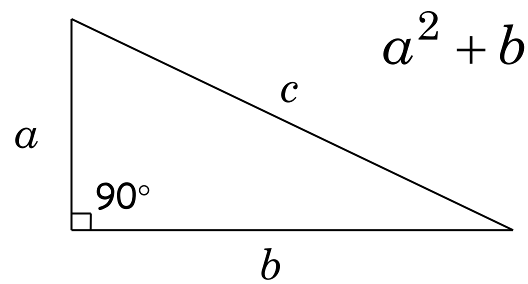
When you square the legs of a right triangle and add them together, you get the square of the hypotenuse.

As a formula, we can state it this way:

If a and b are the legs of a right triangle and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

The Pythagorean Theorem



$$a^2 + b^2 = c^2$$

The sum of the squares of the legs is equal to the square of the hypotenuse.

Solving Right Triangles

EXAMPLE 2: **The legs of a right triangle are 6 and 8. Find the hypotenuse.**

Solution: We begin by writing the Pythagorean Theorem. Then we plug in the known values, and finally determine the hypotenuse of the triangle.

$a^2 + b^2 = c^2$ (the Pythagorean Theorem)

$\mathbf{6}^2 + \mathbf{8}^2 = c^2$ (substitute the known values)

$36 + 64 = c^2$ (square each leg)

$100 = c^2$ (simplify)

What number, when squared, results in 100? A little experimentation yields the solution 10 (since $10^2 = 100$). Our conclusion:

The hypotenuse is 10

Note: The equation $100 = c^2$ also has the solution $c = -10$ [since $(-10)^2 = 100$]. But a negative length makes no sense, so we stick with the positive solution, $c = 10$.

<u>EXAMPLE 3:</u> **Find the hypotenuse of a right triangle whose legs are 5 and 7.**

<u>Solution:</u> This is very similar to Example 2.

$a^2 + b^2 = c^2$ (the Pythagorean Theorem)

$\mathbf{5}^2 + \mathbf{7}^2 = c^2$ (substitute the known values)

$25 + 49 = c^2$ (square each leg)

$74 = c^2$ (simplify)

Is there a whole number whose square is 74? No, there's not, because $8^2 = 64$, which is too small, while $9^2 = 81$, which is too big. We see that the solution for c is somewhere between 8 and 9. But where between 8 and 9?

Finding a number whose square is 74 is the kind of problem that has plagued and enticed mathematicians, scientists, and philosophers for literally thousands of years. They'd really be irked if they knew that we can find an excellent approximation of this number using a cheap calculator. Enter the number 74 followed by the ***square root*** key, which is labeled something like $\sqrt{x}$. Thus, the hypotenuse is $\sqrt{74}$ (read: the positive square root of 74), and your calculator should display 8.602325267, or something close. [With fancier calculators, enter the square root key first, then the 74.]

But your calculator doesn't tell the entire story. The fact is, the square root of 74 has an infinite number of digits following the decimal point, and they never have a repeating pattern. Thus, we'll have to round off the answer to whatever's appropriate for the problem. For this problem, we'll round to the third digit past the point.

> This means that $\sqrt{74}$ is an **irrational** number.

> The hypotenuse is 8.602

Homework

33. In each problem, the two legs of a right triangle are given. Find the **hypotenuse**.

a. 3, 4	b. 5, 12	c. 10, 24	d. 30, 16
e. 7, 24	f. 12, 16	g. 30, 40	h. 9, 40
i. 12, 35	j. 20, 21	k. 48, 55	l. 13, 84
m. 17, 144	n. 11, 60	o. 51, 140	p. 24, 143

34. Find the **hypotenuse** of the triangle with the given legs. Use your calculator and round your answers to the hundredths place.

a. 2, 5	b. 4, 6	c. 1, 7	d. 5, 8
e. 2, 6	f. 3, 5	g. 4, 7	h. 7, 8

❑ INEQUALITIES

You must score *between* 80% and 89% to get a B in your math class.

You must be *at least* 18 years of age to vote.

You can be *no taller* than 48 inches to play in the park.

These are all examples of quantities being greater than something and/or less than something. Since they are not equalities, they are called ***inequalities***.

We know that 5 is bigger than 3, which we can write as "5 **>** 3." The symbol "**>**" can also be read as "is larger than" or "is greater than."

But, of course, the fact that 5 is larger than 3 is the same as the fact that 3 is less than 5. This is written "3 **<** 5."

> $>$ means "is greater than"
>
> $<$ means "is less than"
>
> $\geq$ means "is greater than or equal to"
>
> $\leq$ means "is less than or equal to"

The symbol "≥" can be read "is greater than or equal to." For example, $9 \geq 7$ because 9 is indeed greater than or equal to 7. (Actually, it's greater than 7, but that doesn't change the fact that it's greater than or equal to 7.) And believe it or not, $12 \geq 12$ is a true statement — after all, since $12 = 12$, it's certainly the case that 12 is greater than or equal to 12.

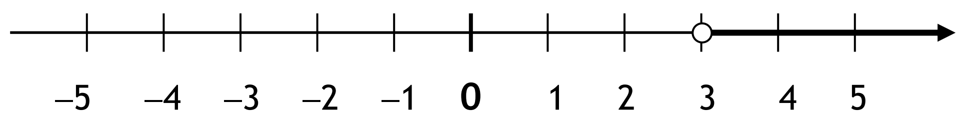
The symbol "≤" is read "less than or equal to." A couple of examples are $6 \leq 10$ and $8 \leq 8$.

❑ *Intervals on the Line*

First Example: Consider all the real numbers greater than 3. One simple way to express this set of numbers is the following ***inequality***:

$\boldsymbol{x > 3}$ Do you see that this inequality can also be written as follows? $3 < x$

We can also ***graph*** this set of numbers on a number line:



Notice that we put an "open dot" at $x = 3$ to show that the 3 is not part of the set of numbers. But the arrow goes infinitely to the right because $x > 3$ is the set of numbers greater than 3.

And a third way to denote this interval is called ***interval notation***, and for this inequality we write:

$\boldsymbol{(3, \infty)}$

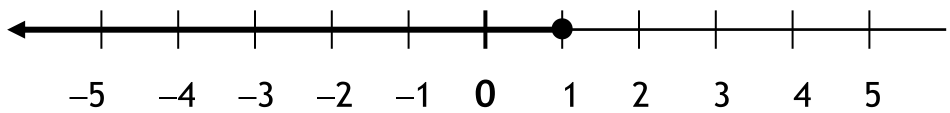
The parenthesis next to the 3 is analogous to the "open dot" on the graph — it means exclude the endpoint. And you always use a parenthesis at the ∞ or $-\infty$ end of an interval because ∞ is not really a number, so you can't possibly include it.

Whether written as an inequality, or a graph on a number line, or in interval notation, note that the numbers 3.01, π, 17, and 200 are part of the set; but the numbers –5, 0, 2.5 and 3 are not part of the set.

Second Example: Now we consider all the real numbers less than or equal to 1. This set of numbers can be written as an inequality like this:

$\boldsymbol{x \le 1}$ Note that this inequality can also be expressed as: $1 \le x$

As a graph on a number line, we write:



The "solid dot" is used to show that $x = 1$ is part of the set of the numbers. And since x must be less than or equal to 1, the arrow goes infinitely to the left. Either as an inequality or a graph, you should see that the numbers –3, –1.1, $\frac{7}{8}$, and 1 are part of the set, while the numbers 1.001 and $\sqrt{2}$ are not part of the set.

As for interval notation, we write

$$(-\infty, 1]$$

The (square) bracket next to the 1 is analogous to the "solid dot" on the graph — it means include the endpoint. And, as mentioned before, always us a parenthesis with $\pm\,\infty$, since you can't ever "get" to ∞.

Third Example: Now it's time for an interval that represents all the numbers between two numbers. Consider the double inequality

$$-2 \le x < 5$$

This can be read as "all real numbers between –2 and 5, including the –2, but excluding the 5."

As for interval notation and a graph, here they are:

$[-2, 5)$

Note that some numbers in the interval are –2, 0, $\sqrt{24}$, and 4.9999. But the numbers –2.1, 5, and 2π are not in the interval.

Fourth Example: Sometimes answers to an inequality problem end up looking something like this:

$$x < 3 \;\; \text{OR} \;\; x \geq 5$$

This means pretty much what it says: x can be less than 3, or it can be greater than or equal to 5. As long as x satisfies (at least) one of the two conditions, it's part of the answer. So some x's that satisfy the inequality are 0, 2.9, 5, and 3π. On the other hand, some numbers that do NOT satisfy it are 3, π, and 4.99.

How do we express this in interval notation? Like this, using the **union** symbol:

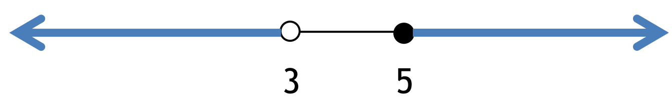
$$(-\infty, 3) \cup [5, \infty)$$

The *union* symbol, $\cup$, just means putting all the elements of the two sets into one big set; for example,

$$\{a, b\} \cup \{a, c, d\} = \{a, b, c, d\}$$

Note that the "overlap" of a is listed only once in the final set.

And as a graph, we write:



Homework

35. T/F:

a. $7 > 3$ b. $-2 < 1$

c. $13 \geq 13$ d. $-9 \leq -9$

e. $12 \geq 9$ f. $-18 \leq -20$

g. $\pi > 0$ h. $-\sqrt{2} \leq 0$

Can you see that the interval $(-\infty, \infty)$ is the same as the entire set of real numbers?

$$(-\infty, \infty) = \mathbb{R}$$

In fact, using the *union* notation, we can also write

$$\mathbb{R} = \text{Rationals} \cup \text{Irrationals}$$

36. Express each statement as an inequality:

a. Your age, a, must be at least 18 years.

b. Your height, h, can be no taller than 48 inches.

c. Your years of experience, y, must exceed 10 years.

d. The number of driving tickets, t, must be fewer than 5.

e. the mean, μ (Greek letter mu), must be at least 75.

f. the standard deviation, σ (Greek letter sigma), must be no more than 10.

g. the energy, E, must be greater than 100.

h. the mass, m, must be less than 3.7.

37. Convert each inequality to *interval notation*:

a. $x > 2$ b. $x \le 5$ c. $-1 \le x < 6$ d. $x < -3$ OR $x > 0$

38. Convert each interval to an *inequality*:

a. $[3, \infty)$ b. $(-\infty, -5)$ c. $(-1, 8]$ d. $(-\infty, -2] \cup (7, \infty)$

Solutions

1.

a.	Real, Rational	b.	Real, Rational	c.	Real, Rational
d.	Real, Rational	e.	Imaginary	f.	Real, Rational
g.	Real, Rational	h.	Real, Irrational	i.	Real, Rational
j.	Real, Irrational	k.	Real, Irrational	l.	Imaginary
m.	Real, Rational	n.	Real, Rational	o.	Real, Rational
p.	Real, Irrational	q.	Real, Irrational	r.	Imaginary
s.	Imaginary	t.	Real, Irrational	u.	Real, Rational

2. $-\pi$, $-\sqrt{3}$, -1, $\sqrt{0}$, $\frac{1}{101}$, $\sqrt{1}$, $\sqrt{5}$, 3.0808. . . , $\frac{11}{3}$, $\sqrt{25}$, 2π, 2^3, 3^2

3. a. -17 b. 0 c. 3.5 d. -8π e. $\sqrt{2}$

4. a. True b. 0 c. positive d. negative e. 0

5. a. -9 b. 3 c. 0 d. $-\pi$ e. $\sqrt{2}$

6. a. $\frac{1}{5}$ b. $\frac{9}{2}$ c. $-\frac{3}{7}$ d. 1 e. Undefined f. π g. $\sqrt{3}$

7. a. False; 0 has no reciprocal. b. Undefined c. negative
d. positive e. 1

8. a. $\frac{1}{14}$ b. $\frac{3}{2}$ c. $-\frac{1}{99}$ d. $-\frac{4}{5}$ e. Undefined

9. a. 5π b. 23.9 c. 0 d. 2 e. $\sqrt{7}$

10. a. T b. 0 c. positive d. positive

11. Opposite & Absolute Value (the reciprocal of 0 does not exist)

12. a. 72 b. 99 c. 0 d. π e. π f. $\sqrt{2}$

13. a. 10 b. 22 c. 0 d. 5π

14. a. 33 b. 0 c. 25 d. 17 or -17 e. 0 f. No solution

15. a. $\frac{9}{4}x$ b. $\frac{17}{2}x$ c. $\frac{3}{16}y$ d. $-\frac{33}{17}a$
e. $-\frac{5}{23}n$ f. $\frac{1}{2}x$ g. $3x$ h. $\frac{2}{3}m$
i. $\frac{3}{5}x$ j. $-\frac{3}{4}z$

16. a. $\frac{3}{4}x+2$ b. $-\frac{3}{2}x+3$ c. $\frac{1}{8}y-\frac{1}{8}$ d. $-\frac{3}{5}n-3$
e. $-\frac{3}{2}w-12$ f. $3x-5$ g. $\frac{3}{5}x+\frac{4}{5}$ h. $\frac{31}{20}x+\frac{17}{20}$

17. a. $-\frac{13}{10}$ b. 0 c. $\frac{3}{2}$ d. $-\frac{2}{15}$ e. $\frac{41}{5}$

f. $-\frac{5}{3}$ g. $-\frac{7}{3}$ h. $\frac{1}{3}$ i. $-\frac{15}{28}$

18. a. $\frac{5}{12}$ b. -1 c. 1 d. $\frac{4}{9}$ e. $-\frac{1}{18}$
f. $-\frac{28}{3}$ g. 6 h. $-\frac{1}{10}$ i. $-\frac{32}{25}$

19. True

20. a. $\frac{4}{9}$ b. $-\frac{1}{8}$ c. $\frac{1}{81}$ d. $-\frac{14}{19}$
e. $-\frac{1}{32}$ f. $\frac{64}{729}$ g. -1

21. a. $\frac{9}{10}$ b. $\frac{3}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{11}{12}$
f. Not a real number g. $\frac{16}{17}$ h. 0

22. a. 230 b. 31 c. 42 d. 46 e. 1 f. 0
g. 5 h. 47 i. 396 j. 2916 k. 97 l. 4096

23. a. $(x + y)^2 = (2 + 1)^2 = 3^2 = 9$
b. $x^2 + y^2 = 10^2 + 5^2 = 100 + 25 = 125$
c. 63 d. 96 e. 44

24. a. 1 b. 3 c. 2 d. 1 e. 2 f. 3 g. 1 h. 2
i. 1 j. 2 k. 1 l. 2 m. 2 n. 3 o. 2 p. 4
q. 2 r. 3 s. 4 t. 1 u. 2

25. a. $\frac{0}{15} = 0$ since $0 \times 15 = 0$, and 0 is the only number that accomplishes this.

b. $\frac{32}{0}$ is undefined because any number times 0 is 0, never 32; thus NO number works.

c. $\frac{0}{0}$ is undefined because any number times 0 is 0; thus EVERY number works.

26. a. 0 b. 0 c. 0 d. 0 e. Undefined f. Undefined
g. Undefined h. Undefined i. Undefined j. Undefined
k. Undefined l. 0

27. a. **28**. b. **29**. b.

30. a. You can't conclude anything — it depends on what's on the bottom. If the bottom is a non-zero number (like 7), then $\frac{0}{7} = 0$. If the bottom is zero, then $\frac{0}{0}$ is undefined.

b. This time we <u>can</u> conclude that the fraction is undefined, since division by 0 is undefined, no matter what's on the top of the fraction.

31. a. $a = \frac{19}{16}$ b. $e = -\frac{61}{24}$ c. $r = \frac{1}{16}$
d. $j = -\frac{41}{101}$ e. $d = \frac{167}{13}$

32. a. $x = d + c$ b. $x = \frac{R-b}{2}$ c. $x = \frac{c}{ab}$
d. $x = Nu$ e. $x = \frac{a}{y+z}$ f. $x = n(c - d)$
g. $x = (m - n)(a + b)$ h. $x = (c + Q)(c - Q)$ i. $x = R(a - b + c)$
j. $x = \frac{A}{b_1 + b_2}$ k. $x = a(m + e)$ l. $x = by - a$
m. $x = \frac{cz+by}{a}$ n. $x = \frac{(h-g)(y+z)+a}{c}$
o. $x = \frac{c(Q+d)-b}{a}$ p. $x = \frac{(m+n)(Q+R)+w-u}{9}$
q. $x = \frac{7y-13}{9}$

33. a. 5 b. 13 c. 26 d. 34 e. 25 f. 20
g. 50 h. 41 i. 37 j. 29 k. 73 l. 85
m. 145 n. 61 o. 149 p. 145

34. a. 5.39 b. 7.21 c. 7.07 d. 9.43
e. 6.32 f. 5.83 g. 8.06 h. 10.63

35. a. T b. T c. T d. T
e. T f. F g. T h. T

36. a. $a \geq 18$ b. $h \leq 48$ c. $y > 10$ d. $t < 5$
e. $\mu \geq 75$ f. $\sigma \leq 10$ g. $E > 100$ h. $m < 3.7$

37. a. $(2, \infty)$ b. $(-\infty, -5]$ c. $[-1, 6)$ d. $(-\infty, -3) \cup (0, \infty)$

38. a. $x \geq 3$ b. $x < -5$ c. $-1 < x \leq 8$ d. $x \leq -2$ OR $x > 7$

“The greatest mistake
you can make in life
is to be continually
fearing you
will make one.”

– **Elbert Hubbard (1856–1915)**