CH 9 – LINEAR MODELING

❑ INTRODUCTION

Sometimes it's not too hard to create a formula from words. For example, the phrase *density is the quotient of mass and volume* translates to the formula $d = \frac{m}{V}$. But this method of formula creation is useful only if we comprehend the words that need to be translated into Algebra.

More often, we're given a *situation* that needs translation. For example, if we buy 12 lbs of cashews at a unit price of \$14/lb, and we wish to determine the total cost, we have to understand that *multiplication* does the trick; no one's going to tell you that — you have to reason it out yourself. So the total cost is

$$12 \text{ lbs} \times \$14/\text{lb} = \underline{\$168}$$

That's fine for one or a few calculations. But if you need to program a calculator, or perhaps a spreadsheet, or maybe a programming

language, you need to teach the machine what to do, and that can only be done with *variables*. So in the cashew problem above, we could let $\boldsymbol{Q}$ stand for the quantity (number of pounds) of cashews, let $\boldsymbol{P}$ be the unit price (the price of 1 pound) of the cashews, and let $\boldsymbol{C}$ be the total cost. Again, if we understand that the key to the formula will be multiplication, we can write a formula that can be entered into a computer (using *****, the computer symbol for multiplication):

$$\boldsymbol{C = P * Q}$$

❑ *CREATING LINEAR MODELS*

Business people are constantly trying to determine their business future. What will interest rates be, what will be our expenses next year, how much could we make if we sell more widgets? These and thousands more are questions whose answers may very well determine the success of a company. This chapter will look at problems that involve predicting future revenues and future insurance costs. You will also study how we use a linear model to convert between temperature scales.

EXAMPLE 1: **To join the Model Railroad Club, a member must pay an up-front fee of $25 to join, and then pay $10 per month for each month in the club.**

a. **Find the total cost for someone to be a club member for 8 months.**

b. **Find a formula for someone to be a club member for *m* months.**

c. **Use your formula to calculate the total to be in the club for 36 months.**

d. **Use your formula to calculate the number of months Sam was in the club if he paid a total of $995.**

Solution: Let's begin with a simple table that shows the cost of membership for various months:

Months	0	1	2	3	4	5	6	7	8
Cost	$25	$35	$45	$55	$65	$75	$85	$95	$105

a. The cost, C, is \$25 plus \$10 per month for 8 months:

$$C = 25 + 10(8) = 25 + 80 = 105$$

Therefore, the cost of 8 months is **\$105**.

b. Just change the 8 in part a. to the variable m:

$$C = 25 + 10m, \text{ or } \mathbf{C = 10m + 25}$$

c. Using the formula in part b., we get a cost of

$$C = 10m + 25 = 10(\mathbf{36}) + 25 = 360 + 25 = \mathbf{\$385}$$

d. $C = 10m + 25 \Rightarrow 995 = 10m + 25 \Rightarrow 970 = 10m$
$\Rightarrow m = 97$. Thus, Sam's been a member for **97 months**.

Homework

1. a. It costs \$2 per mile to take a taxi. If m represents the total miles traveled, write a formula for the total cost, C.

   b. Use your formula to calculate the total cost of a 20-mile trip.

2. a. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. Calculate the total cost of a 21-mile taxi ride.

   b. It costs \$3 for the first mile of a taxi ride, and \$2 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

   c. Use your new formula from part b. to calculate the total cost of a 15-mile trip.

   d. Using the same formula, assume Joanna paid \$35 for a taxi to the airport. How many miles was the taxi ride?

3. a. It costs \$7 for the first mile of a taxi ride, and \$4 per mile for each additional mile. If m represents the total miles traveled, write a formula for the total cost, C.

   b. Michael paid \$95 for a taxi to the airport. How many miles was the taxi ride?

4. More on the taxi ride:

   Let C = the total cost of the taxi ride

   f = the cost of the first mile

   a = the cost of each mile after the first

   m = the total miles traveled

   Write a formula for the total cost of a taxi ride.

5. a. QRS, Inc. sold 5000 widgets last month at a unit selling price of \$45/widget. Calculate the revenue obtained.

   b. STU, Inc. sold w widgets last month at a unit selling price of \$$p$/widget. Create a formula for the revenue (R) obtained.

6. a. WXY, Inc. produced 600 widgets last quarter, where it cost \$50/widget. In addition, fixed costs (rent, utilities, salaries, etc.) totaled \$2,400 last quarter. Calculate the total cost of producing the 600 widgets.

   b. Let w = the number of widgets produced

   c = unit cost to produce one widget

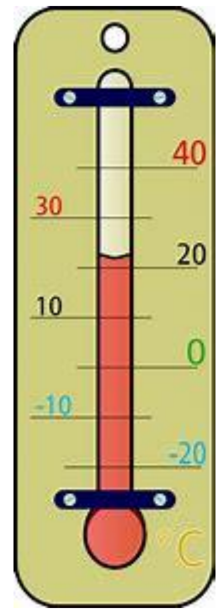
   f = fixed costs (rent, utilities, salaries, etc.)

   E = total expense to produce the w widgets

   Create a formula for the total expense of producing w widgets.

7. Ernie earns a base salary of \$2,000/month and a commission of \$50 for each widget he sells. Find Ernie's total salary during a month in which he sold 75 widgets. Construct a formula which would give Ernie's total salary, S, during a month if he sold w widgets.

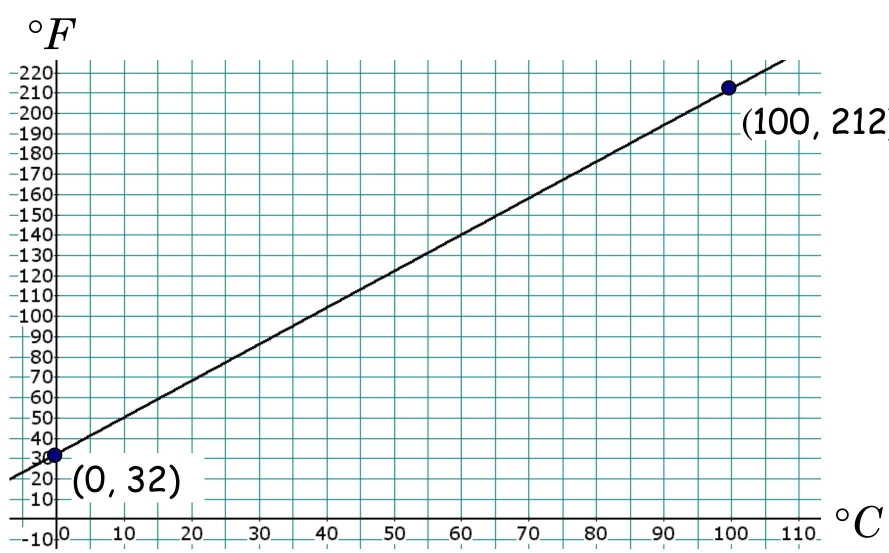
❑ *Temperature Scales*



On the ***Fahrenheit*** temperature scale (proposed in 1724), water freezes at 32°F and boils at 212°F. In 1743, the ***Celsius*** (originally called *centigrade*) scale was created to make the numbers easier to work with. On this new scale, water freezes at 0°C and boils at 100°C. We conclude that 0°C = 32°F and 100°C = 212°F.

Our goal here is to create, from scratch, a formula that converts from degrees Celsius to degrees Fahrenheit. All we need to accomplish this goal are two facts: 0°C = 32°F, 100°C = 212°F, and one assumption: that the relationship is linear (the graph is a straight line).

If we label the horizontal axis $°C$ and the vertical axis $°F$, then the temperature facts above translate into the following two points on the line: (0, 32) and (100, 212).



We begin with a linear formula relating $°C$ to $°F$, where m represents the slope of the line, and b is the "y-intercept."

$F = mC + b$ [note that C is the x-value and F is the y-value]

First, we use the two points that we know are on the line to calculate the slope of the line:

$$m = \frac{\Delta F}{\Delta C} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \mathbf{1.8}$$

So now our equation is

$$F = 1.8C + b$$

Second, we use one of the given points — we'll use (0, 32) — to find b:

$$\mathbf{32} = 1.8(\mathbf{0}) + b \Rightarrow 32 = 0 + b \Rightarrow 32 = b$$

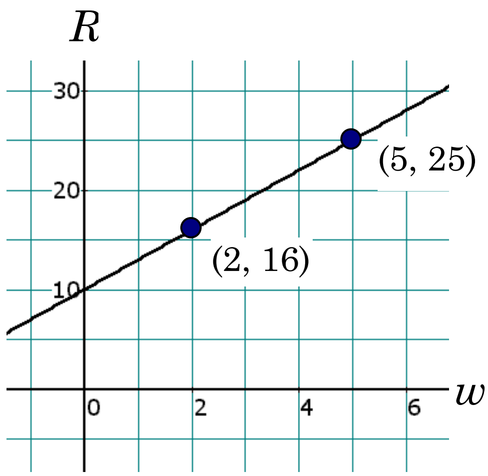
and we have our equation relating the two temperature scales:

$$\boxed{F = 1.8C + 32}$$

❑ *Predicting Revenue*

EXAMPLE 2: **The revenue obtained when 2 widgets were sold was $16, and the revenue obtained when 5 widgets were sold was $25. Project (predict) the revenue if 20 widgets are sold.**

Solution: If the unit selling price is the same for each sale of widgets, then it would be an easy arithmetic problem requiring no algebra. Let's see: 2 widgets for $16 is $8 per widget, while 5 widgets for $25 comes to $5 per widget. They're not the same unit price, so we'll have to assume that some additional source of income is contributing to our revenue amounts. Here's a graph of the given information, where the horizontal w-axis is the number of widgets sold and the vertical R-axis is the revenue obtained:



Let's assume that the graph through these two points is a straight line. This means we are assuming a linear model ($y = mx + b$ from Chapter 6); that is, we assume that the relationship between revenue and widgets sold is given by a linear formula

$$R = mw + b$$

where R = revenue, m is the slope, w is the number of widgets sold, and b is the R-intercept.

Note: There are an infinite number of graphs that pass through the two given points. Let's take the simplest one: a straight line.

First we find the slope of the line. Normally, our slope formula would be $m = \frac{\Delta y}{\Delta x}$, but we're not using x and y-axes; we're using w and R. So the slope is calculated using the two given points:

$$m = \frac{\Delta R}{\Delta w} = \frac{25-16}{5-2} = \frac{9}{3} = \mathbf{3}$$

The equation of the line at this point is then

$$R = 3w + b$$

To find b (the R-intercept), we pick either point on the line and place it into the previous equation; let's choose (2, 16):

$$\mathbf{16} = 3(\mathbf{2}) + b \Rightarrow 16 = 6 + b \Rightarrow b = 10$$

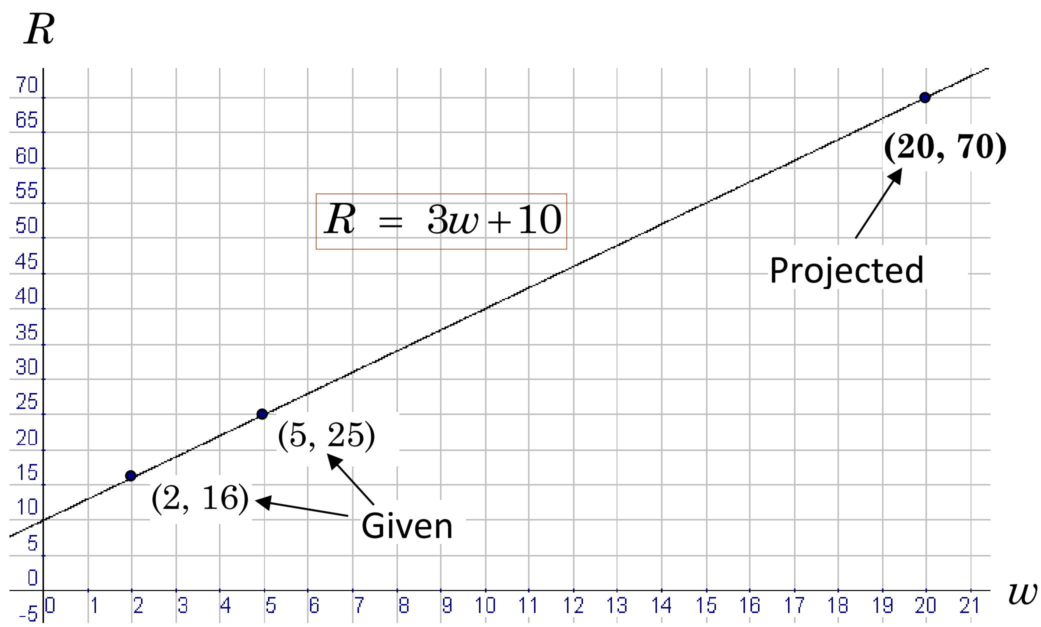
Our line equation is therefore

$$\underline{R = 3w + 10}$$

Now we can *project the revenue* for any number of widgets. If 20 widgets are sold, the revenue will be $R = 3(\mathbf{20}) + 10 = 60 + 10 =$

$70.00

The following graph summarizes this entire problem:



Homework

8. The revenue obtained when 10 widgets were sold was \$155, and the revenue obtained when 13 widgets were sold was \$176. Project the revenue if 100 widgets are sold.

9. When 7 widgets were sold, \$152 in revenue was obtained, and when 13 widgets were sold, the revenue was \$218. If 85 widgets are sold, project the revenue.

10. The revenue obtained when 100 widgets were sold was \$2,200.75, and the revenue obtained when 120 widgets were sold was \$2,470.75. Project the revenue if 250 widgets are sold.

11. The revenue obtained when 20 widgets were sold was \$360, and the revenue obtained when 36 widgets were sold was \$568. Project the revenue if 100 widgets are sold.

12. When 10 widgets were sold, \$575 in revenue was obtained, and when 21 widgets were sold, the revenue was \$1,015. If 50 widgets are sold, project the revenue.

13. When 50 widgets were sold, \$3000 in revenue was obtained, and when 51 widgets were sold, the revenue was \$3,052. If 200 widgets are sold, project the revenue.

❑ *PREDICTING INSURANCE COSTS*

Management is trying to project the cost of medical insurance premiums for the employees of the company.

<u>EXAMPLE 3:</u> **In 2019, there were 23 employees and the total medical insurance premiums were \$22,050. In 2020, the company had 40 employees and the premiums were \$36,500. Project the medical insurance premiums for the year 2021, when there should be 85 employees.**

<u>Solution:</u> A logical first guess is that the amount paid each year <u>per employee</u> is the same. If this is the case, we'll finish this problem quickly.

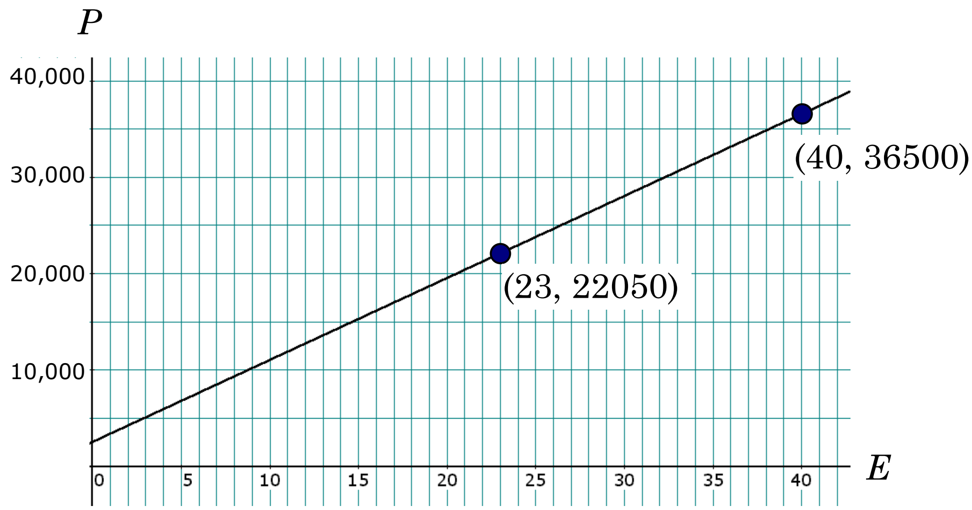
2019: The cost per employee was $\frac{\$22,050}{23 \text{ emp}} = \$958.70/\text{employee}$

2020: The cost per employee was $\frac{\$36,500}{40 \text{ emp}} = \$912.50/\text{employee}$

No such luck — whatever formula relates the number of employees to the total of the medical insurance premiums is more complicated than this (just like the previous example with the revenue from selling widgets). So we'll assume the linear formula

$$P = mE + b \qquad \text{(same kind of formula as revenue)}$$

where P = medical premiums, E = number of employees, and m and b are to be determined.



First, find the slope of the line connecting the points (23, 22050) and (40, 36500):

$$m = \frac{\Delta P}{\Delta E} = \frac{36500 - 22050}{40 - 23} = \frac{14450}{17} = 850$$

The equation of the line at this point is then

$$P = 850E + b$$

To find b (the P-intercept), we pick either point on the line and place it into the previous equation; let's choose (23, 22050):

$$22050 = 850(23) + b \Rightarrow 22050 = 19550 + b \Rightarrow b = 2500$$

Our line equation is therefore

$$\underline{P = 850E + 2{,}500}$$

Finally we get to the question at hand: the projection for the year 2021. Placing $E = 85$ employees into the formula gives

$$P = 850(85) + 2{,}500 = 72{,}250 + 2{,}500 = 74{,}750$$

Thus, the projected medical insurance premiums for 2021 is

$$\boxed{\$74{,}750}$$

Homework

14. Look back at the previous example on predicting insurance costs. The final formula was a little more complicated that we first imagined: $P = 850E + 2{,}500$. Try to interpret this formula. Remembering that E represents the number of employees in the insurance plan, what might the 850 and the 2,500 represent?

15. Last year there were 53 employees and the total medical insurance premiums were $45,710. This year the company has 100 employees and the premiums were $81,900. Project the medical insurance premiums for next year, when there should be 150 employees.

16. Last year there were 132 employees and the total medical insurance premiums were $40,204. This year the company has 101 employees and the premiums were $30,997. Project the medical insurance premiums for next year, when there should be 125 employees.

17. Last year there were 180 employees and the total medical insurance premiums were $37,200. This year the company has 190 employees and the premiums were $39,100. Project the medical insurance premiums for next year, when there should be 225 employees.

18. Last year there were 90 employees and the total medical insurance premiums were $23,300. This year the company has 100 employees and the premiums were $25,800. Project the medical insurance premiums for next year, when there should be 125 employees.

Practice Problems

19. a. Suppose that the DVC Library currently has 7000 books, and is buying 150 more books per year. How many books will the library have 4 years from now?
    b. Write a formula that will give the number of books the library will have y years from now.
    c. Use your formula to calculate the number of years it will take for the library collection to reach 10,000 books.

20. You must pay $75 up front to join the Painting Club, and then pay $35 per month for each month you're a member of the club. Create a formula that will give the total cost, C, to be in the club for m months.

21. The revenue obtained when 2 widgets were sold was $16, and the revenue obtained when 5 widgets were sold was $28. Project (predict) the revenue if 50 widgets are sold. Assume the linear model $R = mw + b$, where w = widgets sold and R = revenue.

22. Last year there were 83 employees and the total medical insurance premiums were $25,599. This year the company has 200 employees and the premiums were $61,050. Project the medical insurance premiums for next year, when there should be 300 employees.

Solutions

1. a. $C = 2m$ b. $C = 2m = 2(20) = \mathbf{\$40}$

2. a. $3 + 2(20) = 3 + 40 = \mathbf{\$43}$

b. $C = 3 + 2(m - 1)$, or $\boldsymbol{C = 2(m - 1) + 3}$

c. $C = 2(m - 1) + 3 = 2(15 - 1) + 3 = 2(14) + 3 = 28 + 3 = \mathbf{\$31}$

d. $C = \$35$, so we have to solve the equation $35 = 2(m - 1) + 3$ for m.

17 miles

3. a. $C = 4(m - 1) + 7$ b. 23 miles

4. $C = f + a(m - 1)$

5. a. \$225,000 b. $R = pw$

6. a. $600(50) + 2{,}400 = \mathbf{\$32{,}400}$

b. $E = cw + f$

7. $\$2{,}000 + \$50(75) = \mathbf{\$5{,}750}$

In general, $S = 2{,}000 + 50(w)$, or $\boldsymbol{S = 50w + 2{,}000}$

8. \$785.00 **9.** \$1,010.00 **10.** \$4,225.75

11. \$1,400 **12.** \$2,175 **13.** \$10,800

14. There are probably lots of valid interpretations, but here's what I see: Since \$850 is being multiplied by the number of employees, I think the 850 is the premium for each employee. In that case, the \$2,500 represents something like a flat fee to initially set up the insurance plan. That's how I see it — what do you see?

15. \$120,400 **16.** \$38,125 **17.** \$45,750 **18.** \$32,050

19. a. $7{,}000 + 4(150) = \underline{7{,}600 \text{ books}}$
b. $B = 150y + 7{,}000$
c. $10{,}000 = 150y + 7{,}000 \Rightarrow y = \underline{20 \text{ years}}$

20. The cost would be \$75 plus \$35 for each month in the club.
That's \$75 + \$35 × number of months in the club.
That's $C = 75 + 35m$, which should be written $\underline{C = 35m + 75}$

21. Find the equation of the line connecting (2, 16) and (5, 28), using w and R instead of x and y. You should get $R = 4w + 8$. Now project the revenue if 50 widgets are sold: 208 widgets

22. \$91,350

“The only good is knowledge and the only evil ignorance.”

– Σοχρατεσ (Socrates)