CH 14 – INEQUALITIES AND ABSOLUTE VALUE EQUATIONS

❑ INTRODUCTION

Inequality	Interval
$x > 4$	$(4, \infty)$
$x \ge -2$	$[-2, \infty)$
$x < 0$	$(-\infty, 0)$
$x \le 13$	$(-\infty, 13]$
$-5 < x \le 2$ which means $x > -5$ AND $x \le 2$, which means that x is *between* -5 and 2, excluding the -5, but including the 2.	$(-5, 2]$
$x \le -1$ OR $x > 3$	$(-\infty, -1] \cup (3, \infty)$

❑ LINEAR INEQUALITIES

To solve an inequality such as $\frac{-2x+7}{3} \ge -4$, we have to perform the operations necessary to isolate the x. We will accomplish this goal by using the usual *do the same thing to each side* rule, but since we're talking about an inequality — not an equation — we have to be very careful. The following experiment should illustrate the potential problems, and how we can resolve them.

Let's perform six experiments. Consider the true statement

$$4 < 6$$

i. Add 10 to each side: $4 + \mathbf{10} < 6 + \mathbf{10} \Rightarrow 14 < 16$ ✓

ii. Subtract 3 from each side: $4 - \mathbf{3} < 6 - \mathbf{3} \Rightarrow 1 < 3$ ✓

iii. Multiply each side by 5: $4(\mathbf{5}) < 6(\mathbf{5}) \Rightarrow 20 < 30$ ✓

iv. Divide each side by 2: $\frac{4}{2} < \frac{6}{2} \Rightarrow 2 < 3$ ✓

v. Multiply each side by –3: $4(-3) < 6(-3) \Rightarrow -12 < -18$ ☹

vi. Divide each side by –2: $\frac{4}{-2} < \frac{6}{-2} \Rightarrow -2 < -3$ ☹

What can we deduce from these six calculations? The first two show that adding or subtracting the same number in an inequality is legal — they lead to a true inequality, with no issues to worry about.

The next two calculations indicate that multiplying or dividing by a <u>positive</u> number always leads to an inequality without problems.

The last two cases, however — multiplying or dividing by a <u>negative</u> number — have led to false statements. So in these two scenarios, we must <u>reverse</u> the order of the inequality sign in order to maintain a true statement.

These numerical experiments are by no means a complete proof of the principle we are about to state, but they're convincing enough.

The Basic Principle of Inequalities

If you <u>multiply</u> or <u>divide</u> each side of an inequality by a <u>negative</u> number, you must <u>reverse</u> the inequality symbol.

EXAMPLE 1: **Solve each inequality:**

A. $\boldsymbol{x + 3 > 4}$

Subtracting 3 from each side gives $\boldsymbol{x > 1}$, which is **(1, ∞)**.
Note: The inequality symbol was not reversed.

B. $\boldsymbol{-2n - 9 \le 13}$

Add 9 to each side to get $-2n \le 22$.
(The inequality symbol was not reversed.)
Divide each side by -2: $\boldsymbol{n \ge -11}$, which is **[–11, ∞)**.
This time the inequality symbol was reversed.

C. $\boldsymbol{\frac{u}{5} + 3 < -4}$

Subtracting 3 gives $\frac{u}{5} < -7$.
Multiplying by 5 gives $\boldsymbol{u < -35}$, which is **(–∞, –35)**.
Neither operation required reversing the inequality symbol

D. $\dfrac{\mathbf{-2x-5}}{\mathbf{-3}} \geq \mathbf{11}$

Multiplying each side of the inequality by –3, and reversing the inequality sign gives:

$$-2x - 5 \leq -33$$

Adding 5 to each side of the inequality does not require reversing the inequality sign:

$$-2x \leq -28$$

Dividing each side of the inequality by –2, and remembering to reverse the inequality sign, we get the solution:

$\boldsymbol{x \geq 14}$, which is $\mathbf{[14, \infty)}$.

Note: Although the final inequality symbol ($\geq$) is the same as the one in the given problem, notice that we reversed it twice to get that answer. If you get the right answer on an exam without reversing it twice, it will probably be marked wrong.

Homework

1. Solve each inequality:

a. $x + 7 > -10$ b. $x - 3 \leq 5$ c. $2x \geq 14$

d. $-3x < -42$ e. $\dfrac{x}{8} > -3$ f. $\dfrac{x}{-5} \geq 10$

2. a. Solve for y: $6(y - 5) - (y + 1) \leq 12y + 21$

b. Solve for u: $-2(7 + 3u) - (1 - u) > 2u - 10$

3. a. Solve for x: $\frac{-3x+5}{-2}+9 > 16$ Note: The process is more important than the answer. Do you know what this means?

b. Solve for a: $\frac{7a-5}{-3}-12 < 12$

c. Solve for y: $\frac{9-y}{-2} \ge -2$

d. Solve for n: $\frac{-14-2n}{12} \le -1$

❑ *QUADRATIC INEQUALITIES*

For linear inequalities, the Basic Principle of Inequalities stated above is sufficient. But now consider the inequality

$$x^2 < 25$$

It's tempting to merely take the square root of each side of the inequality:

$$\sqrt{x^2} < \sqrt{25}$$

and conclude that $x < 5$.

This can't be the solution, though, and here's why: Let $x = -10$, which is certainly less than 5. Check this solution against the original problem:

$$x^2 < 25 \quad \Rightarrow \quad (-10)^2 < 25 \quad \Rightarrow \quad 100 < 25, \text{ NO!}$$

So that was a dead end. Let's try something else. Recall from your previous Algebra courses that 25 actually has <u>two</u> square roots, 5 <u>and</u> –5. [This is because $5^2 = 25$ <u>and</u> $(-5)^2 = 25$.] If we do that, we'd get

$$x < \pm 5$$

I don't even know what that means, so that kills the "taking square root" methods.

But there is a way to do this kind of problem, as well as absolute-value inequalities like $|2x - 6| - 8 \geq 0$. It's a technique called the Boundary Point Method.

The Boundary Point Method

1) Change the inequality to an equation and then solve that equation.

2) The solutions of that equation are called the ***boundary points***.

3) Plot the boundary points on the real line. This will create one or more intervals on the line.

4) Now choose a ***test point*** from each interval (any convenient number in the interval) and plug it into the original inequality. If the test point satisfies the inequality, then the entire interval in which the test point lies is a part of the solution of the inequality. If the test point does not satisfy the inequality, then the associated interval is not part of the solution.

5) Last, check the boundary points themselves in the inequality.

Now back to the inequality that stymied us at the top of this section of the chapter: $x^2 < 25$.

EXAMPLE 2: **Solve for x: $x^2 < 25$**

Solution: We solve this kind of problem using the Boundary Point Method. Solve the associated equation:

$$x^2 = 25$$

$$\Rightarrow \quad x^2 - 25 = 0$$

$$\Rightarrow \quad (x+5)(x-5) = 0$$

$$\Rightarrow \quad x + 5 = 0 \text{ OR } x - 5 = 0$$

$$\Rightarrow \quad x = -5, 5$$

Thus, –5 and 5 are the two *boundary points*. Now plot these two points on the real line.

Notice that our line is broken up into three intervals: $(-\infty, -5)$, $(-5, 5)$, and $(5, \infty)$.

Now for some *test points* — any point in a given interval will do the trick. We'll choose –7 for the left interval, 2 for the center interval, and 9 for the right interval. In other words,

$$\mathbf{-7} \in (-\infty, -5) \qquad \mathbf{2} \in (-5, 5) \qquad \mathbf{9} \in (5, \infty)$$

The symbol ∈ means "is an element of" , or, "is a member of"

Testing each test point in the original problem:

–7: $(-7)^2 < 25 \Rightarrow 49 < 25$ NO

2: $2^2 < 25 \Rightarrow 4 < 25$ YES

9: $9^2 < 25 \Rightarrow 81 < 25$ NO

The last step is to check the boundary points; I'll leave it to you to determine that they do NOT work. [You'll get "equal to," not "less than."]

One interval worked, while the other two failed. Our final answer is that center interval, without the boundary points:

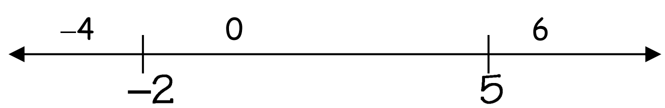
$\boxed{(-5, 5)}$ which is double inequality: $-5 < x < 5$

<u>EXAMPLE 3:</u> **Solve for x: $x^2 - 3x - 10 \geq 0$**

<u>Solution:</u> Again, by using the Boundary Point Method, we first we find the boundary points by solving the associated equation:

$$x^2 - 3x - 10 = 0 \Rightarrow (x + 2)(x - 5) = 0 \Rightarrow x = -2 \text{ or } x = 5$$

Now we mark these boundary points on a number line, and choose a test point in each interval to test in the original inequality $x^2 - 3x - 10 \geq 0$:



$-4 \in (-\infty, -2)$ $0 \in (-2, 5)$ $6 \in (5, \infty)$

$(-4)^2 - 3(-4) - 10 = 16 + 12 - 10 = 18$, which is ≥ 0. Thus, the interval $(-\infty, -2)$ <u>is</u> part of the solution .

$0^2 - 3(0) - 10 = 0 - 0 - 10 = -10$, which is not ≥ 0. So, $(-2, 5)$ is <u>not</u> part of the solution.

$6^2 - 3(6) - 10 = 36 - 18 - 10 = 8$, which is ≥ 0. Therefore, $(5, \infty)$ <u>is</u> part of the solution.

Last, you can check the boundary points yourself — they work in the original inequality. Our final answer is the combination of the left interval and the right interval:

$$(-\infty, -2] \cup [5, \infty)$$

EXAMPLE 4: **Solve for w: $9w - w^2 \geq 14$**

Solution: Solve the associated equation: $9w - w^2 = 14$. Rearranging the terms and multiplying through by –1 gives the equation $w^2 - 9w + 14 = 0$, whose solutions you can determine to be 2 and 7.

These two boundary points yield the three intervals to check: $(-\infty, 2)$, $(2, 7)$ and $(7, \infty)$. Choose a test point in each interval and substitute that test point into the original inequality. You'll find that the only interval that works is the center one: $(2, 7)$.

Finally, check the boundary points themselves. They both work in the original inequality. The final answer, then, is $2 \leq w \leq 7$, which we write as

$$[2, 7]$$

EXAMPLE 5: **Solve for y: $y^2 - 12 > -4y$**

Solution: To solve this inequality, convert it to an equation, write it in standard quadratic form, factor, and you'll find that the boundary points are –6 and 2. Pick some test points and see that the intervals $(-\infty, -6)$ and $(2, \infty)$ work, but the interval $(-6, 2)$ doesn't. Also, the boundary points themselves don't work. The final answer is therefore

$$(-\infty, -6) \cup (2, \infty)$$

Homework

Solve each quadratic inequality:

4. $a^2 - 9a - 10 > 0$
5. $2x^2 \le 5 - 9x$
6. $y^2 + 10y + 25 \le 0$
7. $z^2 > 12z - 36$
8. $x^2 \ge 9$
9. $2x^2 - 15x + 7 < 0$
10. $5a - a^2 \ge 6$

❑ *THE MEANING OF ABSOLUTE VALUE*

The Prologue gave us our first encounter with the notion of absolute value, while Chapter 2 taught us about the graphs of absolute values. We recall that we use a pair of vertical bars to represent absolute value; the absolute value of x is written $|x|$.

If a number is greater than or equal to 0, then its absolute value is that same number. If a number is less than 0 (which means it's a negative number), then its absolute value is the opposite of that number (which will be a positive number, right?). We can write this in the following way:

$$\text{If } x \ge 0, \text{ then } |x| = x$$
$$\text{If } x < 0, \text{ then } |x| = -x$$

This definition ensures that the absolute value of a quantity is never negative.

Here are some examples:

$$|9| = 9 \qquad |0| = 0 \qquad |-13| = 13$$

An Alternative Definition of Absolute Value

Consider a number line and ask, "What is the distance between a given number and 0? The answer to that question is the ***absolute value*** of that number.

What is the distance between 9 and 0? It's 9, so $|9| = 9$.

What is the distance between 0 and 0? The distance is 0, so $|0| = 0$.

And what is the distance between –13 and 0? Since it's 13, $|-13| = 13$.

Note

The absolute value of a quantity is either positive or zero.

If $\boldsymbol{x}$ is any real number, then $|\boldsymbol{x}| \geq \mathbf{0}$.

In other words, the absolute value of a quantity is never negative. To illustrate this point, although you may not be able to calculate

$$\left|\sin^2(\pi/6) + \ln(e-1)\right|$$

until pre-calculus, you should still be able to understand that the answer to this problem must be greater than or equal to zero. The expression, no matter what it all means, is definitely not negative.

EXAMPLE 6: **Evaluate each expression:**

A. $|7-2| = |5| = \mathbf{5}$

B. $|3^2-15| = |9-15| = |-6| = \mathbf{6}$

C. $-|2-7| = -|-5| = \mathbf{-5}$

D. $|-3-4|-|10-2| = |-7|-|8| = 7-8 = \mathbf{-1}$

E. $|3^3-9\cdot 3| = |27-27| = |0| = \mathbf{0}$

Homework

11. The absolute value of any real number is ______.

12. True/False:

    a. Every real number has an absolute value.
    b. There is a number whose absolute value is 0.
    c. There is a number whose absolute value is negative.
    d. There are two different numbers whose absolute value is 9.

13. Simplify each expression:

    a. $|4-14|$ b. $|2(-3)-7|$ c. $-|-9|$ d. $|2^0-\pi^0|$

❑ *ABSOLUTE VALUE EQUATIONS*

EXAMPLE 7: **Solve for x: $|x| = 12$**

Solution: What can you take the absolute value of and get a result of 12? Well, the absolute value of 12 is 12, so x could be 12. But –12 also has an absolute value of 12, so x could be –12, too. In other words, since $|12| = 12$ and $|-12| = 12$, it appears that this equation has two solutions, 12 and –12.

$$\boxed{x = \pm 12}$$

EXAMPLE 8: **Solve for n: $|n| = -5$**

Solution: This equation is a statement that the absolute value of some number is –5. But the absolute value of any number is greater than or equal to zero; that is, the absolute value of any number can never be negative. Thus, there is no number n that will work in this equation. Our conclusion:

$$\boxed{\text{No solution}}$$

EXAMPLE 9: **Solve for w: $|w| = 0$**

Solution: What number has an absolute value of 0? There's only one such number, and it's 0: $|0| = 0$. Therefore,

$$\boxed{w = 0}$$

EXAMPLE 10: **Solve for x: $|8-2x| = 20$**

Solution: Here's what we ask ourselves: "What has an absolute value of 20?" There are two numbers that have an absolute value of 20, and they are 20 and –20. This means that the quantity inside the absolute value sign can be either 20 or –20. In other words, the quantity 8 – 2x can be either 20 or –20. This gives us two equations to solve:

$8-2x = \mathbf{20}$	$8-2x = \mathbf{-20}$
$-2x = 12$	$-2x = -28$
$x = -6$	$x = 14$

Our absolute value equation has two solutions:

$$x = -6, 14$$

Check:

$\underline{x=-6:}$ $|8-2x| = |8-2(\mathbf{-6})| = |8+12| = |20| = 20$ ✓

$\underline{x=14:}$ $|8-2x| = |8-2(\mathbf{14})| = |8-28| = |-20| = 20$ ✓

Homework

14. Solve each absolute value equation:

a. $|t| = 4$ b. $|n| = 0$ c. $|R| = -1$

d. $|x+1| = 9$ e. $|x-3| = 5$ f. $|2x+8| = 0$

g. $|2-5x| = -3$ h. $|2x+8| = 10$ i. $|3y-6| = 9$

j. $|2a+1| = 19$ k. $|7-5y| = 3$ l. $\left|7x+\sqrt{7}\right| = -\frac{\pi}{2}$

m. $|t|-7 = 4$ n. $|n|-\pi = -\pi$ o. $|R|+7 = 3$

p. $|x-4| = 9$ q. $|x+20| = 5$ r. $|2x-17| = 0$

s. $|2-5x|-2 = -3$ t. $|9-2x| = 10$ u. $|3y-20| = 9$

v. $|2a+3| = 21$ w. $|5-7y| = 5$ x. $\left|\sqrt{7}\,x+\sqrt{7}\right| = -\sqrt{7}$

Practice Problems

15. a. What is the distance between –200 and 0 on the number line?
    b. What is the distance between 7π and 0 on the number line?
    c. What is the distance between 0 and 0 on the number line?
    d. What is the distance between $-7\sqrt{11}$ and 0 on the number line?

16. Evaluate: $|7|-|-4|+|\pi-\pi|$

17. a. If $x \ge 0$, then $|x| =$ ______
    b. If $x < 0$, then $|x| =$ ______

18. Which is the best statement regarding $|x|$, where x is any real number?
    a. $|x| < 0$ b. $|x| > 0$ c. $|x| \le 0$ d. $|x| \ge 0$

Solve each inequality or equation:

19. $-2n - 9 \geq -12$

20. $\frac{8+3x}{-5} < -10$

21. $-2(3y - 9) + 1 \leq 10 - (3y - 4) - y$

22. $|x| = 44$

23. $|y| + 17 = 15$

24. $|z-1| = 0$

25. $|2x+9| = 10$

26. $|7n-20| = 13$

27. $2a - 9 < 9a + 7$

28. $-5x + 5 \geq -9x - 10$

29. $\frac{10-2x}{-4} < -14$

30. $\frac{-8t+2t-10}{-3} \leq -9$

Solutions

1. a. $x > -17$ b. $x \leq 8$ c. $x \geq 7$
d. $x > 14$ e. $x > -24$ f. $x \leq -50$

2. a. $y \geq -\frac{52}{7}$ b. $u < -\frac{5}{7}$

3. a. $x > \frac{19}{3}$ b. $a < -\frac{67}{7}$ c. $y \geq 5$ d. $n \geq -1$

4. $(-\infty, -1) \cup (10, \infty)$ **5.** $\left[-5, \frac{1}{2}\right]$ **6.** $\{-5\}$

7. $(-\infty, 6) \cup (6, \infty)$; or $\mathbb{R} - \{6\}$ **8.** $(-\infty, -3] \cup [3, \infty)$

9. $\left(\frac{1}{2}, 7\right)$ **10.** $[2, 3]$

11. ≥ 0

12. a. T b. T c. F d. T

13. a. 10 b. 13 c. −9 d. 0

14. a. $t = \pm 4$ b. $n = 0$ c. No solution d. $x = 8, -10$
e. $x = 8, -2$ f. $x = -4$ g. No solution h. $x = 1, -9$
i. $y = 5, -1$ j. $a = 9, -10$ k. $y = \frac{4}{5}, 2$ l. No solution
m. $t = \pm 11$ n. $n = 0$ o. No solution p. $x = 13, -5$
q. $x = -15, -25$ r. $x = \frac{17}{2}$ s. No solution t. $x = \frac{-1}{2}, \frac{19}{2}$
u. $y = \frac{29}{3}, \frac{11}{3}$ v. $a = 9, -12$ w. $y = 0, \frac{10}{7}$ x. No solution

15. a. 200 b. 7π c. 0 d. $7\sqrt{11}$

16. $7 - 4 + 0 = 3$ **17**. a. x b. $-x$ **18**. d.

19. $n \le \frac{3}{2}$ **20**. $x > 14$ **21**. $y \ge \frac{5}{2}$ **22**. $x = \pm 44$

23. No solution **24**. $z = 1$ **25**. $z = \frac{1}{2}, -\frac{19}{2}$

26. $n = \frac{33}{7}, 1$ **27**. $a > -\frac{16}{7}$ **28**. $x \ge -\frac{15}{4}$

29. $x < -23$ **30**. $t \le -\frac{37}{6}$

"*Effort* only fully releases its reward after a person refuses to quit."

– Napoleon Hill