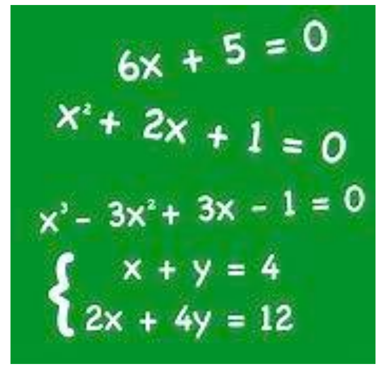
CH 25 – MORE EQUATIONS

❑ INTRODUCTION

This chapter is a collection of additional equations. The first few sections are polynomial equations that require factoring to solve. (You can never be too good at factoring.) The last sections will teach you how to solve radical equations — these are equations with square roots or cube roots in them.



❑ CUBIC EQUATIONS USING THE GCF

<u>EXAMPLE 1:</u> **Solve the cubic equation: $30x^3 + 99x^2 = 21x$**

<u>Solution:</u> Since factoring is the way we solved quadratic equations earlier in the course, let's give factoring a try here. The factoring method requires one side of the equation to be 0:

$$30x^3 + 99x^2 - 21x = 0$$

Another theme in this chapter is that factoring always begins with an attempt to factor out the GCF, which in this case is $3x$:

$$3x(10x^2 + 33x - 7) = 0$$

Further factoring of the trinomial in the parentheses gives the final factorization of the left side our equation:

$$3x(5x - 1)(2x + 7) = 0$$

Noting that $3x$ is one of the factors, we have three factors whose product is 0; we therefore set <u>each</u> of the three factor to 0:

$$3x = 0 \quad \text{or} \quad 5x - 1 = 0 \quad \text{or} \quad 2x + 7 = 0$$

Solving each of these three linear equations gives us three solutions to our cubic equation:

$$\boxed{x = 0, \frac{1}{5}, -\frac{7}{2}}$$

Homework

1. Solve each cubic equation:

a. $x^3 + 3x^2 + 2x = 0$

b. $4n^3 - 18n^2 + 8n = 0$

c. $x^3 = 16x$

d. $3y^3 = -30y^2 - 75y$

e. $a^3 + 9a = 0$

f. $30x^3 + 25x^2 - 30x = 0$

2. Solve for x: $x^2(x + 1)(2x - 3)(x + 7)^3(x^2 - 4)(x^2 - 5x + 6) = 0$

❑ *CUBIC EQUATIONS USING GROUPING*

<u>EXAMPLE 2:</u> **Solve the cubic equation: $x^3 + 5x^2 = 9x + 45$**

<u>Solution:</u> The first step is to bring the terms on the right side of the equation to the left:

$$x^3 + 5x^2 - 9x - 45 = 0$$

Factor the GCF in the first pair of terms and the last pair of terms:

$$x^2(x + 5) - 9(x + 5) = 0$$

Pull out the GCF, $x + 5$:

$$(x + 5)(x^2 - 9) = 0$$

Continue the factoring of the difference of squares:

$$(x + 5)(x + 3)(x - 3) = 0$$

Set each factor to 0:

$$x + 5 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$

Solving each linear equation gives:

$$x = -5 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 3$$

And now we have our three solutions:

$$\boxed{x = -5, -3, 3}$$

❑ *QUARTIC EQUATIONS*

<u>EXAMPLE 3:</u> **Solve the quartic equation: $x^4 - 26x^2 + 25 = 0$**

<u>Solution:</u> The factoring we learned in Chapter 19 is exactly what we need here:

$$x^4 - 26x^2 + 25 = 0$$

$$\Rightarrow \quad (x^2 - 1)(x^2 - 25) = 0$$

$$\Rightarrow \quad (x + 1)(x - 1)(x + 5)(x - 5) = 0$$

$$\Rightarrow \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 5 = 0 \quad \text{or} \quad x - 5 = 0$$

$$\Rightarrow \quad x = -1 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -5 \quad \text{or} \quad x = 5$$

$$\boxed{x = \pm 1, \pm 5}$$

EXAMPLE 4: **Solve for n: $2n^4 - 15n^2 = 27$**

Solution: The factoring technique requires one side of the equation to be 0, so our first step is to make that happen, by subtracting 27 from each side of the equation:

$$2n^4 - 15n^2 - 27 = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n^2 - 9) = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n + 3)(n - 3) = 0$$

$$\Rightarrow \quad 2n^2 + 3 = 0 \text{ or } n + 3 = 0 \text{ or } n - 3 = 0$$

Let's first try to solve the first equation:

$$2n^2 + 3 = 0 \Rightarrow 2n^2 = -3 \Rightarrow n^2 = -\frac{3}{2} \Rightarrow n = \pm\sqrt{-\frac{3}{2}},$$

which are numbers that are NOT in the set of real numbers, $\mathbb{R}$. In this class, we conclude that the equation $2n^2 + 3 = 0$ has No Solution. The other two equations should be easy for you by now. The final solution is

$$\boxed{x = \pm 3}$$

EXAMPLE 5: **Solve for a: $a^4 = -13a^2 - 36$**

Solution:

$$a^4 = -13a^2 - 36$$

$$\Rightarrow \quad a^4 + 13a^2 + 36 = 0$$

$$\Rightarrow \quad (a^2 + 9)(a^2 + 4) = 0$$

$$\Rightarrow \quad a^2 + 9 = 0 \text{ or } a^2 + 4 = 0$$

I hope it's clear that neither of these last two equations has a solution in $\mathbb{R}$, the set of real numbers. We're done:

No Solution

Homework

3. Solve each equation:

a. $x^3 + x^2 - 16x = 16$ b. $n^4 = 13n^2 - 36$

c. $2a^4 - 49a^2 = 25$ d. $x^4 + 17x^2 + 16 = 0$

❑ *RADICAL EQUATIONS*

An equation such as $\sqrt{x-4} - 5 = 0$, with the variable in the radical sign, is called a ***radical equation***. Just as subtraction is undone by addition, a radical is removed by raising to a power.

Here are the major steps to solve a radical equation:

1. Isolate the radical.
2. Raise each side of the equation to an appropriate power.
3. Solve the resulting equation for potential solutions.
4. Check every potential solution in the <u>original</u> equation.

<u>EXAMPLE 1</u>: **Solve for z: $\sqrt{z} = 7$**

<u>Solution</u>: The radical is already isolated, so on to step 2. Since the radical is a <u>square root</u>, we will <u>square</u> each side of the equation:

$\sqrt{z} = 7$ (the original equation)

$\Rightarrow \left(\sqrt{z}\right)^2 = 7^2$ (square each side of the equation)

$\Rightarrow \boxed{z = 49}$ (squaring undoes square rooting)

Put 49 back into the original equation, and you'll see that we have a valid solution.

<u>EXAMPLE 2</u>: **Solve for x: $\sqrt{x-4} - 5 = 0$**

<u>Solution</u>: Our plan is to isolate and square.

Start with the original equation: $\sqrt{x-4} - 5 = 0$

First, isolate the radical: $\sqrt{x-4} = 5$

Second, to remove the square root sign, square each side of the equation: $\left(\sqrt{x-4}\right)^2 = 5^2$

Simplify each side of the equation: $x - 4 = 25$

Now solve for x: $\underline{x = 29}$

Now it's time to check our solution. Let's put our candidate, 29, into the <u>original</u> equation, and perform the arithmetic separately on each side of the equation:

$\sqrt{\mathbf{29}-4}-5$	0
$\sqrt{25}-5$	
$5-5$	
0	✓

29 checks out; therefore, the solution is $\boxed{x = 29}$

EXAMPLE 3: **Solve for *a*:** $\sqrt{\boldsymbol{a}^2+\mathbf{15}}+\mathbf{10} = \mathbf{2}$

Solution: We isolate the radical, square each side, solve for a, and then CHECK the solutions.

$$\sqrt{a^2+15}+10 = 2 \quad \text{(the original equation)}$$
$$\Rightarrow \sqrt{a^2+15} = -8 \quad \text{(isolate the radical)}$$
$$\Rightarrow \left(\sqrt{a^2+15}\right)^2 = (-8)^2 \quad \text{(square each side)}$$
$$\Rightarrow a^2 + 15 = 64 \quad \text{(simplify each side)}$$
$$\Rightarrow a^2 - 49 = 0 \quad \text{(subtract 64 from each side)}$$
$$\Rightarrow (a-7)(a+7) = 0 \quad \text{(factor)}$$
$$\Rightarrow a-7=0 \text{ or } a+7=0 \quad \text{(set each factor to 0)}$$
$$\Rightarrow \underline{a = 7 \text{ or } a = -7} \quad \text{(solve for } a\text{)}$$

That's two possible solutions, 7 and –7. Let's check out the 7:

$\sqrt{\mathbf{7}^2+15}+10$	2
$\sqrt{49+15}+10$	
$\sqrt{64}+10$	
$8+10$	
18	✗

The two sides of the equation do <u>not</u> balance; therefore, 7 is <u>not</u> a solution of the radical equation. Using –7 for *a* will result in the same calculations. So, what do we have here? We solved an equation and got two potential solutions, 7 and –7, but neither of them panned out. That's it — it's over — nothing works.

No Solution

<u>EXAMPLE 4:</u> **Solve for *y*:** $\sqrt[3]{2y+7}-3 = 0$

<u>Solution:</u> Step 1: Isolate the radical.
Step 2: Remove the cube root by *cubing* each side of the equation:

$\sqrt[3]{2y+7} = 3$ (add 3 to each side of the equation)

$\Rightarrow \left(\sqrt[3]{2y+7}\right)^3 = 3^3$ (raise each side to the third power)

$\Rightarrow 2y+7 = 27$ (simplify each side of the equation)

$\Rightarrow 2y = 20$ (subtract 7 from each side)

$\Rightarrow \underline{y = 10}$ (divide each side by 2)

Let's check this solution:

$\sqrt[3]{2(\mathbf{10})+7}$	3
$\sqrt[3]{20+7}$	
$\sqrt[3]{27}$	
3	✓

Therefore the solution of the radical equation is $\boxed{y = 10}$

Homework

Solve and CHECK each equation:

4. $\sqrt{x} = 10$

5. $\sqrt{y} = 5$

6. $\sqrt{t} = 0$

7. $\sqrt{a} = -3$

8. $\sqrt{x+1} = 3$

9. $\sqrt{c-5} = 10$

10. $\sqrt{x+5} - 2 = 3$

11. $\sqrt{2y-1} + 3 = 6$

12. $\sqrt{8-n} - 4 = 0$

13. $\sqrt{7z+100} + 10 = 20$

14. $\sqrt{x^2+11} = 6$

15. $\sqrt{w^2-13} - 6 = 0$

16. $\sqrt{a^2+8} - 1 = 2$

17. $\sqrt{q^2-17} + 7 = 15$

18. $\sqrt{R^2+11} + 6 = 0$

19. $\sqrt[3]{3y-25} = 5$

20. $\sqrt[4]{2x+10} - 2 = 0$

21. $\sqrt[5]{3a+2} = 2$

22. $\sqrt[4]{x+1} + 3 = 0$

23. $\sqrt[3]{5w+2} + 2 = 0$

EXAMPLE 5: **Solve for x:** $\sqrt{11-x}+5 = x$

Solution: To isolate the radical, we need to subtract 5 from each side of the equation:

$$\sqrt{11-x} = x-5$$

Since we now have a square root to undo, we will square each side of the equation:

$$\left(\sqrt{11-x}\right)^2 = (x-5)^2$$

Now we simplify each side of the equation:

$$11 - x = x^2 - 10x + 25$$

Recognizing this equation as quadratic, we'll put it into standard form by adding x to each side and subtracting 11 from each side:

$$0 = x^2 - 9x + 14$$

Factoring produces

$$0 = (x-7)(x-2)$$

Setting each factor to 0 and solving gives us two candidates for a solution:

$\underline{x = 7 \text{ or } x = 2}$

We now check each potential solution in the original equation:

$x = 7$		$x = 2$	
$\sqrt{11-\mathbf{7}}+5$	**7**	$\sqrt{11-\mathbf{2}}+5$	**2**
$\sqrt{4}+5$		$\sqrt{9}+5$	
$2 + 5$		$3 + 5$	
7	✓	8	✗

What does all this mean? Easy: the 7 works and the 2 doesn't. So, even though we got two solutions when we solved the radical equation, only one of them worked. The 2, which we obtained — but failed to work in the original equation — is called an *extraneous* solution. (By the way, just because one of the solutions failed to work does not mean that we made any mistakes when we solved the problem.) Therefore, the final solution of the equation is

$$\boxed{x = 7}$$

Homework

24. $\sqrt{x+1} - x + 5 = 0$

25. $\sqrt{n+3} = n - 3$

26. $\sqrt{t+8} - 6 = t$

27. $\sqrt{v-1} - v = -7$

28. $\sqrt{2x+2} = x - 3$

29. $\sqrt{r+11} - r - 5 = 0$

30. $\sqrt{h+2} - h = 0$

31. $\sqrt{5x+4} - x = -4$

Practice Problems

Solve and check each equation:

32. $30x^3 = 2x^2 + 4x$

33. $n^3 + 5 = 5n^2 + n$

34. $x^4 + 900 = 109x^2$

35. $a^4 + 3a^2 - 4 = 0$

36. $t^4 + 49 + 50t^2 = 0$

37. $w^3 + 2w^2 - 12w - 9 = 0$ [Hard]
Hint: To factor, divide by $w - 3$.

Solve and check each equation:

38. $\sqrt{n} = 0$

39. $\sqrt{w} - 7 = 0$

40. $\sqrt{y} + 3 = 0$

41. $\sqrt{7x + 14} = 7$

42. $\sqrt[3]{5x + 14} - 10 = -6$

43. $\sqrt[4]{x + 9} + 3 = 0$

44. $\sqrt[5]{10x - 12} + 2 = 0$

45. $\sqrt{x^2 - 9x + 20} = 0$

46. $\sqrt{7y + 16} - y = -2$

47. $\sqrt{5 - 5x} - 1 = -x$

Solutions

1. a. $x = 0, -1, -2$ b. $n = 0, \frac{1}{2}, 4$ c. $x = 0, 4, -4$

d. $y = 0, -5$ e. $a = 0$ f. $x = 0, \frac{2}{3}, -\frac{3}{2}$

2. There are quite a few solutions.

3. a. $x = \pm 4, -1$ b. $n = \pm 2, \pm 3$ c. $a = \pm 5$ d. No solution

4. $x = 100$ **5.** $y = 25$ **6.** $t = 0$ **7.** No solution

8. $x = 8$ **9.** $c = 105$ **10.** $x = 20$ **11.** $y = 5$

12. $n = -8$ **13.** $z = 0$ **14.** $x = \pm 5$ **15.** $w = \pm 7$

16. $a = \pm 1$ **17.** $q = \pm 9$ **18.** No solution **19.** $y = 50$

20. $x = 3$ **21.** $a = 10$ **22.** No solution **23.** $w = -2$

24. $x = 8$ **25.** $n = 6$ **26.** $t = -4$ **27.** $v = 10$

28. $x = 7$ **29.** $r = -2$ **30.** $h = 2$ **31.** $x = 12$

32. $x = 0, -\frac{1}{3}, \frac{2}{5}$ **33.** $n = 5, \pm 1$ **34.** $x = \pm 3, \pm 10$

35. $a = \pm 1$ **36.** No solution

37. After you divide the cubic by $w - 3$, the other factor should be a quadratic, but it is NOT factorable. So the only solution is $w = 3$. Note: The solutions provided by the quadratic can be determined four or five chapters from now.

38. $n = 0$ **39.** $x = 49$ **40.** No solution **41.** $x = 5$

42. $x = 10$ **43.** No solution **44.** $x = -2$ **45.** $x = 4, 5$

46. $y = 12$ **47.** $x = -4, 1$

Edith Hamilton:

"To be able to be caught up into the world of thought – that is educated."