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# CH 27 – RADICALS, PART II

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## □ PROVING THE THEOREMS FROM CHAPTER 24

The chapter on fractional exponents showed us that  $x$  to the power  $1/n$  is equal to the  $n$ th root of  $x$ :

$$x^{1/n} = \sqrt[n]{x}$$

This way of expressing radicals using exponents allows us to prove a pair of rules we took for granted in Chapter 24.

THEOREM: Assume that  $x$  and  $y$  represent non-negative numbers. Then

$$\sqrt[n]{xy} = \sqrt[n]{x} \sqrt[n]{y}$$

The root of a product  
is the product of the roots.

Proof: Our technique will be to convert the left side of the formula to exponent form, use a law of exponents, and then convert back to radical form to arrive at the right side of the formula.

$$\begin{aligned} & \sqrt[n]{xy} \\ = & (xy)^{1/n} && \text{(definition of fractional exponent)} \\ = & x^{1/n} y^{1/n} && \text{(law of exponents)} \\ = & \sqrt[n]{x} \sqrt[n]{y} && \text{(definition of fractional exponent)} \end{aligned}$$

**THEOREM:** If  $x$  represents a non-negative number (that is, if  $x \geq 0$ ), then

$$\sqrt[n]{x^n} = x$$

The  $n$ th root cancels  
the  $n$ th power.

**Proof:** Our method of proof is similar to the previous theorem.

$$\sqrt[n]{x^n} = (x^n)^{1/n} = x^{n \cdot \frac{1}{n}} = x^1 = x$$

## □ **ADDING AND SUBTRACTING RADICALS**

We have a simple radical rule that states that  $\sqrt{a}\sqrt{b} = \sqrt{ab}$ . How about a rule like  $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ ? Seems reasonable; let's perform an experiment. Let  $a = 16$  and  $b = 9$ .

$$\text{Left side: } \sqrt{a} + \sqrt{b} = \sqrt{16} + \sqrt{9} = 4 + 3 = 7$$

$$\text{Right side: } \sqrt{a+b} = \sqrt{16+9} = \sqrt{25} = 5$$

We get two different results; we thus conclude that  $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ .

So maybe we can't add  $\sqrt{7}$  and  $\sqrt{11}$  together, but we can add  $2\sqrt{7}$  and  $3\sqrt{7}$  together. This is simply the *combining of like terms*. After all, we may not be very comfortable with a number like  $\sqrt{7}$ , but certainly 2 of them plus 3 more of them should total 5 of them:  $2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$ .

Alternatively, we can apply the distributive property to the sum  $2\sqrt{7} + 3\sqrt{7}$  by factoring out the GCF:  $\sqrt{7}(2+3)$ , again giving  $5\sqrt{7}$ .

Thus, to add (or subtract) radicals, we can use either the concept of like terms or the idea of factoring.

**EXAMPLE 1: Simplify each radical expression:**

A.  $3\sqrt{11} + 5\sqrt{11} = 8\sqrt{11}$

They're like terms, so we can add them up, just like  
 $3x + 5x = 8x$ .

Or, factor out the GCF:  $\sqrt{11}(3 + 5) = 8\sqrt{11}$

B.  $18\sqrt[3]{10} - 20\sqrt[3]{10} = -2\sqrt[3]{10}$

C.  $\sqrt{3} + \sqrt{5}$  cannot be worked out. They're unlike terms, and no amount of simplification will turn them into like terms.

D.  $3\sqrt{20} + 10\sqrt{45}$  It might appear that these terms cannot be combined, but don't jump to conclusions — a little simplification first will yield an answer:

$$\begin{aligned} & 3\sqrt{20} + 10\sqrt{45} \\ = & 3\sqrt{4 \cdot 5} + 10\sqrt{9 \cdot 5} \\ = & 3 \cdot 2\sqrt{5} + 10 \cdot 3\sqrt{5} \\ = & 6\sqrt{5} + 30\sqrt{5} && \text{[Like Terms]} \\ = & 36\sqrt{5} \end{aligned}$$

E.  $\sqrt{7} + \sqrt[3]{7}$  can't be simplified. Even though the radicands (the 7's) are the same, the roots are different, and therefore they are unlike terms and cannot be added together.

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## Homework

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1. Simplify each radical expression:

a.  $2\sqrt{x} + 10\sqrt{x}$

b.  $3\sqrt{7} - 2\sqrt{2}$

c.  $10\sqrt[3]{n} + 3\sqrt[3]{n}$

d.  $6\sqrt{x} - 2\sqrt[3]{x}$

e.  $9\sqrt{8} + 10\sqrt{50}$

f.  $\sqrt[3]{24} + \sqrt[3]{81}$

g.  $3\sqrt{80} - 5\sqrt{45}$

h.  $10\sqrt[3]{189} + \sqrt[3]{56}$

i.  $\sqrt[3]{z^2} + 2\sqrt[3]{z^2}$

j.  $\sqrt{12} + 2\sqrt{75}$

k.  $2\sqrt{20} + 2\sqrt{45}$

l.  $2\sqrt{275} - 3\sqrt{99}$

## □ MULTIPLYING RADICALS

Using the formula  $\sqrt[n]{x} \sqrt[n]{y} = \sqrt[n]{xy}$ , we can multiply “like roots.”

**EXAMPLE 2:** Simplify each radical expression:

A.  $\sqrt[5]{3} \cdot \sqrt[5]{7} = \sqrt[5]{21}$

B.  $\sqrt{7}\sqrt{7} = \sqrt{49} = 7$

C.  $\sqrt[3]{17} \times \sqrt[3]{3} = \sqrt[3]{51}$

D.  $\sqrt{2}\sqrt{8} = \sqrt{16} = 4$

E.  $\sqrt{14}\sqrt{2} = \sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$

F.  $\sqrt{2}(1 + \sqrt{2}) = \sqrt{2} + \sqrt{2}\sqrt{2} = \sqrt{2} + 2$

G.  $\sqrt{6}(\sqrt{10} - \sqrt{6}) = \sqrt{6}\sqrt{10} - \sqrt{6}\sqrt{6} = \sqrt{60} - 6 = 2\sqrt{15} - 6$

$$\begin{aligned} \text{H. } (\sqrt{3} + 1)(\sqrt{3} - 5) &= \sqrt{3}\sqrt{3} - 5\sqrt{3} + 1\sqrt{3} - 5 \\ &= 3 - 4\sqrt{3} - 5 = -2 - 4\sqrt{3} \end{aligned}$$

I.  $(1 + \sqrt{7})^2 = (1 + \sqrt{7})(1 + \sqrt{7}) = 1 + \sqrt{7} + \sqrt{7} + 7 = 8 + 2\sqrt{7}$

$$\begin{aligned} \text{J. } (3 + \sqrt{10})(3 - \sqrt{10}) &= 9 - 3\sqrt{10} + 3\sqrt{10} - \sqrt{10}\sqrt{10} \\ &= 9 - 10 = -1 \end{aligned}$$

## Homework

2. Find the product:

a.  $\sqrt{14}\sqrt{6}$

b.  $\sqrt[3]{3} \times \sqrt[3]{9}$

c.  $\sqrt{2}\sqrt{7}$

d.  $\sqrt{171}\sqrt{171}$

e.  $\sqrt{14}\sqrt{2}$

f.  $\sqrt[3]{10} \sqrt[3]{12.5}$

g.  $\sqrt[4]{2} \sqrt[4]{8}$

h.  $\sqrt{x}\sqrt{yz}$

i.  $\sqrt[3]{2} \sqrt[3]{4}$

3. Find the product:

a.  $\sqrt{n}(1 + \sqrt{n})$

b.  $(x + \sqrt{y})(x - \sqrt{y})$

c.  $(2 + \sqrt{3})^2$

d.  $(\sqrt{p} - q)^2$

e.  $(\sqrt{7} - 2)(\sqrt{7} + 2)$

f.  $(x - \sqrt{n})^2$

g.  $(\sqrt{a} + \sqrt{c})(\sqrt{a} - \sqrt{c})$

h.  $2(\sqrt{3} + \sqrt{4})$

i.  $(\sqrt{x} + \sqrt{x})^2$

### □ ***DIVIDING RADICALS***

Just as the square root of a product is the product of the square roots ( $\sqrt{ab} = \sqrt{a}\sqrt{b}$ ), the square root of a quotient is the quotient of the square roots.

**THEOREM:** Assume  $a$  and  $b$  are positive real numbers. Then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\begin{aligned}
 \text{Proof: } & \sqrt{\frac{a}{b}} \\
 &= \left(\frac{a}{b}\right)^{1/2} && \text{(convert radical to fractional exponent)} \\
 &= \frac{a^{1/2}}{b^{1/2}} && \text{(a law of exponents)} \\
 &= \frac{\sqrt{a}}{\sqrt{b}} && \text{(convert back to radical form)}
 \end{aligned}$$

Before we get to the heart of dividing radicals, let's recall the little formula  $(\sqrt[n]{x})^n = x$ . Check out the following examples:

$$\begin{aligned}
 (\sqrt{u})^2 &= u & (\sqrt[3]{w})^3 &= w & (\sqrt[4]{xy})^4 &= xy \\
 \sqrt{c}\sqrt{c} &= (\sqrt{c})^2 = c & \sqrt{17} \times \sqrt{17} &= (\sqrt{17})^2 = 17
 \end{aligned}$$

Now consider the number  $\frac{1}{\sqrt{2}}$ . It's just a fraction, and a calculator

gives an *approximate* value of 0.707106781. But now consider the scenario where you have no calculator, but you need a decimal version of the fraction. Even if you could look up the positive square root of 2 somehow — and find that it's about 1.414213562 — you'd still be stuck with the following long division problem:

$$1.414213562 \overline{) 1.000000000}$$

This is a killer long division problem — it would take a really long time to get a few digits of the answer.

But now watch this trick: I'm going to take the fraction  $\frac{1}{\sqrt{2}}$  and

multiply it by the fraction  $\frac{\sqrt{2}}{\sqrt{2}}$  (which, of course, equals 1). Here we go:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \left[ \frac{\sqrt{2}}{\sqrt{2}} \right] = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We've shown that  $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ . [Double-check with your calculator.]

What's the purpose of all this? Consider the long division problem created by the second fraction,  $\frac{\sqrt{2}}{2}$ :

$$2 \overline{)1.414213562}$$

This is much simpler to carry out than the previous long division problem, and yet it yields the same answer. This is the historical reason that students, especially in the sciences, would convert  $\frac{1}{\sqrt{2}}$  into  $\frac{\sqrt{2}}{2}$  — it made calculations much simpler.

Are we going to perform this kind of maneuver — removing radicals from the denominator of a fraction — in our Algebra course? Yes, and for two reasons. One is historical; it's been done like this for centuries. As Tevya from *Fiddler on the Roof* might say, "We do it because it's our tradition!" A second reason: It's a lot easier when we look up answers in the "back of the book" if we can all agree on what form to leave our answers. We thus have a basic agreement in math: Do not leave radicals in the denominator of a fraction. Converting a fraction with a radical in the denominator into an equivalent fraction without a radical in the denominator is called **rationalizing the denominator**.

I'll let you in on a little secret: If you take Business Calculus, there will be problems where you will want to leave radicals in the denominator. Indeed, you will actually remove radicals from the numerator.

### EXAMPLE 2:

A. Rationalize the denominator, which is a **monomial**:

$$\frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}} \left[ \frac{\sqrt{x}}{\sqrt{x}} \right] = \frac{3\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{3\sqrt{x}}{x}$$

B. Rationalize the denominator, which is a **binomial**:

For this example, the trick is in figuring out what, exactly, we should multiply the top and the bottom of the fraction by.

$$\frac{\sqrt{2}}{5-\sqrt{10}} = \frac{\sqrt{2}}{5-\sqrt{10}} \left[ \frac{5+\sqrt{10}}{5+\sqrt{10}} \right] = \frac{5\sqrt{2} + \sqrt{20}}{25-10} = \frac{5\sqrt{2} + 2\sqrt{5}}{15} = 1$$

**Notes:** Many students try multiplying top and bottom by  $\sqrt{10}$ , figuring that it would remove the  $\sqrt{10}$ . Try it and you'll see this procedure fail.

Other students think they're more clever and multiply top and bottom by  $5 - \sqrt{10}$ . Go for it; you'll still have radicals in the bottom.

The quantity  $5 + \sqrt{10}$ , which does eradicate the radical from the bottom, is called the **conjugate** of  $5 - \sqrt{10}$ .

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## Homework

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4. Simplify so that there are no radicals in the denominator:

- |                           |                                    |                           |  |
|---------------------------|------------------------------------|---------------------------|--|
| a. $\frac{2}{\sqrt{8}}$   | b. $\frac{x^2}{\sqrt{x}}$          | c. $\frac{3}{5+\sqrt{2}}$ | d. $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$ |
| e. $\frac{1}{\sqrt{3}}$   | f. $\frac{6}{\sqrt{9}}$            | g. $\frac{8}{\sqrt{2}}$   | h. $\frac{\sqrt{x}}{\sqrt{y}}$                       |
| i. $\frac{5}{1+\sqrt{5}}$ | j. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ | k. $\frac{7}{3\sqrt{2}}$  | l. $\frac{x}{n\sqrt{x}}$                             |
| m. $\frac{7}{1+\sqrt{7}}$ | n. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$ | o. $\frac{a}{b\sqrt{c}}$  | p. $\frac{7+\sqrt{103}}{7+\sqrt{103}}$               |



5. Two of the following are valid, while two of them are bogus. Which are which?

a.  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

b.  $\sqrt{a-b} = \sqrt{a} - \sqrt{b}$

c.  $\sqrt{ab} = \sqrt{a}\sqrt{b}$

d.  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

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## Practice Problems

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6. a. Find the two fourth roots of 256.  
 b. Find all the cube roots of -1000.  
 c. Find all the square roots of -121.

7. Simplify each expression:

a.  $\sqrt{7}\sqrt{5}$       b.  $\sqrt{7} + \sqrt{5}$       c.  $\frac{\sqrt{7}}{\sqrt{5}}$       d.  $\sqrt{7} - \sqrt{5}$

e.  $\sqrt{2} + \sqrt[3]{2}$       f.  $\sqrt{28} + \sqrt{175}$       g.  $(\sqrt{h} + 7)^2$       h.  $(3 - \sqrt{t})^2$

8. Simplify so that there are no radicals in the denominator:

a.  $\frac{4}{\sqrt{35} - 5}$       b.  $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}$       c.  $\frac{12}{\sqrt{9} + 3}$

9. Simplify each expression:

a.  $\sqrt{18}\sqrt{8}$       b.  $\sqrt{7} - \sqrt{7}$       c.  $\frac{1 + \sqrt{7}}{\sqrt{7}}$       d.  $\sqrt{50} - \sqrt{8}$

e.  $\frac{3}{\sqrt{28} + 5}$       f.  $\frac{\sqrt{a} + b}{\sqrt{a} - b}$       g.  $\frac{3 - \sqrt{25}}{\sqrt{25} - 3}$

10. Let  $r_1 = \frac{3+\sqrt{11}}{4}$  and  $r_2 = \frac{3-\sqrt{11}}{4}$ .

Prove a.  $r_1 + r_2 = \frac{3}{2}$     b.  $r_1 \times r_2 = -\frac{1}{8}$

11. True/False:

a. 36 has two square roots.

b.  $\sqrt{49} = \pm 7$

c. If  $x \leq 0$ , then  $\sqrt{x}$  is not a real number.

d. 64 has exactly one cube root.

e.  $-\sqrt[4]{20}$  is a real number.

f.  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

g.  $\sqrt{ab} = \sqrt{a}\sqrt{b}$

h.  $\sqrt[3]{18} = 3\sqrt{2}$

i.  $\sqrt{20} + \sqrt{5} = 5$

j.  $\sqrt{20} + \sqrt{5} = 3\sqrt{5}$

k.  $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$

l.  $\frac{7}{\sqrt{n}} = 7\sqrt{n}$

m.  $\frac{1}{\sqrt{n}+1} = \frac{\sqrt{n}-1}{n-1}$

n.  $\sqrt[3]{x^2} + \sqrt[3]{x} = x$

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# Solutions

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1. a.  $12\sqrt{x}$     b. As is    c.  $13\sqrt[3]{n}$     d. As is    e.  $68\sqrt{2}$   
 f.  $5\sqrt[3]{3}$     g.  $-3\sqrt{5}$     h.  $32\sqrt[3]{7}$     i.  $3\sqrt[3]{z^2}$     j.  $12\sqrt{3}$   
 k.  $10\sqrt{5}$     l.  $\sqrt{11}$

2. a.  $\sqrt{84} = 2\sqrt{21}$     b.  $\sqrt[3]{27} = 3$     c.  $\sqrt{14}$   
 d. 171    e.  $\sqrt{28} = 2\sqrt{7}$     f.  $\sqrt[3]{125} = 5$   
 g.  $\sqrt[4]{16} = 2$     h.  $\sqrt{xyz}$     i.  $\sqrt[3]{8} = 2$

3. a.  $\sqrt{n} + n$     b.  $x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$   
 c.  $4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$     d.  $(\sqrt{p} - q)(\sqrt{p} - q) = p - 2q\sqrt{p} + q^2$   
 e.  $7 - 4 = 3$     f.  $x^2 - 2x\sqrt{n} + n$     g.  $a - c$   
 h.  $2(\sqrt{3} + 2) = 2\sqrt{3} + 4$     i.  $(2\sqrt{x})^2 = 4x$

4. a.  $\frac{2}{\sqrt{8}} \left[ \frac{\sqrt{8}}{\sqrt{8}} \right] = \frac{2\sqrt{8}}{8} = \frac{4\sqrt{2}}{8} = \frac{\sqrt{2}}{2}$     b.  $x\sqrt{x}$   
 c.  $\frac{3}{5 + \sqrt{2}} \left[ \frac{5 - \sqrt{2}}{5 - \sqrt{2}} \right] = \frac{15 - 3\sqrt{2}}{25 - 2} = \frac{15 - 3\sqrt{2}}{23}$     d.  $\frac{x + 2\sqrt{xy} + y}{x - y}$   
 e.  $\frac{\sqrt{3}}{3}$     f. 2    g.  $4\sqrt{2}$     h.  $\frac{\sqrt{xy}}{y}$   
 i.  $\frac{5}{1 + \sqrt{5}} \left[ \frac{1 - \sqrt{5}}{1 - \sqrt{5}} \right] = \frac{5 - 5\sqrt{5}}{1 - 5} = \frac{5 - 5\sqrt{5}}{-4} = \frac{5 - 5\sqrt{5}}{-4} \left[ \frac{-1}{-1} \right] = \frac{5\sqrt{5} - 5}{4}$

The last step is optional. It was done just to remove the negative sign from the bottom of the fraction.

j.  $-3 - 2\sqrt{2}$     k.  $\frac{7}{3\sqrt{2}} \left[ \frac{\sqrt{2}}{\sqrt{2}} \right] = \frac{7\sqrt{2}}{3 \cdot 2} = \frac{7\sqrt{2}}{6}$

l.  $\frac{\sqrt{x}}{n}$     m.  $\frac{7\sqrt{7} - 7}{6}$     n.  $\frac{a^2 + 2a\sqrt{b} + b}{a^2 - b}$     o.  $\frac{a\sqrt{c}}{bc}$     p. 1

5. c. and d. are valid — a. and b. are bogus.
6. a.  $\pm 4$       b.  $-10$       c. None
7. a.  $\sqrt{35}$       b. As is      c.  $\frac{\sqrt{35}}{5}$       d. As is  
 e. As is      f.  $7\sqrt{7}$       g.  $h+14\sqrt{h}+49$       h.  $9-6\sqrt{t}+t$
8. a.  $\frac{2\sqrt{35}+10}{5}$       b.  $\frac{a-2\sqrt{ab}+b}{a-b}$       c. 2
9. a. 12      b. 0      c.  $\frac{\sqrt{7}+7}{7}$       d.  $3\sqrt{2}$   
 e.  $2\sqrt{7}-5$       f.  $\frac{a+2b\sqrt{a}+b^2}{a-b^2}$       g. -1
10. Do the arithmetic and it should work out.
11. a. T    b. F    c. F    d. T    e. T    f. F    g. T    h. F  
 i. F    j. T    k. T    l. F    m. T    n. F

“Every job is a self-portrait of  
 the person who did it. Autograph  
 your work with excellence.”

– Unknown