CH 30 – THE QUADRATIC FORMULA

❑ *INTRODUCTION*

We recently finished solving quadratic equations by Completing the Square. This method always works, whether the solutions are integers, fractions, or radicals; it even works to tell us that there are no solutions in the real numbers. But it's quite boring and not the easiest thing in the world to do.

There must be some way to complete the square on a generic quadratic equation, (using a, b, and c — rather than specific numbers — as the coefficients) and end up with a formula that can then be used to more easily solve any quadratic equation. Moreover, this completing the square will also make it much easier to program a computer to solve quadratic equations.

❑ *DERIVING THE QUADRATIC FORMULA*

Here we go. Start with a quadratic equation in standard form:

$$ax^2 + bx + c = 0$$ [we assume $a > 0$, for if it's not, we can multiply through by -1 and make $a > 0$.]

The technique of completing the square requires that we always have a leading coefficient of 1, and so we divide each side of the equation by a (which we know is not zero — otherwise we wouldn't have a quadratic equation in the first place):

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$ [Divide each side of the equation by a]

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ [Split the fraction into three fractions]

Now bring the constant to the right side of the equation by subtracting $\frac{c}{a}$ from each side of the equation:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

It's now time to compute the *magic number*, the quantity which will complete the square. We start with the coefficient of the linear term, in this case $\frac{b}{a}$, take half of it, and then square that result:

$$\frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}, \text{ and then } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} = \boxed{\frac{b^2}{4a^2}}$$

← **The Magic Number**

We now add the magic number to each side of the equation:

$$x^2 + \frac{b}{a}x + \boxed{\frac{b^2}{4a^2}} = -\frac{c}{a} + \boxed{\frac{b^2}{4a^2}}$$

Next, we factor the trinomial on the left side of the equation:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{[Check by squaring } x + \frac{b}{2a}\text{]}$$

Notice that the variable x occurs just once, buried amidst the parentheses on the left side of the equation; now we can isolate the x.

Now we'll combine the two fractions on the right side of the equation into a single fraction. Since the LCD of the denominators is $4a^2$, we need to multiply the top and the bottom of the first fraction by $4a$:

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a}\left[\frac{\mathbf{4a}}{\mathbf{4a}}\right] + \frac{b^2}{4a^2}$$

or,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$$

Adding the fractions yields

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

The next step is to take the square root of each side of the equation, remembering that every positive number has <u>two</u> square roots, denoted

$\pm\sqrt{\;}$. We will also use the commutative property for addition to switch the two terms in the numerator of the right-hand fraction:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

To isolate the *x* (which is the whole point of this effort), we'll bring the term $\frac{b}{2a}$ to the right side of the equation by subtracting it from each side of the equation:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Now it's appropriate to split the radical into two separate radicals:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

You may recall from Chapter 24 that $\sqrt{a^2} = |a|$. But since we assumed that $a > 0$ at the beginning of this problem, the absolute value sign disappears.

Finally, we combine the two fractions into a single fraction (can you believe that the two fractions already have the same denominator?):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We summarize this procedure in the following statement:

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by **The Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In order to apply the Quadratic Formula to solve a quadratic equation, we need practice in determining the values of a, b, and c in the equation. Study the following examples carefully:

Quadratic Equation	*a*	*b*	*c*
$2x^2 + 7x + 5 = 0$	2	7	5
$5y^2 - 9y = 0$	5	–9	0
$-8z^2 + 17 = 0$	–8	0	17
$7n^2 = 0$	7	0	0
$(2x + 7)(x - 4) = 3x - 7$	2	–4	–21

To justify this last result, let's convert the equation to standard quadratic form:

$$(2x + 7)(x - 4) = 3x - 7$$
$$\Rightarrow \quad 2x^2 - 8x + 7x - 28 = 3x - 7$$
$$\Rightarrow \quad 2x^2 - x - 28 = 3x - 7$$
$$\Rightarrow \quad 2x^2 - 4x - 21 = 0$$

Homework

1. Which symbol in the Quadratic Formula is responsible for possibly giving us two solutions to a quadratic equation?

2. How many real solutions of the quadratic equation will there be if the quantity inside the square root sign (called the radicand) is 100?

3. How many real solutions of the quadratic equation will there be if the radicand is 0?

4. How many real solutions of the quadratic equation will there be if the radicand is –9?

5. Consider the possibility that the value of a is 0 in the quadratic equation $ax^2 + bx + c = 0$. Does the Quadratic Formula apply? First, consider what happens to the quadratic equation if $a = 0$. Second, analyze what happens if we actually let $a = 0$ in the Quadratic Formula.

❑ *NOTES ON THE QUADRATIC FORMULA*

- The Quadratic Formula can be used to solve <u>any</u> quadratic equation, whether or not it's factorable, and whether or not its solutions are integers (whole numbers and their opposites), fractions, or even radicals (like $\sqrt{2}$). And if the quadratic equation has no solution in the real numbers, the Quadratic Formula will tell us that, too.

- Because of the $\pm$ sign, there are potentially two solutions. In fact, the two solutions of the quadratic equation $ax^2 + bx + c = 0$ can be written separately:

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

- The Quadratic Formula is a <u>single fraction</u>:

  It is <u>**NOT**</u> $x = -b \pm \dfrac{\sqrt{b^2-4ac}}{2a}$

  If you don't write it as a single fraction at every stage of your solution, either you or your instructor will misread it and you'll be the one to lose points on the problem.

❑ QUADRATIC EQUATIONS WITH RATIONAL SOLUTIONS

EXAMPLE 1: **Solve for x: $x^2 - 9x - 10 = 0$**

Solution: We see that this is a quadratic equation in standard form, and in fact $a = 1$, $b = -9$, and $c = -10$. By the Quadratic Formula, x has two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-10)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{81 + 40}}{2} = \frac{9 \pm \sqrt{121}}{2} = \frac{9 \pm 11}{2}$$

Using the plus sign: $\frac{9+11}{2} = \frac{20}{2} = 10$

Using the minus sign: $\frac{9-11}{2} = \frac{-2}{2} = -1$

Final answer: $\boxed{10, -1}$

Let's check these solutions:

Letting $x = 10$ in the original equation,

$$x^2 - 9x - 10 = \mathbf{10}^2 - 9(\mathbf{10}) - 10 = 100 - 90 - 10 = 0 \ ✓$$

Letting $x = -1$,

$$x^2 - 9x - 10 = (\mathbf{-1})^2 - 9(\mathbf{-1}) - 10 = 1 + 9 - 10 = 0 \ ✓$$

<u>EXAMPLE 2:</u> **Solve for n: $6n^2 + 40 = -31n$**

<u>Solution:</u> The first step is to transform this quadratic equation into standard form. Adding $31n$ to each side (and putting the $31n$ <u>between</u> the $6n^2$ and the 40) produces

$$6n^2 + 31n + 40 = 0.$$

Noting that $a = 6$, $b = 31$, and $c = 40$, we're ready to apply the Quadratic Formula:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{-31 \pm \sqrt{(31)^2 - 4(6)(40)}}{2(6)}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{961 - 960}}{12} = \frac{-31 \pm \sqrt{1}}{12} = \frac{-31 \pm 1}{12}$$

The two solutions are given by

$$\frac{-31+1}{12} = \frac{-30}{12} = -\frac{5}{2}$$

$$\text{and} \quad \frac{-31-1}{12} = \frac{-32}{12} = -\frac{8}{3}$$

$$\boxed{-\frac{5}{2},\ -\frac{8}{3}}$$

It's a lot of work, but you should check these solutions.

<u>EXAMPLE 3:</u> **Solve for y: $7y^2 = 5y$**

<u>Solution:</u> To transform the given equation into the standard form $ax^2 + bx + c = 0$, we need to subtract $5y$ from each side of the equation, resulting in the equation

$$7y^2 - 5y = 0$$

which we can think of as

$$7y^2 - 5y + 0 = 0$$

from which we conclude that $a = 7$, $b = -5$, and $c = 0$. Applying the Quadratic Formula gives

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(7)(0)}}{2(7)} = \frac{5 \pm \sqrt{25 - 0}}{14} = \frac{5 \pm 5}{14}$$

Splitting it into two pieces gives us

$$\frac{5+5}{14} = \frac{10}{14} = \frac{5}{7} \qquad \text{and} \qquad \frac{5-5}{14} = \frac{0}{14} = 0$$

The two solutions are therefore

$$\boxed{\frac{5}{7}, 0}$$

EXAMPLE 4: **Solve for p: $2p^2 = 200$**

Solution: Bringing the 200 over to the left side of the equation gives us our quadratic equation in standard form:

$$2p^2 - 200 = 0$$

which we can view as

$$2p^2 + 0p - 200 = 0$$

Notice that $a = 2$, $b = 0$, and $c = -200$. Substituting these values into the Quadratic Formula yields our two solutions for p:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-0 \pm \sqrt{(0)^2 - 4(2)(-200)}}{2(2)}$$

$$\Rightarrow \quad p = \frac{0 \pm \sqrt{0 - 4(2)(-200)}}{4} = \frac{\pm\sqrt{1600}}{4} = \frac{\pm 40}{4} = \pm 10$$

Thus, the solutions are $\boxed{\pm 10}$

Note: At or near the beginning of this problem we could have divided each side of the equation by 2. The Quadratic Formula, although containing different values for a and c, would have yielded the same answers, ±10. Try it.

EXAMPLE 5: **Solve for u: $28u - 4u^2 = 49$**

Solution: The standard quadratic form requires that the quadratic term (the squared variable) be in front, followed by the term containing the variable, followed by the constant, all set equal to zero; that is, $ax^2 + bx + c = 0$. We achieve this goal with the following steps:

$28u - 4u^2 = 49$ (the given equation)

$\Rightarrow \quad -4u^2 + 28u = 49$ (interchange the terms)

$\Rightarrow \quad -4u^2 + 28u - 49 = 0$ (subtract 49 from each side)

At this point, it's in a good form for the Quadratic Formula, but it's traditional to disallow the leading coefficient (the –4) to be negative. One way to change –4 into 4 is to multiply it by –1; but, of course, we will have to do this to both sides of the equation.

$\Rightarrow \quad -1[-4u^2 + 28u - 49] = -1[0]$ (multiply each side by –1)

$\Rightarrow \quad 4u^2 - 28u + 49 = 0$ (distribute)

Now it's clear that $a = 4$, $b = -28$, and $c = 49$. On to the Quadratic Formula:

$$u = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(4)(49)}}{2(4)} = \frac{28 \pm \sqrt{784 - 784}}{8}$$

$$= \frac{28 \pm \sqrt{0}}{8} = \frac{28 \pm 0}{8} = \frac{28}{8} = \boxed{\frac{7}{2}}$$

Notice that, although all the previous quadratic equations had two solutions, this equation has just one solution. (Or does it have two solutions that are the same?)

EXAMPLE 6: **Solve for z: $z^2 + 3z + 3 = 0$**

Solution: This quadratic equation seems innocent enough, so let's take the values $a = 1$, $b = 3$, and $c = 3$ and place them in their proper places in the Quadratic Formula:

$$z = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 12}}{2} = \frac{-3 \pm \sqrt{-3}}{2}$$

Houston. . .we have a problem. If you try to take the square root of –3, your calculator will indicate an error, showing that it's not a real number. We can't finish this calculation (at least not in this Algebra course), so we must agree that there is simply **NO REAL NUMBER SOLUTION** to the equation $z^2 + 3z + 3 = 0$.

Homework

6. Solve each quadratic equation using the Quadratic Formula:

a. $w^2 - 12w + 35 = 0$

b. $2x^2 - 13x + 15 = 0$

c. $n^2 - 25n + 150 = 0$

d. $6a^2 - 31a + 40 = 0$

e. $x^2 + 2x - 15 = 0$

f. $y^2 - 14y + 49 = 0$

g. $z^2 - 9 = 0$

h. $3h^2 - 17h + 10 = 0$

i. $16u^2 - 8u + 1 = 0$

j. $25w^2 - 4 = 0$

k. $8t^2 - 2t - 3 = 0$

l. $15a^2 + 22a + 8 = 0$

7. Solve each quadratic equation using the Quadratic Formula:

a. $2x^2 + 10x - 28 = 0$	b. $25y^2 = 4$
c. $2z^2 + 2 = -4z$	d. $4a^2 = 3 - 4a$
e. $6u^2 = 47u + 8$	f. $0 = 4t^2 + 8t + 3$
g. $x^2 + x - 42 = 0$	h. $2n^2 + n - 3 = 0$
i. $4x^2 = 1$	j. $a^2 + 12a + 36 = 0$
k. $9k^2 = 9k - 2$	l. $9n^2 - 30n + 25 = 0$
m. $6z^2 = 10z$	n. $49t^2 = 4$
o. $25h^2 = 30h - 9$	p. $6x^2 - 7x - 10 = 0$
q. $0 = 6y^2 + y - 2$	r. $4n^2 = 25n + 21$
s. $16q^2 - 24q + 9 = 0$	t. $64t^2 - 9 = 0$
u. $-n^2 + n + 56 = 0$ [Hint: multiply each side by −1]	
v. $-2x^2 + x + 3 = 0$	w. $-14a^2 = 3 - 13a$

❑ *QUADRATIC EQUATIONS WITH IRRATIONAL (RADICAL) SOLUTIONS*

<u>EXAMPLE 7:</u> **Solve for x: $2x^2 - 3x - 1 = 0$**

<u>Solution:</u> For this quadratic equation $a = 2$, $b = -3$, and $c = -1$. Inserting these values into the Quadratic Formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+8}}{4} = \boxed{\frac{3 \pm \sqrt{17}}{4}}$$

I recommend you check these solutions.

Notice that the radical cannot be simplified, nor can the fraction be reduced. Also, we are not using a calculator to turn the answers into approximate decimals because we want to maintain the exact answer. That's why we left the answers the way we did.

EXAMPLE 8: **Solve for x: $2x^2 - 8x + 3 = 0$**

Solution: The values of a, b, and c for use in the Quadratic Formula are

$a = 2 \qquad b = -8 \qquad c = 3$

So now we state the Quadratic Formula, insert the three values, and then do a lot of arithmetic and simplifying until the final answer is as clean and exact as possible.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(3)}}{2(2)} = \frac{8 \pm \sqrt{64 - 24}}{4}$$

$$= \frac{8 \pm \sqrt{40}}{4} = \frac{8 \pm \sqrt{4 \cdot 10}}{4} = \frac{8 \pm 2\sqrt{10}}{4} = \frac{\cancel{2}(4 \pm \sqrt{10})}{\cancel{2} \cdot 2}$$

We write our final answer as

$$\boxed{\frac{4 \pm \sqrt{10}}{2}}$$

EXAMPLE 9: **Solve for y: $3y^2 = 13$**

Solution: We begin by subtracting 13 from each side of the equation to convert the equation into standard quadratic form:

$3y^2 - 13 = 0$

At this point we see that $a = 3$, $b = 0$, and $c = -13$. Applying the Quadratic Formula:

$$y = \frac{-0 \pm \sqrt{0^2 - 4(3)(-13)}}{2(3)} = \frac{\pm\sqrt{0+156}}{6} = \frac{\pm\sqrt{156}}{6}$$

$$= \frac{\pm\sqrt{4 \cdot 39}}{6} = \frac{\pm\sqrt{4}\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{2 \cdot 3} = \frac{\pm \cancel{2}\sqrt{39}}{\cancel{2} \cdot 3}$$

And we have our final answer (two of them!) in simplest form:

$$\boxed{\frac{\pm\sqrt{39}}{3}} \qquad \text{or,} \ \pm\frac{\sqrt{39}}{3}$$

The next example shows you the power of substitution and the flexibility of the Quadratic Formula for an equation that is not technically a quadratic equation. Don't panic — no homework will be assigned.

<u>EXAMPLE 10:</u> **Solve for x: $x^4 + 5x^2 - 1 = 0$**

<u>Solution:</u> This quartic equation won't factor. But notice that although it's not technically quadratic, we can make it look that way — we'll use a substitution trick. Let $n = x^2$, which upon squaring each side, implies that $n^2 = x^4$. We get

$$n^2 + 5n - 1 = 0 \qquad \text{(after the substitutions)}$$

$$\Rightarrow \quad n = \frac{-5 \pm \sqrt{5^2 - 4(1)(-1)}}{2(1)} \qquad \text{(the Quadratic Formula)}$$

$$\Rightarrow \quad n = \frac{-5 \pm \sqrt{29}}{2} \qquad \text{(a little arithmetic)}$$

$$\Rightarrow \quad x^2 = \frac{-5 \pm \sqrt{29}}{2} \qquad \text{(reverse the substitution)}$$

$$\Rightarrow \quad \boxed{x = \pm\sqrt{\frac{-5 \pm \sqrt{29}}{2}}} \qquad \text{(solve by taking square roots)}$$

Note that there are <u>four</u> possible values for x, one for each combination of the plus and minus signs. It's possible that one or more of the values of x in the answer box might not be real numbers. What you should do now is convert these nested radicals into decimals to see just how many of these four numbers are actually real numbers. When you do, you'll see that the two real solutions are $\pm\sqrt{\dfrac{-5+\sqrt{29}}{2}}$, whose decimal approximations are **± 0.439**.

Homework

8. Solve each quadratic equation using the Quadratic Formula:

a. $x^2 - 5x + 1 = 0$
b. $2n^2 + n - 7 = 0$
c. $3a^2 - 3a - 1 = 0$
d. $5t^2 + 7t + 1 = 0$
e. $5x^2 + 15x - 2 = 0$
f. $t^2 + t - 8 = 0$
g. $5h^2 - 13h = -5$
h. $7u^2 + u = 3$
i. $2k^2 + 7k = 8$
j. $2m^2 - 3 = -9m$

9. Solve each quadratic equation using the Quadratic Formula:

a. $3x^2 - 6x - 1 = 0$
b. $2k^2 = 8k - 6$
c. $2n^2 - 10 = 0$
d. $y^2 = 49$
e. $14d^2 - 4d = 4$
f. $x^2 + 3x + 10 = 0$
g. $5a^2 + 7a - 2 = 0$
h. $u^2 = 40$
i. $3z^2 = 3z$
j. $5x^2 - x + 1 = 0$
k. $x(x - 3) = 7$
l. $y(y + 1) = y(y - 7) + 13$

10. Solve each quadratic equation using the Quadratic Formula:

a. $x^2 + 6x + 1 = 0$
b. $2y^2 + 7y = 2$
c. $-n^2 + 28 = 0$
d. $5d^2 = 7d + 1$
e. $x^2 = -4 - 5x$
f. $x^2 + 2x + 1 = 0$
g. $-3a^2 = 5 - 12a$
h. $2x^2 = 7$
i. $3z^2 = 12z$
j. $n^2 + 9 = 0$
k. $-4n^2 + 12n = 9$
l. $t^2 - 9 = 0$

Practice Problems

11. Derive the Quadratic Formula. That is, start with the generic quadratic equation

$$ax^2 + bx + c = 0,$$

complete the square, and end up with

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

12. Solve each quadratic equation using the Quadratic Formula:

a. $x^2 - x - 56 = 0$
b. $u^2 - 8u = 0$
c. $r^2 = 9$
d. $9k^2 + 49 = -42k$
e. $n^2 + 2n - 80 = 0$
f. $-3t^2 = 11t + 6$
g. $2m^2 - 7m = -6$
h. $8d^2 = 2$
i. $-3a^2 = 7a - 6$
j. $-8h^2 - 8h = 0$
k. $n^2 - 3n + 4 = 0$
l. $2p^2 - 3p - 5 = 0$

m. $-3m^2 - 8m + 3 = 0$

n. $-8z^2 + 2z = 0$

o. $-3w^2 + 3 = 0$

p. $5g^2 = 5g + 2$

q. $-3y^2 + 6y = -3$

r. $12t^2 - 8t - 16 = 0$

s. $3x^2 = -15x + 3$

t. $0 = 21g^2 + 9g - 15$

u. $-5z^2 + 4z = -7$

v. $8q^2 + 4q = 3$

w. $y^2 - 4y = -2$

x. $2x^2 - 2x - 3 = 0$

y. $0 = 14d^2 - 4d - 4$

z. $27x^2 - 6x - 15 = 0$

13. Consider the quantity $b^2 - 4ac$, the radicand inside the square root sign of the Quadratic Formula.

a. If $b^2 - 4ac > 0$, then the equation has ____ real solution(s).
b. If $b^2 - 4ac = 0$, then the equation has ____ real solution(s).
c. If $b^2 - 4ac < 0$, then the equation has ____ real solution(s).

The quantity $b^2 - 4ac$ is called the ***discriminant*** of the quadratic equation $ax^2 + bx + c = 0$.

Solutions

1. The plus/minus sign: ±

2. Since $\sqrt{100} = 10$, and since there's a plus/minus sign in front of this, there will be two solutions.

3. Since $\sqrt{0} = 0$, and since ±0 is just the single number 0, the plus/minus sign basically disappears, leaving a single solution.

4. Since $\sqrt{-9}$ does not exist in this class, the Quadratic Formula has no meaning, and so there are no solutions.

5. First, if $a = 0$, the quadratic equation $ax^2 + bx + c = 0$ becomes $bx + c = 0$, which is NOT quadratic anymore. Indeed, we learned weeks ago how to solve this equation: $bx + c = 0 \Rightarrow bx = -c \Rightarrow x = -\frac{c}{b}$.

Second, if we ignore the above fact and use $a = 0$ in the Quadratic Formula, the denominator becomes $2a = 2(0) = 0$. That is, we're dividing by zero, which is undefined.

However you look at it, the Quadratic Formula with $a = 0$ simply makes no sense.

6.

a. $5, 7$ b. $5, \frac{3}{2}$ c. $10, 15$ d. $\frac{8}{3}, \frac{5}{2}$

e. $3, -5$ f. 7 g. ± 3 h. $5, \frac{2}{3}$

i. $\frac{1}{4}$ j. $\pm\frac{2}{5}$ k. $-\frac{1}{2}, \frac{3}{4}$ l. $-\frac{2}{3}, -\frac{4}{5}$

7.

a. $2, -7$ b. $\pm\frac{2}{5}$ c. -1 d. $\frac{1}{2}, -\frac{3}{2}$

e. $8, -\frac{1}{6}$ f. $-\frac{1}{2}, -\frac{3}{2}$ g. $6, -7$ h. $1, -\frac{3}{2}$

i. $\pm\frac{1}{2}$ j. -6 k. $\frac{1}{3}, \frac{2}{3}$ l. $\frac{5}{3}$

m. $0, \frac{5}{3}$ n. $\pm\frac{2}{7}$ o. $\frac{3}{5}$ p. $2, -\frac{5}{6}$

q. $\frac{1}{2}, -\frac{2}{3}$ r. $7, -\frac{3}{4}$ s. $\frac{3}{4}$ t. $\pm\frac{3}{8}$

u. $8, -7$ v. $-1, \frac{3}{2}$ w. $\frac{3}{7}, \frac{1}{2}$

8.

a. $\frac{5 \pm \sqrt{21}}{2}$ b. $\frac{-1 \pm \sqrt{57}}{4}$ c. $\frac{3 \pm \sqrt{21}}{6}$ d. $\frac{-7 \pm \sqrt{29}}{10}$

e. $\frac{-15 \pm \sqrt{265}}{10}$ f. $\frac{-1 \pm \sqrt{33}}{2}$ g. $\frac{13 \pm \sqrt{69}}{10}$ h. $\frac{-1 \pm \sqrt{85}}{14}$

i. $\frac{-7 \pm \sqrt{113}}{4}$ j. $\frac{-9 \pm \sqrt{105}}{4}$

9. a. $\frac{3 \pm 2\sqrt{3}}{3}$ b. 1, 3 c. $\pm\sqrt{5}$ d. ±7

e. $\frac{1 \pm \sqrt{15}}{7}$ f. No solution g. $\frac{-7 \pm \sqrt{89}}{10}$ h. $\pm 2\sqrt{10}$

i. 0, 1 j. No solution k. $\frac{3 \pm \sqrt{37}}{2}$ l. $\frac{13}{8}$

10. a. $-3 \pm 2\sqrt{2}$ b. $\frac{-7 \pm \sqrt{65}}{4}$ c. $\pm 2\sqrt{7}$ d. $\frac{7 \pm \sqrt{69}}{10}$

e. −1, −4 f. −1 g. $\frac{6 \pm \sqrt{21}}{3}$ h. $\frac{\pm\sqrt{14}}{2}$

i. 0, 4 j. No solution k. $\frac{3}{2}$ l. ±3

11. See the first page of this chapter.

12. a. −7, 8 b. 8, 0 c. ± 3 d. $-\frac{7}{3}$

e. −10, 8 f. $-3, -\frac{2}{3}$ g. $2, \frac{3}{2}$ h. $\frac{1}{2}, -\frac{1}{2}$

i. $-3, \frac{2}{3}$ j. −1, 0 k. No solution l. $\frac{5}{2}, -1$

m. $-3, \frac{1}{3}$ n. $0, \frac{1}{4}$ o. ± 1 p. $\frac{5 \pm \sqrt{65}}{10}$

q. $1 \pm \sqrt{2}$ r. $\frac{1 \pm \sqrt{13}}{3}$ s. $\frac{-5 \pm \sqrt{29}}{2}$ t. $\frac{-3 \pm \sqrt{149}}{14}$

u. $\frac{2 \pm \sqrt{39}}{5}$ v. $\frac{-1 \pm \sqrt{7}}{4}$ w. $2 \pm \sqrt{2}$ x. $\frac{1 \pm \sqrt{7}}{2}$

y. $\frac{1 \pm \sqrt{15}}{7}$ z. $\frac{1 \pm \sqrt{46}}{9}$

13. a. 2 b. 1 c. 0

“The mutual desire of good men is knowledge.”

– icniV aD odranoeL