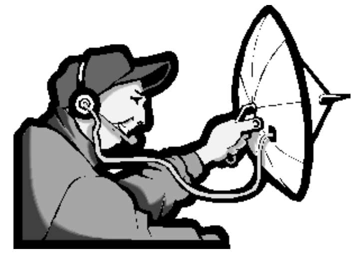
CH 31 – THE PARABOLA

❑ INTRODUCTION



A ***parabola*** (accent on the RAB) is a very special shape used in searchlights and satellite dishes. Even football sports reporters use parabolic reflectors to listen in on comments made by coaches on the sidelines and players in the huddle. In fact, when a football is thrown or punted, its path (neglecting air resistance) is that of a parabola.

❑ GRAPHING A PARABOLA

<u>EXAMPLE 1:</u> **Graph:** $y = x^2 - 4x + 3$

<u>Solution:</u> Let's find some points on our graph by choosing some values of x, and then calculating the corresponding y-values.

If $x = -1$, then $y = (\mathbf{-1})^2 - 4(\mathbf{-1}) + 3 = 1 + 4 + 3 = 8 \Rightarrow$ **(–1, 8)**

If $x = 0$, then $y = (\mathbf{0})^2 - 4(\mathbf{0}) + 3 = 0 - 0 + 3 = 3 \Rightarrow$ **(0, 3)**

If $x = 1$, then $y = (\mathbf{1})^2 - 4(\mathbf{1}) + 3 = 1 - 4 + 3 = 0 \Rightarrow$ **(1, 0)**

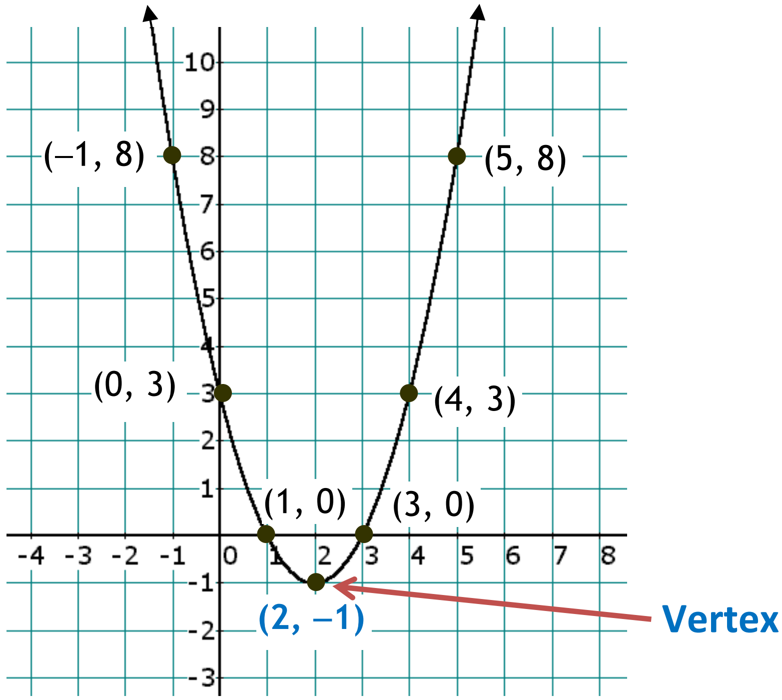
If $x = 2$, then $y = (\mathbf{2})^2 - 4(\mathbf{2}) + 3 = 4 - 8 + 3 = -1 \Rightarrow$ **(2, –1)**

If $x = 3$, then $y = (\mathbf{3})^2 - 4(\mathbf{3}) + 3 = 9 - 12 + 3 = 0 \Rightarrow$ **(3, 0)**

If $x = 4$, then $y = (\mathbf{4})^2 - 4(\mathbf{4}) + 3 = 16 - 16 + 3 = 3 \Rightarrow$ **(4, 3)**

If $x = 5$, then $y = (\mathbf{5})^2 - 4(\mathbf{5}) + 3 = 25 - 20 + 3 = 8 \Rightarrow$ **(5, 8)**

Plotting these seven points leads us to the following graph:



What do we notice about the graph? It's a curve, not a straight line. We notice that x can be any real number (but notice that y never goes below −1). Also note that the graph has one y-intercept and two x-intercepts. In addition, there is no highest point on the parabola, but the lowest point on the parabola is (2, −1), and we call this point the ***vertex*** of the parabola.

We say that the above parabola "opens up." The equation of a parabola is characterized by the fact that one variable is squared while the other variable is raised to the first power.

EXAMPLE 2: **Graph:** $y = -x^2 - 2x - 1$

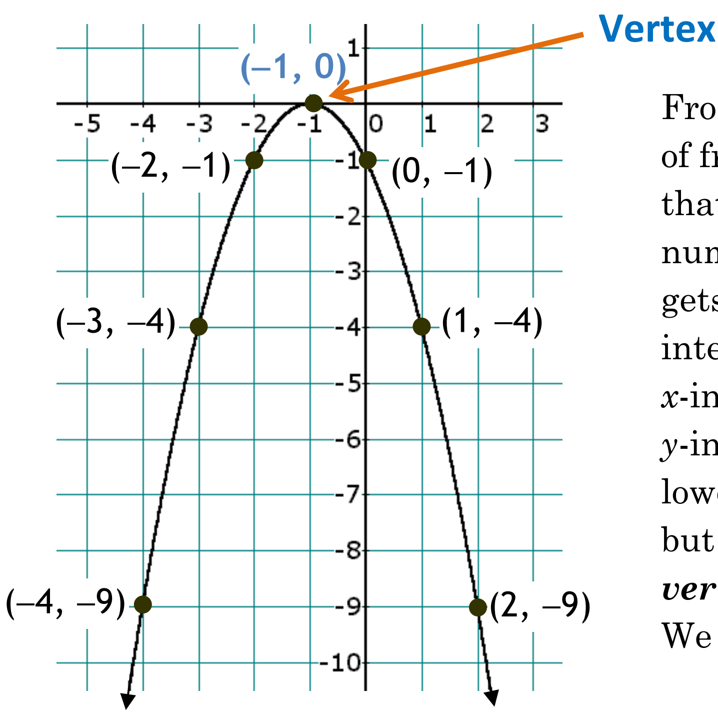
Solution: First, we notice that the quadratic term, the $-x^2$, has a leading negative sign. And we remember (due to the order of operations) that exponents have a higher priority than minus signs, so we know that to evaluate $-x^2$, we square the x first, and apply the minus sign <u>second</u>. We'll calculate two points on the parabola together and leave the rest of the points for you to do.

If $x = 2$, then $y = -(\mathbf{2})^2 - 2(\mathbf{2}) - 1 = -4 - 4 - 1 = -9$. Thus, the point **(2, –9)** is on the graph of our parabola.

If $x = -3$, then $y = -(\mathbf{-3})^2 - 2(\mathbf{-3}) - 1 = -9 + 6 - 1 = -4$. Therefore, the point **(–3, –4)** is on the graph.

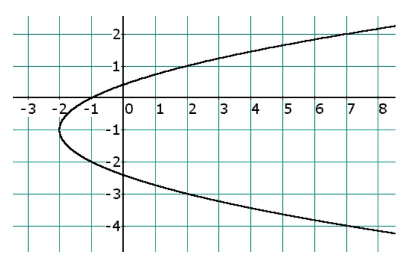
You can show that each of the following points is also on the graph:

(–4, –9)   (–2, –1)   (–1, 0)   (0, –1)   (1, –4)



From the formula (and kind of from the graph) we note that x can be <u>any</u> real number, but that y never gets above 0. As for intercepts, there's one x-intercept and one y-intercept. There is no lowest point on the graph, but the highest point, the ***vertex***, is the point (–1, 0). We say that this parabola

[There are also "sideways" parabolas, but in this chapter only parabolas that open up or down will be discussed. In case you're interested, the up/down parabolas are *functions* while the sideways ones are not.]



Homework

1. In Example 1 we saw that the graph of $y = x^2 - 4x + 3$ was a parabola opening up. Example 2 showed us that the graph of $y = -x^2 - 2x - 1$ was a parabola opening down. Take a guess what property of these equations determines whether the parabola opens up or down.

2. True/False: (Recall that all parabolas in this chapter open up or down.)

   a. Every parabola has a y-intercept.

   b. Every parabola has an x-intercept.

   c. Every parabola has a vertex.

   d. The vertex of a parabola is always the highest point of the parabola.

   e. The vertex of a parabola is always the lowest point of the parabola.

3. Graph each parabola by plotting points. Then use your graph to determine the intercepts and the vertex of your parabola:

   a. $y = 9 - x^2$ b. $y = x^2 + 6x + 5$

   c. $y = x^2 + 2x - 8$ d. $y = -x^2 + 4x - 4$

   e. $y = 0.5x^2$ f. $y = -0.2x^2 + 5$

❑ *Finding Intercepts*

Remember the method for finding the intercepts of a line? Well, an intercept is an intercept, so the rules for finding the intercepts of a parabola (or any graph at all) are identical to the rules we learned before:

> x-intercepts are found by setting y to 0.
>
> y-intercepts are found by setting x to 0.

Also recall that an intercept is a point in the 2-dimensional plane, and should be written as an ordered pair, like (2, 0) or (0, –3).

EXAMPLE 3: **Find the intercepts of $y = x^2 - x - 6$.**

Solution:

x-intercepts: Setting $y = 0$

$$\Rightarrow \mathbf{0} = x^2 - x - 6$$
$$\Rightarrow 0 = (x + 2)(x - 3)$$
$$\Rightarrow x + 2 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow x = -2 \text{ or } x = 3$$

We conclude that

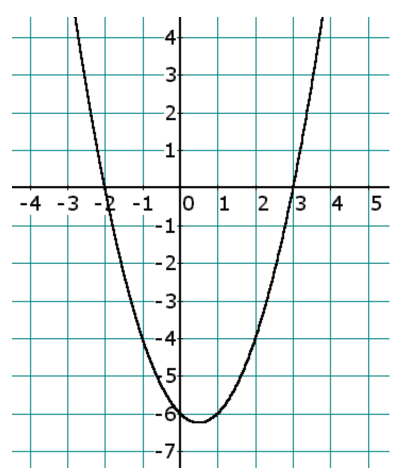
> the x-intercepts are (–2, 0) and (3, 0)

y-intercepts: Setting $x = 0$

$$\Rightarrow y = \mathbf{0}^2 - 6(\mathbf{0}) - 6 = -6$$

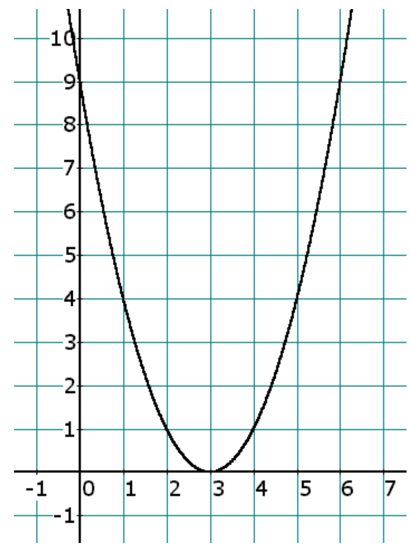
Therefore,

> the y-intercept is (0, –6)



EXAMPLE 4: **Find the intercepts of $y = x^2 - 6x + 9$.**

Solution:



x-intercepts: Setting $y = 0$

$$\Rightarrow \mathbf{0} = x^2 - 6x + 9$$
$$\Rightarrow 0 = (x - 3)(x - 3)$$
$$\Rightarrow x - 3 = 0$$
$$\Rightarrow x = 3$$

only one solution for x.

We conclude that

the x-intercept is (3, 0)

y-intercepts: Setting $x = 0$

$$\Rightarrow y = (\mathbf{0})^2 - 6(\mathbf{0}) + 9 = 9$$

Therefore,

the y-intercept is (0, 9)

EXAMPLE 5: **Find the intercepts of $y = -3x^2 + 5x - 1$.**

Solution:

x-intercepts: Setting $y = 0$

$$\Rightarrow \mathbf{0} = -3x^2 + 5x - 1$$

Bringing all the terms to the left side gives us the following quadratic equation in standard form:

$$3x^2 - 5x + 1 = 0$$

The left side of the equation is not factorable (give it a try!), so we must utilize either the method of completing the square or the

Quadratic Formula. Let's use the Quadratic Formula, where in this case $a = 3$, $b = -5$, and $c = 1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

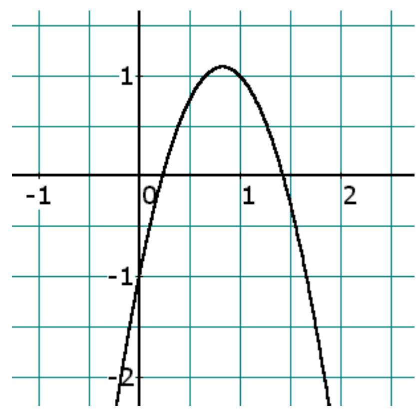
$$= \frac{5 \pm \sqrt{25 - 12}}{6} = \frac{5 \pm \sqrt{13}}{6}$$

These two solutions are certainly correct, and we could even write our two x-intercepts like this:

$$\left(\frac{5+\sqrt{13}}{6}, 0\right), \quad \left(\frac{5-\sqrt{13}}{6}, 0\right)$$

However, this form of the x-intercepts is not very useful for plotting them on a grid or for business and science problems; it's better to convert the two exact *irrational* answers into approximate *rational* answers, and our x-intercepts are roughly

(1.434, 0) (0.232, 0)



y-intercepts: Setting $x = 0 \Rightarrow y = -3(\mathbf{0})^2 + 5(\mathbf{0}) - 1 = -1$. The y-intercept is therefore

(0, –1)

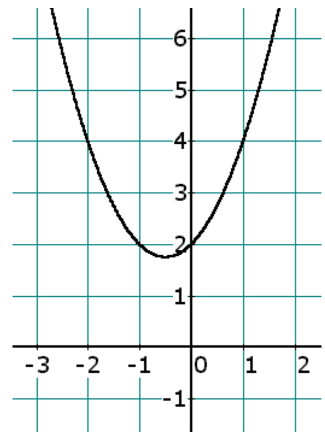
EXAMPLE 6: **Find the intercepts of $y = x^2 + x + 2$.**

Solution: Seems easy enough, but this is a strange one.

x-intercepts: Setting $y = 0$ yields the quadratic equation $x^2 + x + 2 = 0$. First, this quadratic won't factor, but that's okay; we have the Quadratic Formula from the previous chapter to rescue us:

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2}$$

We're in trouble; the –7 in the radical sign indicates that these two solutions for x are not real numbers. This means that wherever these non-real (imaginary) numbers are, they are not on the x-axis (which is just the set of real numbers).



The conclusion of all this? This parabola has

no x-intercept

y-intercepts: Setting $x = 0$ gives $y = 2$, and so the y-intercept is

(0, 2)

Homework

4. Find all the **intercepts** of each parabola — no approximations:

   a. $y = x^2 + 2x - 48$ b. $y = x^2 + 10x + 25$

   c. $y = 2x^2 + 8x + 5$ d. $y = 3x^2 - 6x + 4$

5. Find all the **intercepts** of each parabola, rounding to 3 digits:

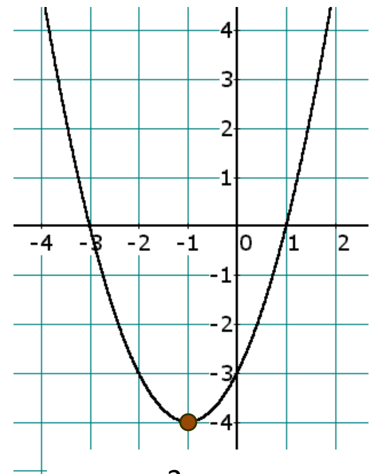
   a. $y = x^2 + 7x + 1$ b. $y = -2x^2 + 5x + 4$

   c. $y = 3x^2 - 6x - 2$ d. $y = 5x^2 + 3x + 1$

❑ *FINDING THE VERTEX*

If a parabola opens up, then the ***vertex*** of the parabola is the point at the bottom, the minimum point of the parabola. If the parabola opens down, then the vertex is the top, the maximum point of the parabola.

The goal is to figure a way to determine the x-coordinate of the vertex, the point on the graph at the bottom of the parabola $y = x^2 + 2x - 3$ shown on the right. Once we find the x-coordinate, we can find the y-coordinate by plugging the x-coordinate into the parabola equation.



$y = x^2 + 2x - 3$

Here's the secret to finding the x-coordinate of the vertex: Go straight up from the vertex to the x-axis. Due to the symmetry of a parabola, that x-value is midway between the x-values of the two x-intercepts; in this case it's midway between $x = -3$ and $x = 1$. Recall from Chapter 5 that the midpoint of a line segment on the x-axis is calculated simply by <u>averaging</u> the two x-values of the endpoints (in this case the x-intercepts). Thus, the x-coordinate of the vertex is

$$x = \frac{-3+1}{2} = \frac{-2}{2} = -1$$

a fact which should be clear by looking at the graph. The parabola's equation is $y = x^2 + 2x - 3$, and so the y-coordinate of the vertex is found by simply placing $x = -1$ into the equation and finding y:

$$y = x^2 + 2x - 3 = (\mathbf{-1})^2 + 2(\mathbf{-1}) - 3 = 1 - 2 - 3 = -4,$$

which is consistent with the picture. We conclude that the vertex of the parabola is the point **(–1, –4)**.

> To find the x-coordinate of the vertex of a parabola, find the <u>average</u> of the x-coordinates of the parabola's x-intercepts.

❑ EXAMPLES AND FURTHER ANALYSIS

EXAMPLE 7: **Find the vertex of $y = -2x^2 - 4x + 30$.**

Solution: According to the rule in the box above, we first calculate the x-coordinates of the x-intercepts of the parabola. This, of course, is accomplished by setting y to 0:

$$\mathbf{0} = -2x^2 - 4x + 30$$

$\Rightarrow$ $2x^2 + 4x - 30 = 0$

$\Rightarrow$ $2(x^2 + 2x - 15) = 0$ (factor out the GCF)

$\Rightarrow$ $2(x + 5)(x - 3) = 0$ (factor the trinomial)

$\Rightarrow$ $(x + 5)(x - 3) = 0$ (divide both sides by 2)

$\Rightarrow$ $x + 5 = 0$ or $x - 3 = 0$ (set each factor to 0)

$\Rightarrow$ $x = -5$ or $x = 3$ (solve each linear equation)

The average of these x-values is $\frac{-5+3}{2} = \frac{-2}{2} = -1$, and so we have $x = -1$ as the x-coordinate of the vertex. The y-value of the vertex is

$$y = -2x^2 - 4x + 30 = -2(-\mathbf{1})^2 - 4(-\mathbf{1}) + 30 = -2 + 4 + 30 = 32$$

and we conclude that the vertex of the parabola is the point

$$\boxed{(-1, 32)}$$

EXAMPLE 8: **Find the vertex of the parabola $y = x^2 - 6x + 9$.**

Solution: Find the x-intercepts by setting y to 0:

$$\mathbf{0} = x^2 - 6x + 9$$

$\Rightarrow$ $0 = (x - 3)(x - 3)$

$\Rightarrow$ $x - 3 = 0$ or $x - 3 = 0$

$\Rightarrow \quad x = 3 \text{ or } x = 3.$

There's only one x-intercept, (3, 0), but it did occur twice! So we'll calculate the average of 3 and 3, which is 3, the x-coordinate of the vertex of the parabola. And the y-coordinate of the vertex is

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 9 = 9 - 18 + 9 = 0$$

We conclude that the vertex of the parabola is

$$\boxed{(3, 0)}$$

<u>EXAMPLE 9:</u> **Find the vertex of $y = 3x^2 - 7x + 3$.**

<u>Solution:</u> Set y to 0 to calculate the x-intercepts. This will be followed by averaging those x-intercepts to find the x-coordinate of the vertex of the parabola.

$$\mathbf{0} = 3x^2 - 7x + 3$$

Factoring will prove fruitless, so it's time for the Quadratic Formula (we could also Complete the Square). The values of a, b, and c are 3, −7, and 3.

$$x = \frac{-(\mathbf{-7}) \pm \sqrt{(\mathbf{-7})^2 - 4(\mathbf{3})(\mathbf{3})}}{2(\mathbf{3})} = \frac{7 \pm \sqrt{49 - 36}}{6} = \frac{7 \pm \sqrt{13}}{6}$$

Thus, the x-coordinates of the x-intercepts are

$$\frac{7 + \sqrt{13}}{6} \text{ and } \frac{7 - \sqrt{13}}{6}$$

It is these two numbers that we average to obtain the x-coordinate of the vertex:

$$x = \frac{\frac{7 + \sqrt{13}}{6} + \frac{7 - \sqrt{13}}{6}}{2} \qquad \text{(average = sum divided by 2)}$$

$$= \frac{\dfrac{7+\sqrt{13}+7-\sqrt{13}}{6}}{2} \qquad \text{(adding the two fractions)}$$

$$= \frac{\frac{14}{6}}{2} = \frac{\frac{7}{3}}{2} = \frac{7}{3} \div 2 = \frac{7}{3} \times \frac{1}{2} = \frac{7}{6}$$

Now we know that the x-coordinate of the vertex is $\frac{7}{6}$, and so the y-coordinate of the vertex is

$$y = 3\left(\mathbf{\frac{7}{6}}\right)^2 - 7\left(\mathbf{\frac{7}{6}}\right) + 3 = 3\left(\frac{49}{36}\right) - 7\left(\frac{7}{6}\right) + 3 = \frac{49}{12} - \frac{49}{6} + 3$$

$$= \frac{49}{12} - \frac{98}{12} + \frac{36}{12} = -\frac{13}{12}$$

We're finally able to state our conclusion: The vertex of the parabola is

$$\boxed{\left(\frac{7}{6}, -\frac{13}{2}\right)}$$

EXAMPLE 10: **Find the vertex of $y = x^2 + 3x + 4$.**

Solution: Seems innocent enough — what could possibly go wrong? (Ever heard of Murphy's Law?) To find x-intercepts, we set y to 0 and obtain the quadratic equation

$$\mathbf{0} = x^2 + 3x + 4$$

Since this quadratic is not factorable, we employ the Quadratic Formula:

$$x^2 + 3x + 4 = 0 \qquad [a = 1;\ b = 3;\ c = 4]$$

$$\Rightarrow \qquad x = \frac{-\mathbf{3} \pm \sqrt{(\mathbf{3})^2 - 4(\mathbf{1})(\mathbf{4})}}{2(\mathbf{1})} = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Gee, these two numbers are not real numbers; recall from the Prologue that we might call them "imaginary" numbers, and that they don't exist anywhere on the x-axis (which is an axis of real numbers). As in Example 6 earlier in this chapter, we deduce that the parabola has no x-intercepts. So how the heck do we average the x-intercepts to find the vertex when those intercepts don't even exist? The answer is, trust me! That is, let's calculate the average of these two "phantom" intercepts and see what happens.

There are two issues to consider if we want to calculate the average of two imaginary numbers. First, the two solutions of the quadratic equation above can be written separately as

$$\frac{-3+\sqrt{-7}}{2} \quad \text{and} \quad \frac{-3-\sqrt{-7}}{2}$$

Second, the average of two numbers (even these imaginary numbers!) is found by dividing their sum by 2 — here goes:

$$x = \frac{\frac{-3+\sqrt{-7}}{2} + \frac{-3-\sqrt{-7}}{2}}{2} \qquad \text{(average = sum divided by 2)}$$

$$= \frac{\frac{-3+\sqrt{-7}-3-\sqrt{-7}}{2}}{2} \qquad \text{(adding the two fractions)}$$

$$= \frac{\frac{-6}{2}}{2} \qquad \text{(combine like terms)}$$

$$= \frac{-3}{2} = -\mathbf{\frac{3}{2}}$$

Thus, assuming you still trust me, we will accept this number to be the x-coordinate of the vertex. The y-coordinate would be

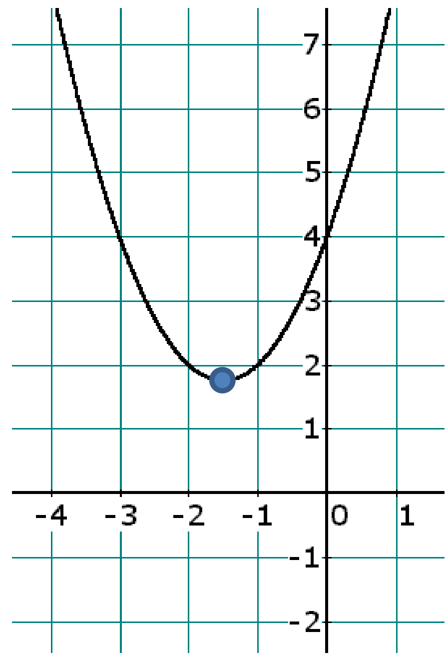
$$y = x^2 + 3x + 4 = \left(-\mathbf{\frac{3}{2}}\right)^2 + 3\left(-\mathbf{\frac{3}{2}}\right) + 4 = \frac{9}{4} - \frac{9}{2} + 4$$

$$= \frac{9}{4} - \frac{18}{4} + \frac{16}{4} = \mathbf{\frac{7}{4}}$$

Our best theory, therefore, is that the vertex of the parabola is

$$\boxed{\left(-\frac{3}{2}, \frac{7}{4}\right)}$$

Let's confirm that this calculated vertex of "the parabola with no x-intercepts" makes some sense by sketching the parabola. It looks pretty reasonable. Notice that the parabola truly has no x-intercepts, just as predicted by the algebra. Moreover, if you look carefully at the vertex (the bottom point) of the parabola, you can estimate that the x-coordinate is between -2 and -1, which the number $-\frac{3}{2}$ is. Also, the y-coordinate of the vertex appears to be a little shy of 2, which the number $\frac{7}{4}$ is. I'm convinced — are you?



Homework

6. Find the **vertex** of each parabola, using the intercepts you calculated in Homework Problem 4:

a. $y = x^2 + 2x - 48$

b. $y = x^2 + 10x + 25$

c. $y = 2x^2 + 8x + 5$

d. $y = 3x^2 - 6x + 4$

❑ *PUTTING IT ALL TOGETHER*

EXAMPLE 11: **Graph $y = x^2 - 6x + 5$ using the concepts of this chapter.**

Solution: The leading coefficient, 1, is positive; thus, the parabola **opens up**.

To find x-intercepts, set y to 0:

$$x^2 - 6x + 5 = \mathbf{0}$$

$$\Rightarrow \quad (x-5)(x-1) = 0$$

$$\Rightarrow \quad x - 5 = 0 \text{ or } x - 1 = 0$$

$$\Rightarrow \quad x = 5 \text{ or } x = 1$$

Therefore, **the x-intercepts are (1, 0) and (5, 0)**.

As for y-intercepts, we set x to 0:

$$y = \mathbf{0}^2 - 6(\mathbf{0}) + 5 = 5$$

Thus, **the y-intercept is (0, 5)**.

To find the x-coordinate of the vertex we calculate the average of the two x-coordinates of the two x-intercepts:

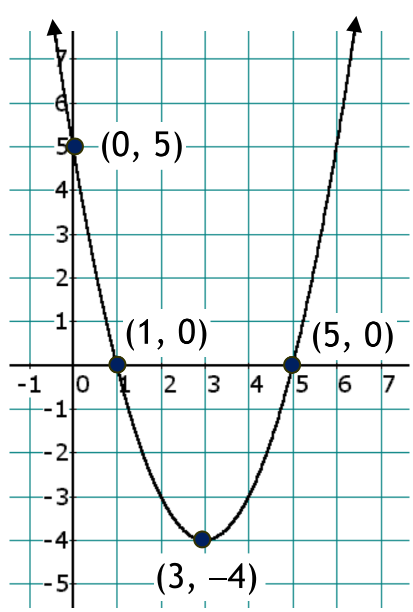
$$x = \frac{1+5}{2} = \frac{6}{2} = 3,$$

which implies that

$$y = \mathbf{3}^2 - 6(\mathbf{3}) + 5 = 9 - 18 + 5 = -4$$

So **the vertex of the parabola is (3, –4)**.

We now have our parabola.



Practice Problems

7. Find the y-intercept, the x-intercept(s), and the vertex of $y = 2x^2 + 7x - 5$. Sketch the parabola.

8. Find the y-intercept, the x-intercept(s), and the vertex of $y = -16x^2 + 24x - 9$. Sketch the parabola.

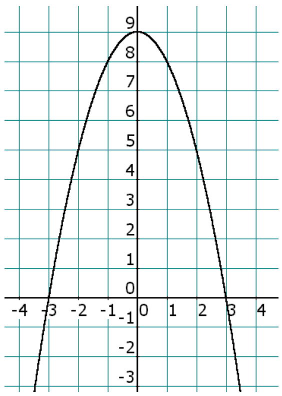
9. Find the y-intercept, the x-intercept(s), and the vertex of $y = x^2 - 3x + 5$. Sketch the parabola.

Solutions

1. The coefficient of the quadratic term in Example 1 is positive, while that of the quadratic term in Example 2 is negative. That's the factor that determines whether a parabola opens up or down. Therefore, the parabola $y = \pi x^2 - 13x + 2$ opens up, and the parabola $y = -0.7x^2 + 99x + 14$ opens down.

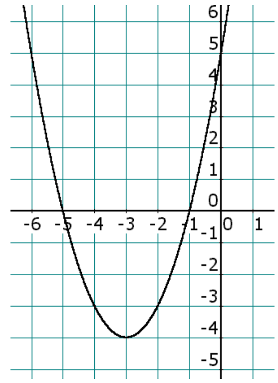
2. a. True b. False c. True d. False e. False

3. a.



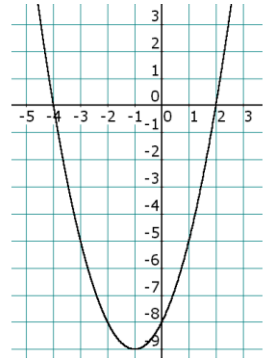
Intercepts:
(–3, 0), (3, 0), (0, 9)
Vertex: (0, 9)

b.



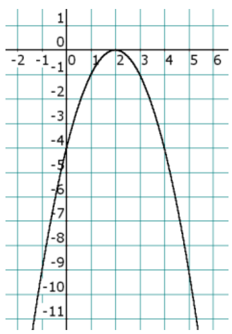
Intercepts:
(–5, 0), (–1, 0), (0, 5)
Vertex: (–3, –4)

c.



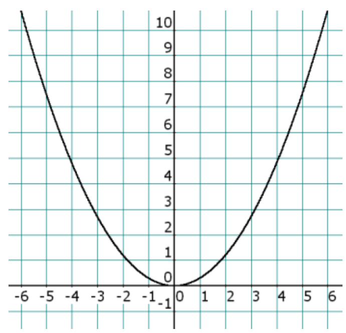
Intercepts:
(–4, 0), (2, 0), (0, –8)
Vertex: (–1, –9)

d.



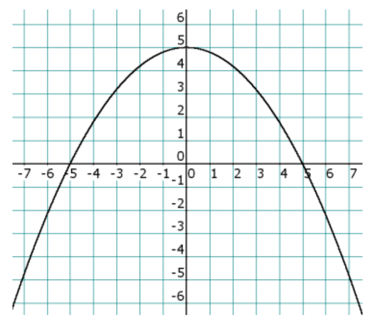
Intercepts:
(2, 0), (0, –4)
Vertex: (2, 0)

e.



Intercepts: (0, 0)
Vertex: (0, 0)

f.



Intercepts:
(–5, 0), (5, 0), (0, 5)
Vertex: (0, 5)

4. a. (–8, 0) (6, 0) (0, –48)

b. (–5, 0) (0, 25)

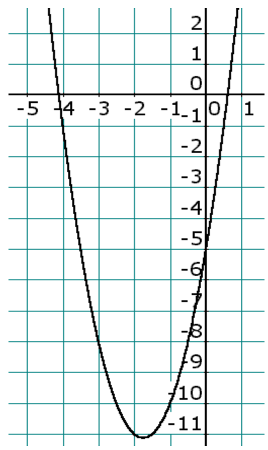
c. $\left(\dfrac{-4+\sqrt{6}}{2},\ 0\right)$ $\left(\dfrac{-4-\sqrt{6}}{2},\ 0\right)$ (0, 5)

d. No x-intercepts (0, 4)

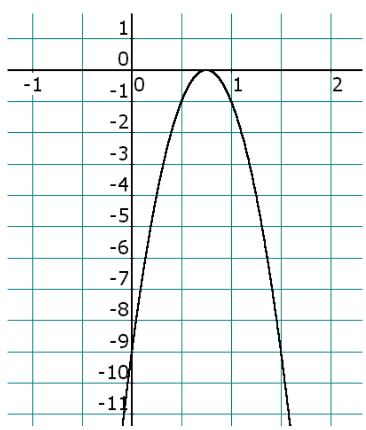
5. a. (–0.146, 0) (–6.854, 0) (0, 1)
b. (3.137, 0) (–0.637, 0) (0, 4)
c. (2.291, 0) (–0.291, 0) (0, –2)
d. No x-intercepts (0, 1)

6. a. (–1, –49) b. (–5, 0) c. (–2, –3) d. (1, 1)

7. y-int: (0, –5) x-int: $\left(\frac{-7+\sqrt{89}}{4}, 0\right)$, $\left(\frac{-7-\sqrt{89}}{4}, 0\right)$ $V\left(-\frac{7}{4}, -\frac{89}{8}\right)$

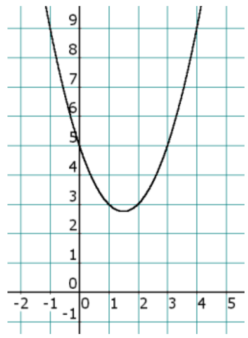


8. y-int: (0, –9) x-int: $\left(\frac{3}{4}, 0\right)$ $V\left(\frac{3}{4}, 0\right)$



Note that the axes are scaled differently. In actuality, the parabola is much skinnier than the one shown.

9. y-int: (0, 5) x-int: None $V\left(\frac{3}{2}, \frac{11}{4}\right)$



"It is our *choices* that show what we truly are, far more than our *abilities*."

– *J.K. Rowling*