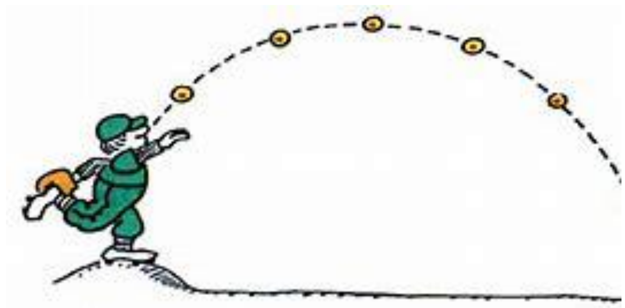
CH 32 – QUADRATIC MODELING

❑ INTRODUCTION



Finding the vertex of a parabola by averaging the x-coordinates of the x-intercepts of the parabola is the same time-consuming process over and over again. Perhaps we can find the vertex of a *generic* parabola, yielding a simple formula that can find the vertex of any parabola quickly.

❑ SHORTCUT TO THE VERTEX

Let's start with a generic parabola (opening up or down) in standard form:

$$y = ax^2 + bx + c$$ [opens up if $a > 0$, and down if $a < 0$]

Our job now is to find the x-intercepts of this parabola. To find x-intercepts, we set y to 0 and solve for x:

$$0 = ax^2 + bx + c$$

Let's first turn the equation around so it's in standard form:

$$ax^2 + bx + c = 0$$

How do we solve this equation for x? Actually, it's quite simple. The solutions of this equation are precisely the solutions provided by the Quadratic Formula. Thus, the two solutions of this equation are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

These two "numbers" are the x-coordinates of the x-intercepts of the parabola. The next step is to find the average of these two numbers; this is done by adding them together and dividing by 2:

$$\frac{\dfrac{-b+\sqrt{b^2-4ac}}{2a}+\dfrac{-b-\sqrt{b^2-4ac}}{2a}}{2}$$

$$=\frac{\dfrac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}}{2} \qquad \text{(common denominator)}$$

$$=\frac{\dfrac{-2b}{2a}}{2} \qquad \text{(the radicals sum to 0)}$$

$$=\frac{\dfrac{-b}{a}}{2} \qquad \text{(reduce the top fraction)}$$

$$=\frac{-b}{a}\cdot\frac{1}{2} \qquad \text{(multiply by reciprocal)}$$

$$=\frac{-b}{2a}$$

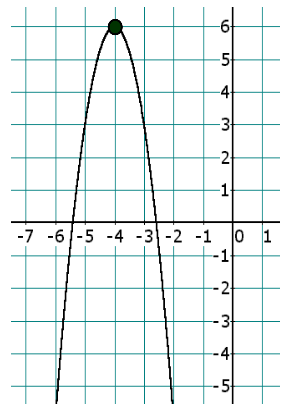
We've proved a theorem:

> The x-coordinate of the **vertex** of the parabola
>
> $$y = ax^2 + bx + c$$
>
> is given by the formula
>
> $$x = \frac{-b}{2a}$$

To find the y-coordinate of the vertex, just plug the x-coordinate of the vertex into the parabola formula.

For example, let's find the vertex of the parabola $y = -3x^2 - 24x - 42$. For this parabola, $a = -3$ and $b = -24$ (we don't the value of c for this problem). So the x-coordinate of the parabola is

$$x = \frac{-b}{2a} = \frac{-(-24)}{2(-3)} = \frac{24}{-6} = -4$$



Now plug $x = -4$ into the parabola formula to find the y-coordinate of the vertex:

$$y = -3x^2 - 24x - 42 = -3(\mathbf{-4})^2 - 24(\mathbf{-4}) - 42$$

$$= -3(16) - 24(-4) - 42 = -48 + 96 - 42 = \mathbf{6}$$

Conclusion: The vertex of the parabola

$y = -3x^2 - 24x - 42$ is $\mathbf{(-4, 6)}$.

> Here's a good way to remember this vertex formula: Take a look at the Quadratic Formula, cross out the radical, and what remains is the *x*-coordinate of the vertex.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Homework

1. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = x^2 + 2x - 48$

b. $y = x^2 + 10x + 25$

c. $y = 2x^2 + 8x + 5$

d. $y = 3x^2 - 6x + 4$

e. $y = -5x^2 + 40x - 20$

f. $y = -3x^2 - 16$

2. Use the formula $x = \frac{-b}{2a}$ to find the **vertex** of each parabola:

a. $y = -2x^2 + 5x - 7$

b. $y = 7x^2 - x - 1$

c. $y = 3x^2 - 3x - 2$ d. $y = 7x^2 + 10$

e. $y = 4x^2 + 7x$ f. $y = -5x^2 - 14x + 10$

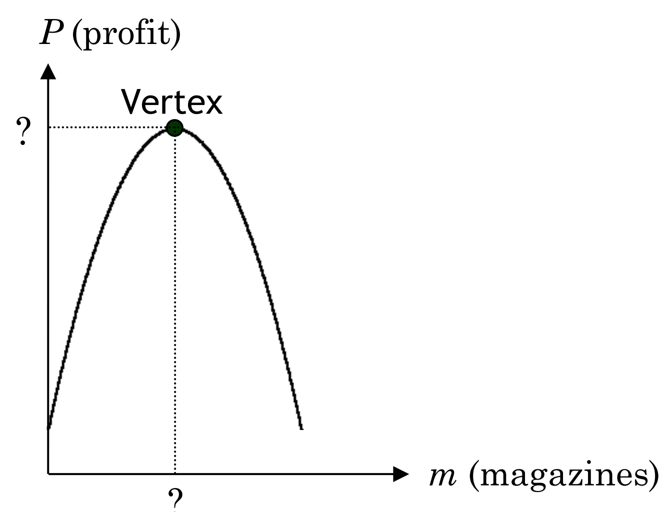
❑ OPTIMIZATION PROBLEMS

EXAMPLE 1: **A publishing company found that the profit they make on their latest magazine is given by the formula**

$$\boldsymbol{P = -2m^2 + 680m + 10200}$$

where P is profit and m is the number of magazines sold. Find the number of magazines that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

Solution: Wow, what a problem! It might help to realize that the variable m is playing the role of the variable x, while the P is playing the role of y. When viewed this way, the profit formula is simply a parabola that opens down (since $-2 < 0$). And where is the maximum point on a parabola that opens down? At its vertex, of course.



Once we find the vertex of the given parabola formula, we can find the number of magazines needed to maximize the profit, and then the dollar amount of the maximum profit itself.

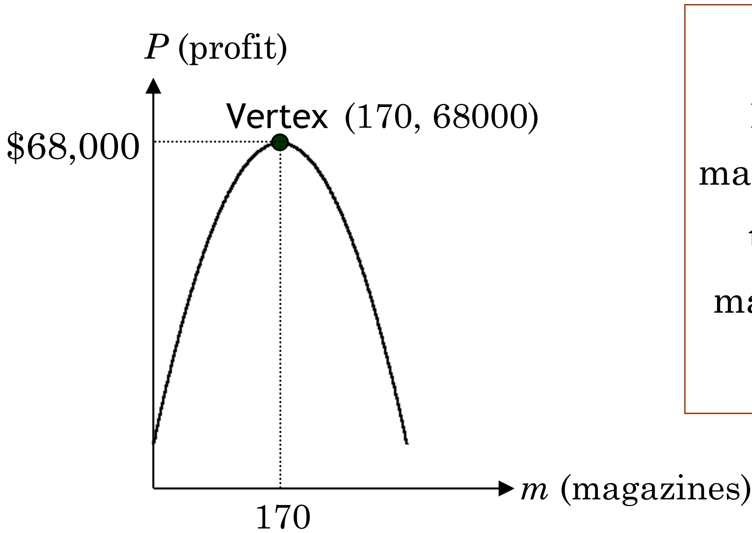
From the given parabola formula $P = -2m^2 + 680m + 10200$, we see that $a = -2$ and $b = 680$. Plugging these two values into our vertex formula gives us the m-coordinate of the vertex:

$$m = \frac{-b}{2a} = \frac{-680}{2(-2)} = \frac{-680}{-4} = 170$$

And since the m-axis is the horizontal axis (the magazine axis), we conclude that the number 170 represents magazines. That is, 170 magazines will produce the maximum profit. To calculate that profit, we merely let $m = 170$ in the profit formula:

$$\begin{aligned}
P &= -2m^2 + 680m + 10200 && \text{(the profit formula)} \\
&= -2(\mathbf{170})^2 + 680(\mathbf{170}) + 10200 && (m = 170) \\
&= -2(28900) + 680(170) + 10200 && \text{(exponents first)} \\
&= -57800 + 115600 + 10200 && \text{(multiplications second)} \\
&= 68000 && \text{(additions last)}
\end{aligned}$$

The vertex of the parabola is therefore the point (170, 68000).



In conclusion, producing **170** magazines will yield the company a maximum profit of **$68,000**.

EXAMPLE 2: **A company found that the cost for manufacturing a surfboard is given by the formula**

$$C = 3s^2 - 270s + 6800$$

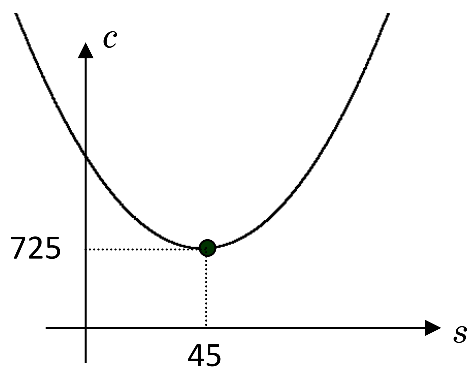
where C is the cost in dollars and s is the number of surfboards produced. Find the number of boards the company needs to produce in order to minimize their cost. What is the minimum cost?

Solution: This problem is very similar to Example 1, except here we are to find the minimum of something, not the maximum. The graph of the cost formula is a parabola opening up (since 3 > 0), so its vertex will be at the bottom of the parabola, and, as before, the vertex tells us everything we need to know.

Since $a = 3$ and $b = -270$, the s-coordinate of the vertex is

$$s = \frac{-b}{2a} = \frac{-(-270)}{2(3)} = \frac{270}{6} = 45$$

Plugging this value of s into the cost formula yields $c = 725$. Therefore, the vertex of the parabola is (45, 725).



What does all this mean? Geometrically, the point (45, 725) is at the bottom of the parabola. Therefore,

45 surfboards will result in a minimum cost of **$725**.

Homework

3. A hard drive company found that the profit they make on their hard drive is given by the formula

$$P = -4h^2 + 200h + 1000,$$

where P is profit and h is the number of hard drives sold. Find the number of hard drives that the company needs to produce in order to maximize their profit. Also determine the maximum profit.

4. A company found that the cost for manufacturing a flash drive is given by the formula

$$C = 7d^2 - 140d + 745,$$

where C is the cost in dollars and d is the number of flash drives produced. Find the number of flash drives the company needs to produce in order to minimize their cost. What is the minimum cost?

5. The path of a baseball follows the parabola

$$y = -12x^2 + 456x + 100$$

At what value of x does the baseball reach its highest point? How high above the x-axis is the highest point?

6. The profit, P, is given by the formula

$$P = -10c^2 + 800c + 10000$$

Find the number of calculators, c, which would maximize the profit, and determine the maximum profit.

7. The cost, C, for manufacturing w widgets is given by

$$C = 7w^2 - 112w + 5000$$

How many widgets will minimize the cost, and what is the cost?

8. The path of a catapulted math teacher follows the parabola $y = -12x^2 + 312x + 250$. Find the point in the x-y plane where the teacher reaches his highest point.

9. A certain mountain has the approximate shape of the parabola $y = -13x^2 + 390x + 200$. Find the coordinates of the mountain peak.

10. A certain valley has the approximate shape of the parabola $y = 7.5x^2 - 1485x - 400$. Find the coordinates of the lowest point of the valley.

❑ *THREE EQUATIONS IN THREE VARIABLES*

To solve the last application in this chapter, we must become adept at solving a system of three equations in three variables.

Solve the system:

$$\mathbf{4x - 2y + 3z = -14} \qquad \text{[Equ 1]}$$
$$\mathbf{6x + y - z = 13} \qquad \text{[Equ 2]}$$
$$\mathbf{-x + 3y - 4z = 24} \qquad \text{[Equ 3]}$$

Start with Equ 1 and Equ 2 to eliminate the x:

$$4x - 2y + 3z = -14 \quad \text{(times 3)} \Rightarrow \quad 12x - 6y + 9z = -42$$
$$6x + y - z = 13 \quad \text{(times -2)} \Rightarrow \quad -12x - 2y + 2z = -26$$

Add: $-8y + 11z = -68$ [Equ 4]

Now use Equ 1 and Equ 3 to eliminate the x:

$$4x - 2y + 3z = -14 \quad \text{(leave it)} \Rightarrow \quad 4x - 2y + 3z = -14$$
$$-x + 3y - 4z = 24 \quad \text{(times 4)} \Rightarrow \quad -4x + 12y - 16z = 96$$

Add: $10y - 13z = 82$ [Equ 5]

Solve the 2×2 system consisting of Equ 4 and Equ 5:

[Equ 4]	$-8y + 11z = -68$	(times 5)	$\Rightarrow$	$-40y + 55z = -340$
[Equ 5]	$10y - 13z = 82$	(times 4)	$\Rightarrow$	$40y - 52z = 328$
	Add:			$3z = -12$
			$\Rightarrow$	$z = -4$

We now calculate y using Equ 4 and our newly found value of z:

$$-8y + 11z = -68$$

$$\Rightarrow \quad -8y + 11(\mathbf{-4}) = -68$$

$$\Rightarrow \quad -8y - 44 = -68$$

$$\Rightarrow \quad -8y = -24$$

$$\Rightarrow \quad y = 3$$

And lastly, we solve for x using Equ 1 and our values of y and z:

$$4x - 2y + 3z = -14$$

$$\Rightarrow \quad 4x - 2(\mathbf{3}) + 3(\mathbf{-4}) = -14$$

$$\Rightarrow \quad 4x - 6 - 12 = -14$$

$$\Rightarrow \quad 4x - 18 = -14$$

$$\Rightarrow \quad 4x = 4$$

$$\Rightarrow \quad x = 1$$

We conclude that the solution of the system of 3 equations in 3 variables is

$$\boxed{x = 1,\ y = 3,\ z = -4}$$

We could CHECK our solution by plugging the values of x, y, and z into ALL THREE of the *original* equations.

Homework

11. Solve each system of equations:

a. $3x + 2y - z = -4$
$2x - 3y + 2z = 21$
$-x + y - 3z = -17$

b. $6h + 3j - 2k = -28$
$7h - 4j + k = -26$
$-2h + 2j - 3k = 2$

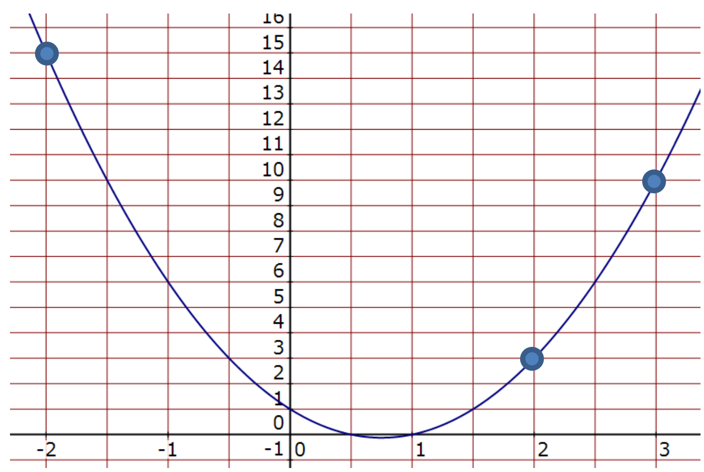
❑ *Finding the Parabola Through Three Points*

EXAMPLE 3: **Find the equation of the parabola which passes through the three points (2, 3), (3, 10), and (–2, 15).**

Solution: Recall that the parabolas in this class have the form

$$y = ax^2 + bx + c$$

To fully reveal the identity of the unknown parabola, we need to find the values of a, b, and c.



Now, what does it mean for a certain point to lie on the graph of some equation? It means that the coordinates of that point (the x and y) should work in the equation.

Therefore, if the point (2, 3) is to lie on the parabola $y = ax^2 + bx + c$, we should be able to plug 2 in for x and 3 in for y — yielding the equation:

$$\mathbf{3} = a \cdot \mathbf{2}^2 + b \cdot \mathbf{2} + \mathrm{c}$$

or, $\underline{3 = 4a + 2b + c}$

Since (3, 10) is on the parabola, its coordinates should also satisfy the parabola equation:

$$\mathbf{10} = a \cdot \mathbf{3}^2 + b \cdot \mathbf{3} + c$$

or, $\underline{10 = 9a + 3b + c}$

And last, using the point (–2, 15) yields the equation:

$$\mathbf{15} = a \cdot (\mathbf{-2})^2 + b \cdot \mathbf{-2} + c$$

or, $\underline{15 = 4a - 2b + \mathrm{c}}$

Let's summarize what we have by taking all three underlined equations and writing them together as a system of three equations in the three variables a, b, and c. We'll also flip each equation around.

$4a + 2b + c = 3$ [Equ 1]

$9a + 3b + c = 10$ [Equ 2]

$4a - 2b + c = 15$ [Equ 3]

We now have a system of 3 equations in 3 unknowns. We solve the system as we did in the previous section.

Start with Equ 1 and Equ 2 and eliminate the a:

$a + b + c = 0$ **(times –9)** $\Rightarrow$ $-9a - 9b - 9c = 0$

$9a + 3b + c = 10$ $\Rightarrow$ $9a + 3b + c = 10$

$-6b - 8c = 10$ [Equ 4]

Now use Equ 1 and Equ 3 and eliminate the a:

$a + b + c = 0$ **(times –4)** $\Rightarrow$ $-4a - 4b - 4c = 0$

$4a - 2b + c = 15$ $\Rightarrow$ $4a - 2b + c = 15$

$-6b - 3c = 15$ [Equ 5]

Solve the 2 by 2 system consisting of Equ 4 and Equ 5:

[Equ 4]	$-6b - 8c = 10$	**(times –1)**	$\Rightarrow$	$6b + 8c = -10$
[Equ 5]	$-6b - 3c = 15$		$\Rightarrow$	$-6b - 3c = 15$
				$5c = 5$
				$c = 1$

Using Equ 4:

$$-6b - 8c = 10$$

$$\Rightarrow \quad -6b - 8(\mathbf{1}) = 10$$

$$\Rightarrow \quad -6b - 8 = 10$$

$$\Rightarrow \quad -6b = 18$$

$$\Rightarrow \quad b = -3$$

(Equ 5 would have worked just as well)

Using Equ 1 from way back, we calculate the value of a:

$$a + b + c = 0$$

$$\Rightarrow \quad a - \mathbf{3} + \mathbf{1} = 0$$

$$\Rightarrow \quad a - 2 = 0$$

$$\Rightarrow \quad a = 2$$

(Equ 2 or 3 would have worked just as well)

Remember, we started with the parabola $y = ax^2 + bx + c$, plugged in the three given points, and determined that $a = 2$, $b = -3$, and $c = 1$. We're done -☺- the parabola we're seeking is

$$y = 2x^2 - 3x + 1$$

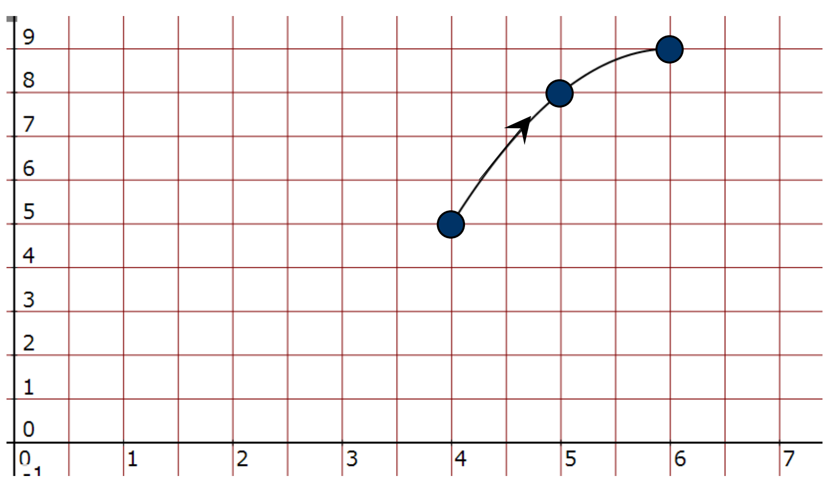
Homework

12. Find the equation of the **parabola** passing through the three given points:

a. (1, 4) (2, 6) (3, 10)
b. (–1, –2) (2, 13) (–3, 8)
c. (3, 13) (1, 3) (–2, 3)
d. (1, 3) (2, –7) (–1, 5)
e. (1, –1) (–1, –7) (2, –10)
f. (1, 17) (2, 38) (–3, 73)
g. (1, 1) (2, –5) (–2, 7)
h. (1, 3) (–1, 13) (2, 22)
i. (1, –5) (–2, –11) (–1, –5)
j. (1, 7) (–2, 52) (–1, 21)

EXAMPLE 4: **A missile is fired from an unknown location on the *x*-axis of the radar grid. Its first three positions are reported as (4, 5), (5, 8), and (6, 9), at which time the radar jams. Assume that the missile follows a parabolic trajectory (path), and that the missile is traveling east. Calculate both the *x*-value from where the missile was fired and the *x*-value where the missile will hit the ground.**

Solution: Here's a graphic of the missile's trajectory, as determined by the three reported positions:



Where was it launched, and where will it land?

Our goal is to "extrapolate" (extend) this curve to a full parabola formula, so we can deduce where the missile came from (pretty important) and where it's going to land (REALLY important).

Beginning with $y = ax^2 + bx + c$ (see Example 3), we get the following three equations, obtained by plugging each of the points on the parabola into the generic parabola equation:

$$y = ax^2 + bx + c$$

(4, 5): $\mathbf{5} = a(\mathbf{4}^2) + b(\mathbf{4}) + c \Rightarrow 16a + 4b + c = 5$

(5, 8): $\mathbf{8} = a(\mathbf{5}^2) + b(\mathbf{5}) + c \Rightarrow 25a + 5b + c = 8$

(6, 9): $\mathbf{9} = a(\mathbf{6}^2) + b(\mathbf{6}) + c \Rightarrow 36a + 6b + c = 9$

We now have a system of three equations in three unknowns, which can be solved using the techniques of this chapter. When you (yes ... you!) solve the system you will determine that $a = -1$, $b = 12$, and $c = -27$, from which we deduce that the parabola that describes the trajectory is

$$y = -x^2 + 12x - 27$$

But we're still not done. (By the way, we will not be blown to bits by the missile while we wade through all these calculations. A computer can solve the problem in a billionth of a second – but remember: someone has to program the computer.) To find the firing point and the landing point on the x-axis, we now calculate the two x-intercepts of the parabola, which we do by setting y to 0:

$0 = -x^2 + 12x - 27$

$\Rightarrow \quad x^2 - 12x + 27 = 0$

$\Rightarrow \quad (x - 3)(x - 9) = 0$

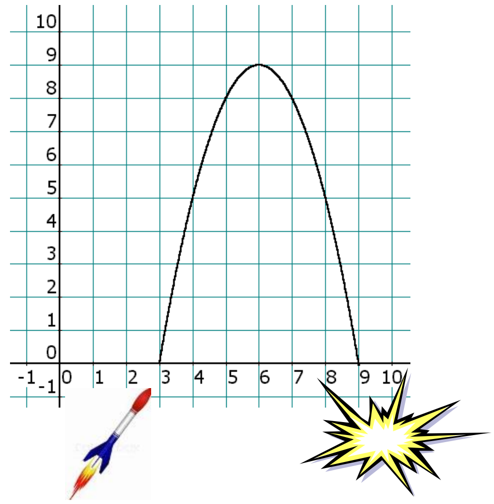
$\Rightarrow \quad x - 3 = 0 \text{ or } x - 9 = 0$

$\Rightarrow \quad x = 3 \text{ or } x = 9 \Rightarrow$

The x-intercepts are **(3, 0)** and **(9, 0)**.

We're now ready to answer the question at hand:

The missile was fired from $\boldsymbol{x = 3}$, and will land at $\boldsymbol{x = 9}$.



Homework

13. Solve the missile trajectory problem (Example 4) if the missile was spotted at the points (4, 14), (5, 18), (7, 20) before the radar failed.

14. Solve the missile trajectory problem if the missile was spotted at the points (7, 10), (8, 12) and (9, 12).

15. Solve the missile trajectory problem if the missile was spotted at the points (1, 0), (3, 8) and (5, 8).

16. Solve the missile trajectory problem if the missile was spotted at the points (3, 3), (4, 4) and (5, 3).

Practice Problems

17. Using the short formula from this chapter, find the vertex of the parabola

$$y = -5x^2 + 70x - 90.$$

18. The profit is given by the formula

$$P = -5b^2 + 750b + 10000$$

where P is profit and b is the number of books published. Find the number of books which would maximize the profit, and determine the maximum profit.

19. The cost, C, for manufacturing t trumpets is given by

$$C = 6t^2 - 276t + 5000$$

How many trumpets will minimize the cost, and what is the cost?

20. The path of a football follows the parabola $y = -7x^2 + 126x + 250$. Find the point in the x-y plane where the football reaches its highest point.

21. Solve the system of equations:

$$\begin{aligned} 4a + 3b - 2c &= 8 \\ -a - b + c &= -3 \\ 5a + 4b - c &= 9 \end{aligned}$$

22. Solve the missile trajectory problem for the given three points:

a. (2, 5) (3, 8) (5, 8)

b. (4, 6) (5, 10) (6, 12)

c. (3, 8) (4, 12) (5, 12)

d. (3, 24) (4, 28) (5, 20)

Solutions

1. a. $(-1, -49)$ b. $(-5, 0)$ c. $(-2, -3)$
d. $(1, 1)$ e. $(4, 60)$ f. $(0, -16)$

2. a. $\left(\frac{5}{4}, -\frac{31}{8}\right)$ b. $\left(\frac{1}{14}, -\frac{29}{8}\right)$ c. $\left(\frac{1}{2}, -\frac{11}{4}\right)$
d. $(0, 10)$ e. $\left(-\frac{7}{8}, -\frac{49}{16}\right)$ f. $\left(-\frac{7}{5}, \frac{99}{5}\right)$

3. 25 hard drives; maximum profit = \$3,500

4. 10 flash drives; minimum cost = \$45

5. $x = 19$; height = 4,432

6. 40 calculators; maximum profit = \$26,000

7. 8 widgets; minimum cost = \$4552

8. $(13, 2278)$ **9.** $(15, 3125)$ **10.** $(99, -73907.5)$

11. a. $x = 2, y = -3, z = 4$ b. $h = -4, j = 0, k = 2$

12. a. $y = x^2 - x + 4$ b. $y = 2x^2 + 3x - 1$ c. $y = x^2 + x + 1$
d. $y = -3x^2 - x + 7$ e. $y = -4x^2 + 3x$ f. $y = 7x^2 + 10$
g. $y = -x^2 - 3x + 5$ h. $y = 8x^2 - 5x$ i. $y\ -2x^2 - 3$
j. $y = 8x^2 - 7x + 6$

13. Fired from $x = 2$; will land at $x = 11$

14. Fired from $x = 5$; will land at $x = 12$

15. Fired from $x = 1$; will land at $x = 7$

16. Fired from $x = 2$; will land at $x = 6$

17. V(7, 155)

18. 75 books; $38,125

19. 23 trumpets; $1826

20. (9, 817)

21. $a = 0$, $b = 2$, $c = -1$

22. a. Fired from $x = 1$; will land at $x = 7$
b. Fired from $x = 3$; will land at $x = 10$
c. Fired from $x = 2$; will land at $x = 7$
d. Fired from $x = \frac{5}{3}$; will land at $x = 6$

"The purpose of learning is growth, and our minds, unlike our bodies, can continue growing as we continue to live."

– Mortimer Adler