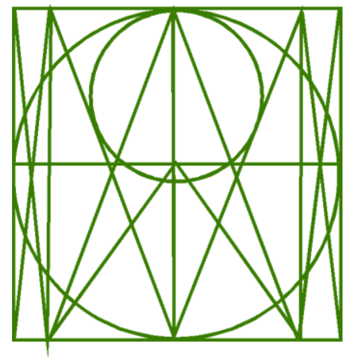
CH 37 – RATIONAL FUNCTIONS

❑ INTRODUCTION

Remember what we call a number like $\frac{2}{7}$? This is called a *rational number* because it is the *ratio* of two integers. In a like manner, a *rational function* is the ratio of two special functions called ***polynomial functions***. Since a rational function is essentially a fraction, we will have to avoid dividing by zero, which means the domain of a such a function may not be all real numbers.



❑ POLYNOMIAL FUNCTIONS

Each of the following is a polynomial function:

$y = 7$ (a *linear* function – it's a horizontal line)

$y = -3x + \sqrt{2}$ (a *linear* function – it's a line with slope = –3)

$y = 2x^2 - x + 9$ (a *quadratic* function – it's a parabola)

$f(t) = \sqrt[4]{2}\,t^3 - t^2$ (a *cubic* function)

$P(w) = -\pi w^4 + 5w^2 + 8$ (a *quartic* function)

$Q(a) = \frac{2}{3}a^5 - 4a + 1$ (a *quintic* function)

The key to any ***polynomial function*** is that all the exponents on the input variable come from the set of whole numbers: $\{0, 1, 2, 3, \ldots\}$. The coefficients (the numbers in front of the variables), on the other hand, can come from anywhere in $\mathbb{R}$, the set of real numbers.

Consider the quartic (4th degree) polynomial function

$$y = -2\pi x^4 + \tfrac{9}{10}x^3 - 17x^2 + \sqrt{2}$$

First look at the exponents; they are all whole numbers. Even the last term, $\sqrt{2}$, can be written as $\sqrt{2}\,x^0$, and so even the exponent on this last term is a whole number. Thus, all the exponents on the x's (4, 3, 2, and 0) come from the whole numbers, while all the coefficients (-2π, $\frac{9}{10}$, -17, $\sqrt{2}$) come from $\mathbb{R}$. Considering the definition of polynomial function, the given function is indeed a polynomial function.

Each of the following is <u>not</u> a polynomial function:

$y = \dfrac{1}{x}$	($\frac{1}{x} = x^{-1}$ and -1 is not a whole number)
$y = \sqrt{x}$	($\sqrt{x} = x^{1/2}$ and $\frac{1}{2}$ is not a whole number)
$f(x) = \dfrac{1}{\sqrt[3]{x}}$	($\frac{1}{\sqrt[3]{x}} = x^{-1/3}$ and $-\frac{1}{3}$ is not a whole number)
$g(x) = \lvert x-1 \rvert$	(no absolute values allowed around the x)
$E(x) = 2^x$	(since x is in the exponent, it can be any number)
$T(x) = \sin x$	(it's on your calculator, but it's not a polynomial function)
$y = \log x$	(a log function can never be a polynomial function)
$x^2 + y^2 = 25$	(it's a circle – it's not a function of any kind)

Homework

1. Explain why $y = \pi x^5 - \sqrt{2}\,x^3 + \frac{1}{4}x - 17.5$ <u>is</u> a polynomial function.

2. Explain why $h(x) = \sqrt[3]{5}x^4 - \sqrt{2x} + \frac{1}{2}$ is <u>not</u> a polynomial function.

3. The highest exponent on the variable in a polynomial function is called its ***degree***. Find the degree of each polynomial function:

   a. $y = \pi$ b. $y = x^3 - 17x^2 + 8$

   c. $y = \sqrt{2}x^8 - \frac{\pi}{2}x^{10}$ d. $y = 7x + 5$

4. a. What is the domain of any polynomial function?

   b. T/F: Every parabola is a polynomial function.

   c. T/F: Every non-vertical line is a polynomial function.

   d. T/F: Every line is a polynomial function.

❑ RATIONAL FUNCTIONS

A ***rational function*** is defined to be the *ratio* of two *polynomial functions*. If P and Q are polynomial functions, then $R = \frac{P}{Q}$ is a rational function. A typical example of a rational function is $y = \frac{3.9x^2 + 7x - 9}{x + 8}$.

<u>EXAMPLE 1:</u> **Graph:** $\boldsymbol{y = \frac{1}{x - 2}}$

<u>Solution:</u> Since y is the ratio of a constant polynomial function and a linear polynomial function, we know that y is a rational function.

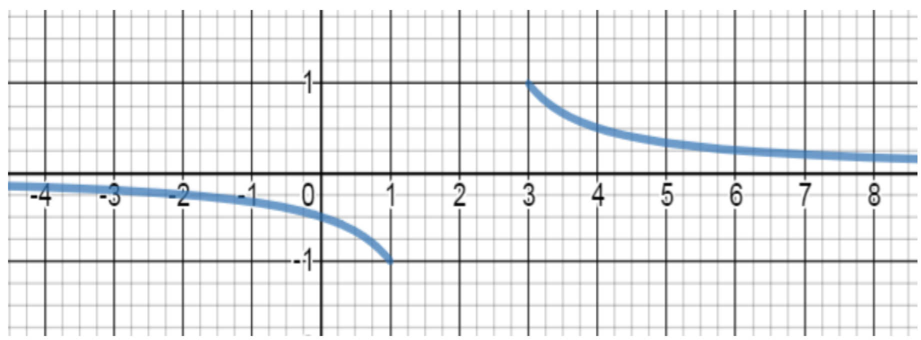
We begin our analysis of this rational function by determining the **domain**. In order that the fraction be defined, we must <u>not</u> divide by zero. What value of x makes the denominator zero? The value $x = 2$ will. (Just set $x - 2 = 0$ and solve for x.) Therefore, the domain is all real numbers except 2; that is, the domain is $\mathbb{R} - \{2\}$.

Intercepts come next. If $x = 0$, then $y = \frac{1}{0-2} = -\frac{1}{2}$. Thus, **(0, -½)** is the y-intercept. To find an x intercept, set $y = 0$. This gives $0 = \frac{1}{x-2} \Rightarrow 0(x-2) = \frac{1}{x-2}(x-2) \Rightarrow 0 = 1$, which has no solution. Thus, there are **no** x-intercepts.

Now for some ordered pairs that satisfy the formula $y = \frac{1}{x-2}$:

x	y
-3	$-\frac{1}{5}$
-2	$-\frac{1}{4}$
-1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	-1
2	Und.
3	1
4	$\frac{1}{2}$
5	$\frac{1}{3}$
6	$\frac{1}{4}$
7	$\frac{1}{5}$

If we plot these points and connect them with a smooth curve, we would get the following graph:

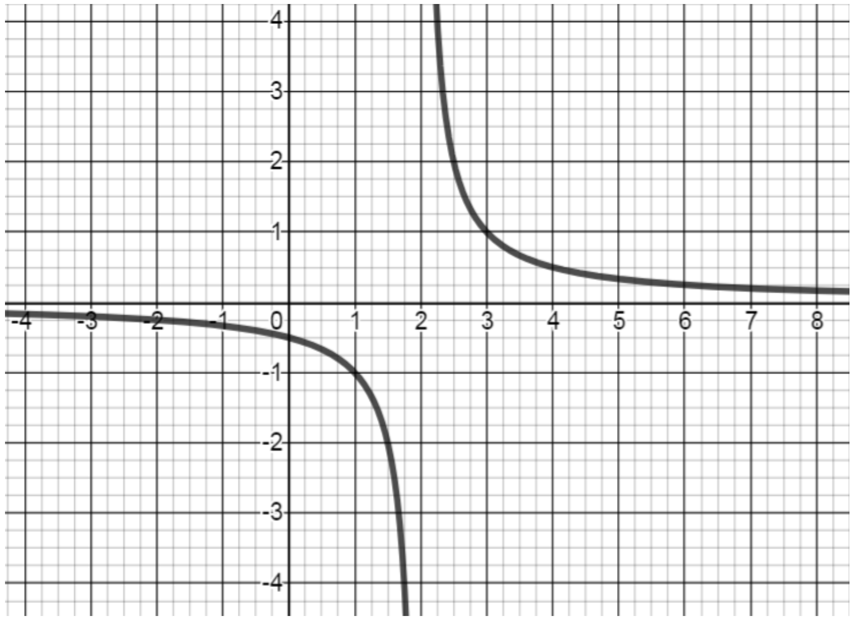


What some students do at this point is to lazily connect the points (3, 1) and (1, –1) with a straight line. Talk about jumping to conclusions! Our domain of $\mathbb{R} - \{2\}$ implies that x cannot be 2 in this function; the straight-line trick won't work. So we agree that a major chunk of the graph is missing. How do we get a better picture of the graph? We try some x-values that are near 2:

$(1\frac{1}{2}, -2)$ $(1\frac{3}{4}, -4)$ $(1\frac{7}{8}, -8)$

$(2\frac{1}{2}, 2)$ $(2\frac{1}{4}, 4)$ $(2\frac{1}{8}, 8)$

Adding these points to our previous attempt at a graph gives us a much better picture:



This graph has some real cool **limits**. Suppose we let x approach ∞. The y-values are positive (the curve is above the x-axis), but are getting smaller and smaller, approaching zero. Thus, as $\boldsymbol{x \to \infty, y \to 0}$. [This can be read: "*As x grows infinitely large, y is getting closer and closer to 0.*"]

Now let x approach $-\infty$. The y-values are negative but are rising toward zero. Therefore, as $\boldsymbol{x \to -\infty, y \to 0}$.

The number 2 seems to be an interesting x-value. Although x can never be 2 in this function, it looks like the curve is getting closer and closer to the vertical line $x = 2$. In fact, if we let x approach 2 from the right, the curve is growing taller and taller, and so we have the limit: **As $\boldsymbol{x \to 2}$ (from the right), $\boldsymbol{y \to \infty}$.** [This can be read: "*As x gets closer and closer to 2, approaching 2 from the right (meaning values larger than 2), y is growing infinitely large.*"]

Now let x approach 2 from the left. This time the curve is dropping rapidly, toward negative infinity. This observation yields the limit: **As $\boldsymbol{x \to 2}$ (from the left), $\boldsymbol{y \to -\infty}$.**

Let's summarize the four limits we've deduced:

- As $x \to \infty$, $y \to 0$.
- As $x \to -\infty$, $y \to 0$.
- As $x \to 2$ (from the right), $y \to \infty$.
- As $x \to 2$ (from the left), $y \to -\infty$.

Do you see that as you move far to the right or far to the left, the curve gets closer and closer to the x-axis? We say that the line $\boldsymbol{y = 0}$ (which is the x-axis) is a ***horizontal asymptote***.

Now look at the region of the graph near $x = 2$. The curve gets closer and closer to the vertical line $x = 2$ (in fact, on both sides of the vertical line). We call the line $\boldsymbol{x = 2}$ a ***vertical asymptote***.

<u>EXAMPLE 2:</u> **Graph:** $\boldsymbol{y = \dfrac{2x-1}{x+2}}$

<u>Solution:</u> First we find the **domain**. Recall that this function will be undefined when the denominator is zero, which occurs when $x = -2$. Thus, the domain is $\mathbb{R} - \{-2\}$.

Now let's explore the **intercepts**:

If $x = 0$, then $y = \dfrac{2(0)-1}{0+2} = -\dfrac{1}{2}$. There's a y-intercept at $\mathbf{(0, -\frac{1}{2})}$.
If $y = 0$, then $0 = \dfrac{2x-1}{x+2} \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. So $\mathbf{(\frac{1}{2}, 0)}$ is an x-intercept.

It's time for some more ordered pairs for this function. Use your calculator to verify each of the following:

(–1, –3) (1, 0.33) (3, 1) (5, 1.29) (10, 1.58)
(15, 1.71) (20, 1.77) (100, 1.95) (1000, 1.995)

What's happening as x grows very large? It appears that y is approaching 2. That is, **as $x \to \infty$, $y \to 2$**.

Now we'll let x go the other direction:

(–3, 7) (–5, 3.67) (–10, 2.63) (–20, 2.28)

(–100, 2.05) (–1000, 2.01)

These points show that as $\boldsymbol{x \to -\infty, y \to 2}$.

Finally, here are some ordered pairs for x's near –2 (the only real number not in the domain):

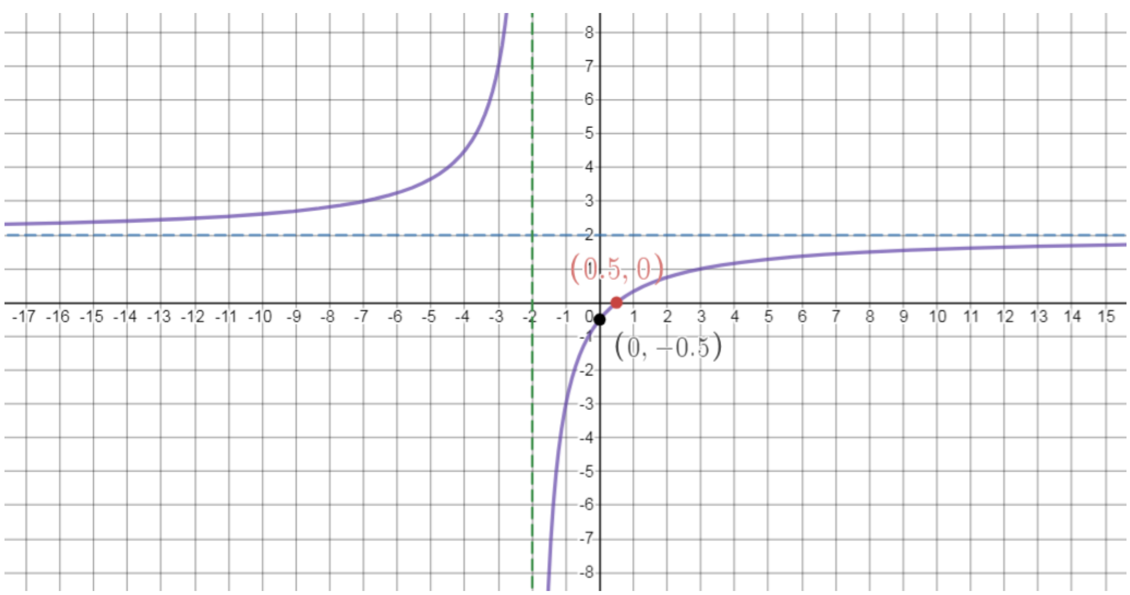
(–1.5, –8) (–1.9, –48) (–1.99, –498)

Thus, **as $\boldsymbol{x \to -2}$ (from the right), $\boldsymbol{y \to -\infty}$**.

(–2.5, 12) (–2.1, 52) (–2.01, 502)

Therefore, **as $\boldsymbol{x \to -2}$ (from the left), $\boldsymbol{y \to \infty}$**.

Plotting as many of the calculated points as possible, including the two intercepts, and considering the four limits we've found, the following graph (the two curvy pieces) emerges:



We can now be reasonably sure of the **asymptotes** (denoted by the dashed lines). Either by recalling the limits described above or by looking at the graph, we conclude that there's **a vertical asymptote at $\boldsymbol{x = -2}$** and **a horizontal asymptote at $\boldsymbol{y = 2}$**.

EXAMPLE 3: **Graph:** $y = \dfrac{4}{1 + x^2}$

Solution: Why is this function rational? Because it's the *ratio* $\dfrac{P}{Q}$ of two polynomial functions: the constant polynomial $P(x) = 4$ and the quadratic polynomial $Q(x) = 1 + x^2$.

To find the **domain**, set the denominator to zero to see what's not in the domain: $1 + x^2 = 0$. This equation has no solution in ℝ, since solving it leads to $x = \pm\sqrt{-1}$, which are not real numbers. In fact, for any value of x, the quantity $1 + x^2$ is at least 1 (why?), so it certainly can't be zero. Since the denominator can never be zero, there's nothing to be excluded from the domain, and therefore the domain is ℝ. We can also figure that the graph will not have a **vertical asymptote**, since the denominator can never be 0.

Notice that if we put in some positive value of x, we'll get a certain y-value. Now look at what will happen if we put $-x$ (the *opposite* of x) into the formula. Since $(-x)^2$ is equal to x^2, we will get the same y-value. This implies that the left side of the graph will be the mirror image of the right side. We say that the graph possesses ***y-axis symmetry*** (or is *symmetric with respect to the y-axis*).

Now we seek the **intercepts**. Set $x = 0$ to get $y = 4$, and so the y-intercept is **(0, 4)**. Now set $y = 0$, giving

$$0 = \frac{4}{1+x^2} \Rightarrow 0(1+x^2) = \frac{4}{1+x^2}(1+x^2) \Rightarrow 0 = 4$$

This absurd result indicates that the equation has no solution; hence, there are no x-intercepts.

It's time for some additional ordered pairs for this function:

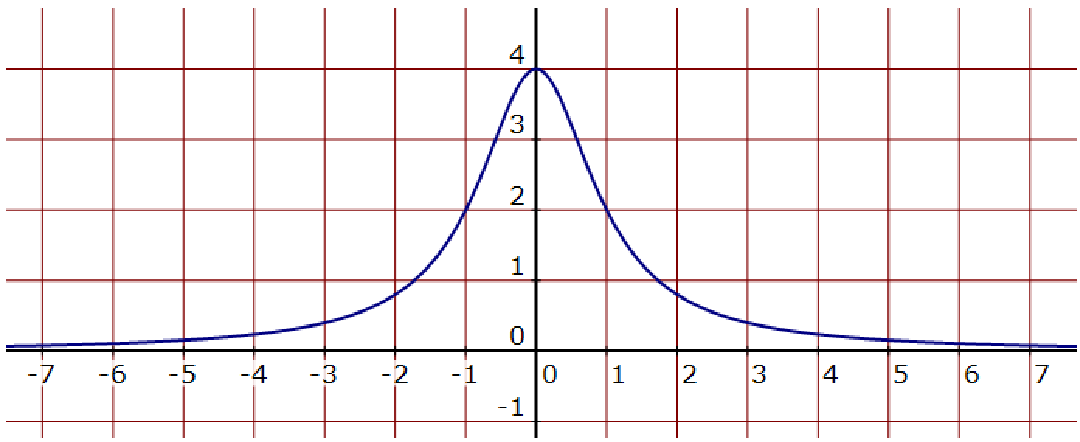
(1, 2) (2, 0.8) (3, 0.4) (4, 0.24) (10, 0.04) (200, 0.0001)

These points suggest the limit: **As $x \to \infty$, $y \to 0$**. This implies that **$y = 0$ is a horizontal asymptote**.

Here are some more ordered pairs, designed to see what happens as we approach the *y*-axis from the right:

(0.75, 2.56) (0.5, 3.2) (0.25, 3.76) (0.1, 3.96) (0.02, 3.998)

If we plot all the points calculated so far, and if we recall the *y*-axis symmetry, we get the following graph:



We determined at the outset that the *domain* of this rational function is $\mathbb{R}$. Is it clear from the graph that this is indeed the case?

Homework

5. Consider the rational function in Example 2. Without referring to the graph, prove that *y* can have the value 2.01, but *y* can never have the value 2.

6. Find the domain:

a. $y = \frac{2x+7}{9}$ b. $f(x) = \frac{2x-3}{4+x}$

c. $g(x) = \frac{3x}{2x-10}$ d. $R(x) = \frac{x-1}{-7x+4}$

e. $f(x) = \frac{x^2-9}{x^2-100}$ f. $g(x) = \frac{8x-16}{x^2+25}$

7. Find the intercepts:

a. $f(x) = \frac{x-4}{x-2}$ b. $y = \frac{3}{5x-15}$

c. $y = \frac{2x+1}{x-3}$ d. $g(x) = \frac{5-x}{6x+1}$

8. Find the asymptotes:

a. $R(x) = \frac{8x+1}{4x-4}$ b. $y = \frac{2x-3}{2x+1}$

c. $y = \frac{3x-7}{x+2}$ d. $h(x) = \frac{2x+7}{4x-4}$

9. Find the domain, the intercepts, and the asymptotes of

$$y = \frac{1}{4+x^2}.$$

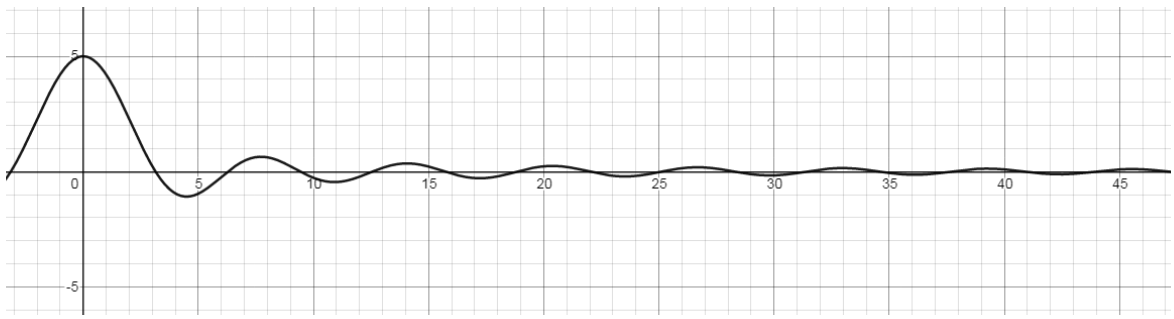
10. Perform a complete analysis of the function $y = \frac{2}{x-3}$.

11. Perform a complete analysis of the function $y = \frac{3x-5}{x-2}$.

12. Perform a complete analysis of the function $y = \frac{2}{2+x^2}$.

13. Perform a complete analysis of the function $y = \frac{-1}{x+1}$.

14. Consider the graph



Explain why the horizontal line $y = 0$ (that is, the x-axis) is a horizontal asymptote for the curve.

Practice Problems

15. a. Explain why $f(x) = \sqrt{7}x^{10} + \pi x^7 - 6x - 1$ is a polynomial function. What is its degree?

    b. Explain why $y = 3x^5 - \sqrt{x} + \pi$ is not a polynomial function.

16. a. A horizontal line (is, is not) a polynomial function.

    b. The function $y = \frac{1}{x}$ (is, is not) a polynomial function.

    c. What is the degree of the polynomial function $y = 7x - \pi$?

    d. Is a circle a polynomial function?

17. Consider the rational function $y = \frac{7}{2x-8}$.

    a. Find the domain.

    b. Find all the intercepts.

    c. Find all the asymptotes.

    d. Calculate y if $x = 4.1$.

18. Find all the intercepts and asymptotes of $r(x) = \dfrac{8x+6}{2x-3}$, and graph.

19. Graph $y = \dfrac{-2x-2}{x-1}$.

As $x \to 1$ (from the right), $y \to$ ___.
As $x \to 1$ (from the left), $y \to$ ___.
As $x \to \infty$, $y \to$ ___.
As $x \to -\infty$, $y \to$ ___.

20. Graph $y = \dfrac{5}{2+x^2}$. Discuss domain, symmetry, and asymptotes.

As $x \to \infty$, $y \to$ ___.
As $x \to -\infty$, $y \to$ ___.
As $x \to 0$ (from the right), $y \to$ ___.
As $x \to 0$ (from the left), $y \to$ ___.

21. True/False:

a. $y = \sqrt[3]{7}x^{10} - \pi x^3 + \sqrt{2}$ is a polynomial function.

b. $y = \dfrac{1}{x^5}$ is a polynomial function.

c. The graph of $f(x) = \dfrac{1}{2x+10}$ has a vertical asymptote at $x = -5$.

d. The graph of $g(x) = \dfrac{10x+9}{5x-11}$ has a horizontal asymptote at $y = 10$.

e. The domain of the function $y = \dfrac{6}{1+x^2}$ is $\mathbb{R} - \{\pm 1\}$.

f. For the graph of $y = \dfrac{3x+1}{x-\pi}$, as $x \to \infty$, $y \to 3$.

Solutions

1. All coefficients are from $\mathbb{R}$, and all exponents are from $\mathbb{W}$ (the whole numbers).

2. The middle term is $\sqrt{2}\,x^{1/2}$, and $\frac{1}{2} \notin \mathbb{W}$.

3. a. 0 b. 3 c. 10 d. 1

4. a. $\mathbb{R}$ b. False c. True d. False

5. $y = \dfrac{2x-1}{x+2} \Rightarrow 2.01 = \dfrac{2x-1}{x+2} \Rightarrow 2.01x + 4.02 = 2x - 1 \Rightarrow x = -502.$
So, (–502, 2.01) is on the graph, and indeed y can be 2.01.

Now let's pretend that y could be 2; then
$2 = \dfrac{2x-1}{x+2} \Rightarrow 2x+4 = 2x-1 \Rightarrow 4 = -1 \Rightarrow$ No solution. Thus, there is no x which will make $y = 2$.

6. a. $\mathbb{R}$ b. $\mathbb{R} - \{-4\}$ c. $\mathbb{R} - \{5\}$ d. $\mathbb{R} - \left\{\frac{4}{7}\right\}$ e. $\mathbb{R} - \{\pm 10\}$ f. $\mathbb{R}$

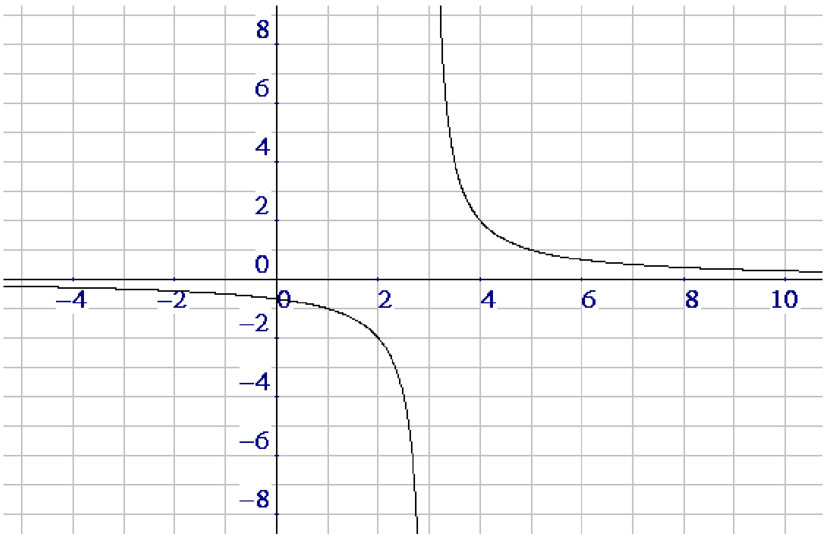
7. a. (4, 0) (0, 2) b. $(0, -\frac{1}{5})$ c. $(-\frac{1}{2}, 0)$ $(0, -\frac{1}{3})$ d. (5, 0) (0, 5)

8. a. $x = 1$ $y = 2$ b. $x = -\frac{1}{2}$ $y = 1$ c. $x = -2$ $y = 3$ d. $x = 1$ $y = \frac{1}{2}$

9. Since the only way the formula can be messed up is by dividing by 0, and since the denominator can never be zero (verify this yourself), the domain is $\mathbb{R}$.
Setting $x = 0$ gives a y-value of 1/4, so the y-intercept is $(0, \frac{1}{4})$. If you set $y = 0$, you'll get no solution for y. Thus, there is no x-intercept.

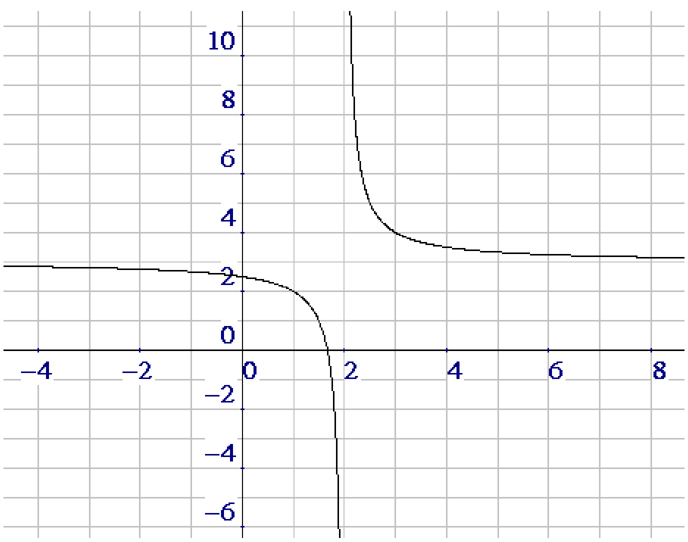
There are no vertical asymptotes, since the denominator is never zero. Letting x approach either ∞ or $-\infty$, y approaches 0. Thus, a horizontal asymptote is $y = 0$ (the x-axis).

10.



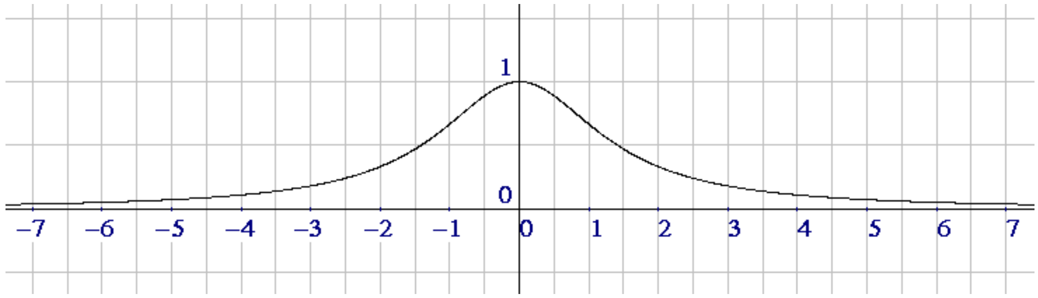
Domain $= \mathbb{R} - \{3\}$
x-int: none
y-int: $(0, -\frac{2}{3})$
As $x \to 3$ (from the right), $y \to \infty$
As $x \to 3$ (from the left), $y \to -\infty$
As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$
vert asy: $x = 3$
horiz asy: $y = 0$

11.



Domain $= \mathbb{R} - \{2\}$
x-int: $(\frac{5}{3}, 0)$
y-int: $(0, \frac{5}{2})$
As $x \to 2$ (from the right), $y \to \infty$
As $x \to 2$ (from the left), $y \to -\infty$
As $x \to \infty$, $y \to 3$
As $x \to -\infty$, $y \to 3$
vert asy: $x = 2$
horiz asy: $y = 3$

12.



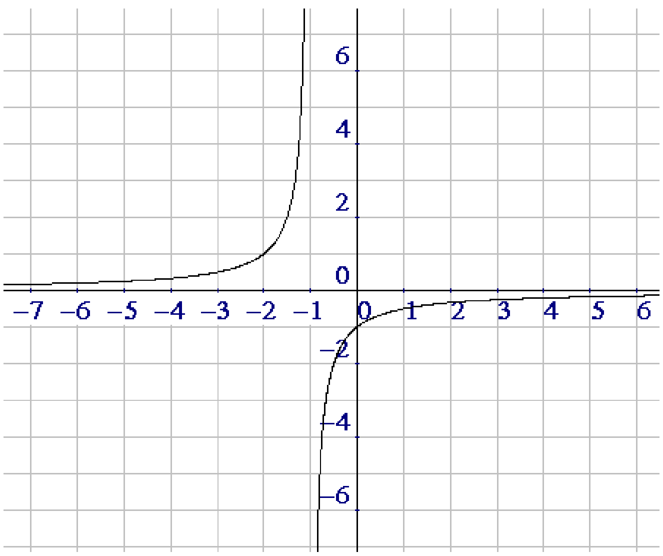
Domain = $\mathbb{R}$ x-int: none y-int: (0, 1)

As $x \to \infty$, $y \to 0$ As $x \to -\infty$, $y \to 0$

vert asy: none   horiz asy: $y = 0$

maximum point at (0, 1)

13.



Domain = $\mathbb{R} - \{-1\}$

x-int: none

y-int: (0, −1)

As $x \to -1$ (from the right),

$y \to -\infty$

As $x \to -1$ (from the left),

$y \to \infty$

As $x \to \infty$, $y \to 0$

As $x \to -\infty$, $y \to 0$

vert asy: $x = -1$

horiz asy: $y = 0$

14. Because of the limit: As $x \to \infty$, $y \to 0$. Even though the graph intersects its own horizontal asymptote infinitely often, the curve nevertheless continues to get closer and closer to the x-axis (the line $y = 0$), and this is ultimately what is meant by a horizontal asymptote.

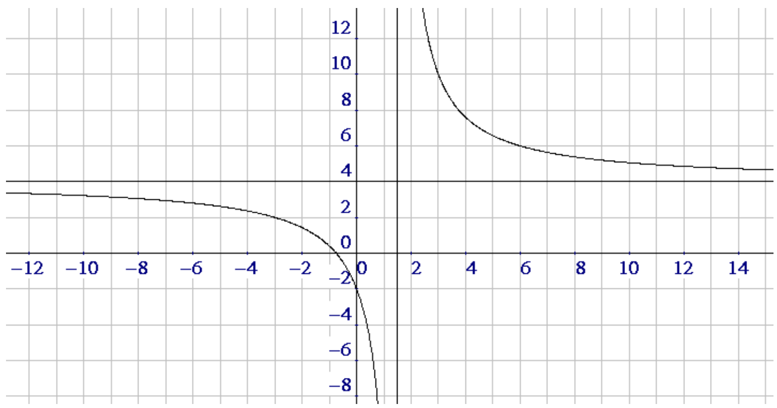
15. a. f is a polynomial because the coefficients are real numbers and the exponents (the 10, 7 and 1) are whole numbers. Its degree is 10.

b. Look at the middle term; it can be written as $x^{1/2}$, a term whose exponent is <u>not</u> from the whole numbers.

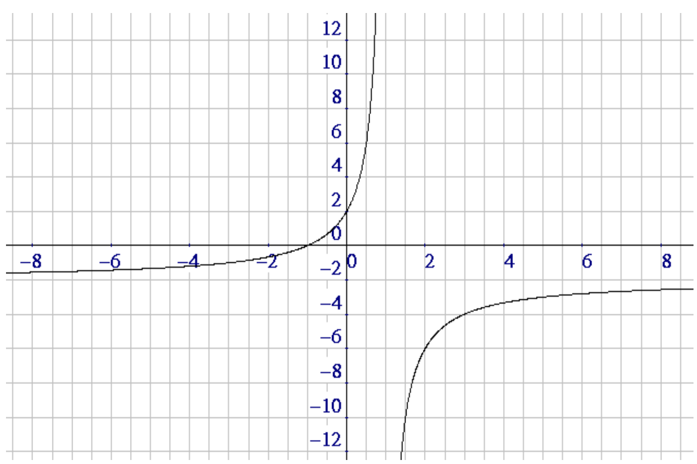
16. a. is b. is not ($1/x = x^{-1}$) c. 1 d. It's not even a function, let alone the special function called a polynomial.

17. a. $\mathbb{R} - \{4\}$ b. $(0, -7/8)$ c. $x = 4$ and $y = 0$ d. 35

18. Intercepts: $(0, -2)$ and $(-\frac{3}{4}, 0)$; vert asy: $x = \frac{3}{2}$; horiz asy: $y = 4$

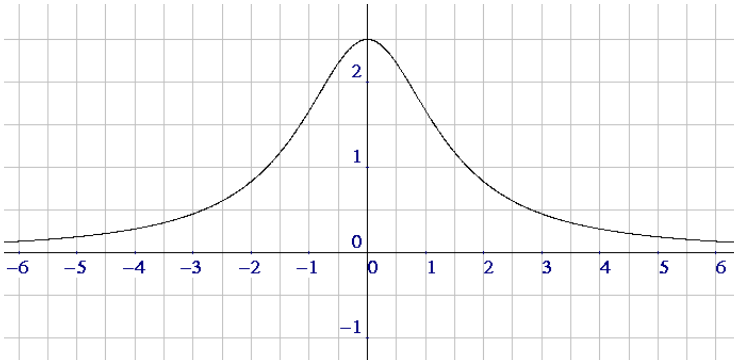


19.



Limits: $-\infty$; ∞; -2; -2

20.



Domain = $\mathbb{R}$

y-axis symmetry

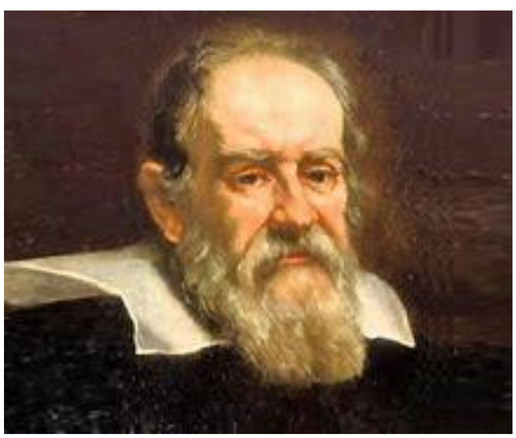
No vert asy

Horiz asy: $y = 0$

Limits: 0; 0; $\frac{5}{2}$; $\frac{5}{2}$

21. a. T b. F c. T d. F e. F f. T

" The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language."



Galileo Galiliei