THE LAWS OF LOGS

❑ *THE TWO CANCELING RULES*

Carefully study the next five examples:

$$\log_9 9^2 = \log_9 81 = 2$$

$$\log_{10} 10^5 = \log_{10} 100{,}000 = 5$$

$$\log_5 5^3 = \log_5 125 = 3$$

$$\log_2 2^1 = \log_2 2 = 1$$

$$\log_e e^0 = \log_e 1 = 0$$

Something's going on here. In each problem the final answer matches the exponent at the front of the problem. For example,

$$\log_9 9^{\boxed{2}} = \boxed{2}$$

Also notice that the base of the log matches the base of the expression that we're taking the log of. For instance,

$$\log_{\boxed{10}} \boxed{10}^5 = 5$$

Using these two insights we might now see that $\log_{12} 12^7 = 7$. We can now generalize this whole discussion into the first of two canceling rules:

$$\boxed{\log_b b^x = x}$$

We call it a canceling rule because the log function (which is done second) cancels out the exponential function (which is done first), leaving just the x.

To motivate the second canceling rule, consider the following four calculations:

$$10^{\log_{10} 1000} = 10^3 = 1000$$

$$2^{\log_2 32} = 2^5 = 32$$

$$5^{\log_5 125} = 5^3 = 125$$

$$7^{\log_7 1} = 7^0 = 1$$

For each example notice that the base of the entire question matches the base of the log -- for example, $\boxed{7}^{\log_{\boxed{7}} 1}$. We also see that in every case the final answer matches the number we're taking the log of -- for instance, $2^{\log_2 \boxed{32}} = 2^5 = \boxed{32}$. In a nutshell,

$$\boxed{b^{\log_b x} = x}$$

This is a canceling rule because the exponential function (which is done second) cancels out the log function (which is done first), leaving just the x.

<u>EXAMPLE 1:</u> **Use the canceling rules to simplify each expression:**

A. $\log_3 3^n = n$

B. $\log 10^{x+y} = x + y$

C. $7^{\log_7 (ab)} = ab$

D. $e^{\ln(e+4)} = e + 4$

E. $\log_4 5^n$ cannot be simplified by a canceling rule, since the base of the exponential (the 5) does not match the base of the log (the 4).

F. $8^{\log 7}$ also cannot be simplified by a canceling rule, since the base of the exponential (the 8) does not match the base of the log (the 10).

Homework

1. In the second canceling rule, explain why we must restrict x to the positive real numbers.

2. Simplify each expression:

a. $\log 10$	b. $\ln e$	c. $\log 10^{u+v}$	d. $\ln e^{xyz}$
e. $\log_9 9^{500}$	f. $10^{\log(xy)}$	g. $e^{\ln(\log 7)}$	h. $2^{\log_2(\ln e)}$
i. $3^{\log_3(a-b)}$	j. $\log_2 2^R$	k. $\log e^x$	l. $e^{\log Q}$

3. A common student error is to figure that $\log_b(x+y) = \log_b x + \log_b y$. We need to dispel this myth right now. Let $b = 2$, so that we're working with base 2. Then let both x and y equal 8. Show that the left side of the formula results in 4, whereas the right side comes out 6.

4. Prove that the conjecture $\log(xy) = (\log x)(\log y)$ is false. Hint: Let $x = 100$ and $y = 1000$ and work out each side.

❑ TWO LAWS OF LOGS

$$\log_b(xy) = \log_b x + \log_b y$$
$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

The log of a product is the sum of the logs.

The log of a quotient is the difference of the logs.

EXAMPLE 2: **Use the two Laws of Logs to expand each expression:**

A. $\log(ABC) = \log A + \log B + \log C$

B. $\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln z = \ln x + \ln y - \ln z$

C. $\log_2\left(\frac{a}{bc}\right) = \log_2 a - \log_2(bc) = \log_2 a - (\log_2 b + \log_2 c)$
$= \log_2 a - \log_2 b - \log_2 c$

❑ THE LOG OF A POWER

We've already seen this rule:

$$\log_b x^n = n \log_b x$$

To calculate the log of a power, bring down the power as a coefficient.

❑ SUMMARY OF LOGS

We can summarize our knowledge of logs in the following six statements -- one definition, two cancellation rules, and three laws:

Statement	Example
$y = \log_b x$ means $b^y = x$	$2 = \log 100$ means $10^2 = 100$
$\log_b b^x = x$	$\ln e^T = T$
$b^{\log_b x} = x$	$2^{\log_2 99} = 99$
$\log_b(xy) = \log_b x + \log_b y$	$\log(100x) = 2 + \log x$
$\log_b \frac{x}{y} = \log_b x - \log_b y$	$\ln \frac{x}{e} = \ln x - 1$
$\log_b x^n = n \log_b x$	$\log_5 \sqrt[7]{Q} = \frac{1}{7}\log_5 Q$

EXAMPLE 3: **Use one of the Laws of Logs to expand each expression:**

A. $\log(7x) = \log 7 + \log x$

B. $\ln(xyz) = \ln x + \ln y + \ln z$

C. $\ln(ex) = \ln e + \ln x = 1 + \ln x$

D. $\log_2 \frac{8}{3} = \log_2 8 - \log_2 3 = 3 - \log_2 3$

E. $\log \frac{A}{B} = \log A - \log B$

F. $\log_{12}[(a+b)(a-b)] = \log_{12}(a+b) + \log_{12}(a-b)$

G. $\log(x^3) = 3\log x$

H. $\log_3 \sqrt{x} = \log_3 x^{1/2} = \frac{1}{2}\log_3 x$

I. $\ln \sqrt[3]{w^2} = \ln w^{2/3} = \frac{2}{3}\ln w$

J. $(\ln x)^2$ cannot be simplified because the power is on the ln x, not the x. Note that $\ln x^2 = 2\ln x$. See the difference?

EXAMPLE 4: **Use the Laws of Logs to expand each expression:**

A. $\ln(ax^3) = \ln a + \ln x^3 = \ln a + 3\ln x$

B. $\log\frac{xy}{z} = \log(xy) - \log z = \log x + \log y - \log z$

C. $\ln\frac{a}{bc} = \ln a - \ln(bc) = \ln a - [\ln b + \ln c] = \ln a - \ln b - \ln c$

D. $\log_2\frac{cd^3}{e^4} = \log_2 cd^3 - \log_2 e^4 = \log_2 c + 3\log_2 d - 4\log_2 e$

E. $\ln\left(\frac{\sqrt{x}\ \sqrt[3]{y}}{e^z\ \sqrt[5]{w^3}}\right) = \ln\sqrt{x}\ \sqrt[3]{y} - \ln e^z\ \sqrt[5]{w^3}$

$$= \ln\sqrt{x} + \ln\sqrt[3]{y} - \left[\ln e^z + \ln\sqrt[5]{w^3}\right]$$

$$= \frac{1}{2}\ln x + \frac{1}{3}\ln y - z - \frac{3}{5}\ln w$$

EXAMPLE 5: **Use the Laws of Logs to condense each expression into a single log with coefficient 1:**

A. $\log x + \log y = \log(xy)$

B. $\ln a - \ln b = \ln\frac{a}{b}$

C. $3\log_2 T = \log_2 T^3$

D. $\frac{3}{7}\log n = \log n^{3/7} = \log \sqrt[7]{n^3}$

E. $\frac{1}{2}\ln x + 5\ln y = \ln\sqrt{x} + \ln y^5 = \ln\left(\sqrt{x}\, y^5\right)$

F. $2\log x + \frac{1}{2}\log y + \frac{2}{5}\log z = \log x^2 + \log\sqrt{y} + \log\sqrt[5]{z^2}$

$$= \log\left(x^2\sqrt{y}\,\sqrt[5]{z^2}\right)$$

G. $\ln x + \ln y - \ln z = \ln\dfrac{xy}{z}$

H. $2\log x - \frac{1}{3}\log y = \log x^2 - \log y^{1/3} = \log\dfrac{x^2}{\sqrt[3]{y}}$

I. $\ln a - \ln b - \ln c = \ln\dfrac{a}{b} - \ln c = \ln\dfrac{\frac{a}{b}}{c} = \ln\dfrac{a}{bc}$

Alternate approach:

$$\ln a - \ln b - \ln c = \ln a - (\ln b + \ln c) = \ln a - \ln(bc) = \ln\frac{a}{bc}$$

J. $\frac{2}{3}\log(a+b) - \frac{1}{2}\log(a-b) - 7\log(ab)$

$$= \log\sqrt[3]{(a+b)^2} - \log\sqrt{a-b} - \log(ab)^7$$

$$= \log\frac{\sqrt[3]{(a+b)^2}}{\sqrt{a-b}} - \log a^7 b^7$$

$$= \log\frac{\dfrac{\sqrt[3]{(a+b)^2}}{\sqrt{a-b}}}{a^7 b^7}$$

$$= \log\frac{\sqrt[3]{(a+b)^2}}{a^7 b^7\sqrt{a-b}}$$

Homework

5. Expand each expression:

a. $\ln(ab)$ b. $\log(wyz)$ c. $\log_2(abcd)$

d. $\log\frac{h}{k}$ e. $\log\frac{a+b}{c}$ f. $\log_8\frac{x-y}{u+v}$

g. $\ln x^3$ h. $\log 7^n$ i. $\log_5 5^7$

6. Expand each expression:

a. $\log_7\sqrt{x}$ b. $\log\sqrt[3]{x}$ c. $\ln\sqrt[5]{Q^2}$

d. $\log\frac{1}{x^3}$ e. $\ln\frac{1}{u^{3/2}}$ f. $\log\frac{1}{T^{-4/5}}$

7. Expand each expression:

a. $\ln(a^2b)$ b. $\log(xy^3)$ c. $\ln(a^2b^3c^4)$

d. $\log_6\frac{x^2}{y^3}$ e. $\ln\frac{ab}{c^9}$ f. $\log\frac{x^2y^5}{z^{10}}$

8. Expand each expression:

a. $\log_3\frac{a}{bc}$ b. $\ln\frac{x^2y}{wz^3}$ c. $\log\frac{\sqrt{x}}{yz}$

d. $\log(a^2b\sqrt[7]{c})$ e. $\ln\left[\frac{wx}{yz}\right]^5$ f. $\log_5\frac{a^2\sqrt[3]{y}}{wx}$

9. Condense each expression:

a. $\ln x-\ln 7$ b. $\log y+\log 12$

c. $\frac{1}{3}\ln 4+\ln 2$ d. $\frac{2}{5}\log t-\frac{1}{5}\log t$

e. $\ln(x^2)+\ln x+\ln 7$ f. $\log x-\log y+\log z$

g. $\ln a - \ln b - 3\ln c$ h. $2\log a - \frac{1}{2}\log b - \frac{2}{3}\log c$

10. Evaluate each expression <u>without a calculator</u>:

a. $\log 10^{100}$ b. $e^{\ln 7}$ c. $\log 5 + \log 2$

d. $\ln \frac{e^{10}}{e^2}$ e. $\log_{24} 8 + \log_{24} 3$ f. $\ln x + \ln \frac{1}{x}$

g. $\log 50 + \log 20$ h. $\log_3 \frac{1}{2} + \log_3 162$ i. $\log_5 50 - \log_5 2$

11. True or False:

a. $\log 10^9 = 9$ b. $7^{\log_7 R} = 7^R$

c. $\ln e^{abc} = \ln(abc)$ d. $\ln(xy) = \ln x + \ln y$

e. $\log a^b = (\log a)^b$ f. $\ln \frac{a}{b} = \frac{\ln a}{\ln b}$

g. $\ln(x+y) = \ln x + \ln y$ h. $\log(a-b) = \log a - \log b$

i. $\ln \frac{1}{x} = -\ln x$ j. $\log_{12}(x^a b) = (a\log_{12} x)(\log_{12} b)$

k. $\ln(xy^z) = z\ln(xy)$ l. $\log_3 x - \log_3 y = \log_3 \frac{x}{y}$

m. $\log a + \log b = \log(ab)$ n. $\ln e^t = t$

o. $10^{\log 9} = 9$ p. $\ln \frac{1}{x} = \frac{1}{\ln x}$

q. $e^{\ln(e-3)} = e - 3$ r. $\ln \frac{x}{y} = \ln x - \ln y$

s. $\log(ab^n) = \log a + n\log b$ t. $\ln \frac{a^2}{b^5} = \frac{2\ln a}{5\ln b}$

12. Prove that $\ln(x^8 + 8x^6 + 24x^4 + 32x^2 + 16) = 4\ln(x^2 + 2)$
HARD!

Practice Problems

$y = \log_b x$ means $b^y = x$
$\log_b b^x = x$
$b^{\log_b x} = x$
$\log_b(xy) = \log_b x + \log_b y$
$\log_b \frac{x}{y} = \log_b x - \log_b y$
$\log_b x^n = n \log_b x$

13. $\ln e + \log 1 - \log_3 3 - \log_7 49 =$

14. a. $\log_{12} 12^N =$ b. $9^{\log_9 56} =$
    c. $\ln e^{a-b} =$ d. $\log 10^{25} =$
    e. $\log_3 5^y =$ f. $5^{\log_5(-5)} =$

15. $\log_b b^x \quad b^{\log_b x} =$

16. $\ln(\log 10) =$

17. Expand: $\log\left(\frac{ab}{10}\right)$

18. Expand: $\ln\left(\frac{x^3}{y\sqrt{z}}\right)$

19. Condense: $\ln x + \ln y - \ln z$

20. Condense: $3\log x + \frac{2}{3}\log y - \frac{1}{2}\log z$

21. True/False:

    a. $\ln(a-b) = \ln a - \ln b$

    b. $\log e^t = t$

    c. $\ln\left(\frac{2}{x}\right) = \frac{\ln 2}{\ln x}$

    d. $\ln(ab^{10}) = 10\ln(ab)$

22. True/False:

a. $\log 10^{a+b} = a+b$

b. $\log e^{u-w} = u-w$

c. $6^{\log_6 z} = z$

d. $e^{\log Q} = Q$

e. $\ln(ab) = \ln a + \ln b$

f. $\ln \frac{x}{y} = \frac{\ln x}{\ln y}$

g. $\log ab^y = y \log ab$

h. $\log ab^y = \log a + y \log b$

i. $\ln p - \ln q = \ln \frac{p}{q}$

j. $\log_3 32 - \log_3 8 = \log_3 4$

k. $\ln(x+y+z) = \ln x + \ln y + \ln z$

l. $\log 7^7 = 7 \log 7$

m. $(\ln x)^3 = 3 \ln x$

n. $\frac{2}{3} \log x + 4 \log y = \log\left(\sqrt[3]{x^2}\, y^4\right)$

o. Use your calculator to determine if $\log_3 20 = \frac{\ln 20}{\ln 3}$.

Solutions

1. Because the first operation in $b^{\log_b x}$ is $\log_b x$, whose domain is $(0, \infty)$.

2. a. 1 b. 1 c. $u + v$ d. xyz e. 500 f. xy g. $\log 7$ h. 1
i. $a - b$ j. R k. As is l. As is

3. $\log_2(8+8) = \log_2 16 = 4$, whereas
$\log_2 8 + \log_2 8 = 3+3 = 6$

4. $\log(xy) = \log(100 \cdot 1{,}000) = \log 100{,}000 = \mathbf{5}$, but
$(\log x)(\log y) = (\log 100)(\log 1{,}000) = 2 \cdot 3 = \mathbf{6}$.

5. a. $\ln a + \ln b$ b. $\log w + \log y + \log z$
c. $\log_2 a + \log_2 b + \log_2 c + \log_2 d$ d. $\log h - \log k$
e. $\log(a+b) - \log c$ f. $\log_8(x-y) - \log_8(u+v)$
g. $3\ln x$ h. $n\log 7$ i. 7

6. a. $\frac{1}{2}\log_7 x$ b. $\frac{1}{3}\log x$ c. $\frac{2}{5}\ln Q$
d. $-3\log x$ e. $-\frac{3}{2}\ln u$ f. $\frac{4}{5}\log T$

7. a. $2\ln a + \ln b$ b. $\log x + 3\log y$
c. $2\ln a + 3\ln b + 4\ln c$ d. $2\log_6 x - 3\log_6 y$
e. $\ln a + \ln b - 9\ln c$ f. $2\log x + 5\log y - 10\log z$

8. a. $\log_3 a - \log_3 b - \log_3 c$ b. $2\ln x + \ln y - \ln w - 3\ln z$
c. $\frac{1}{2}\log x - \log y - \log z$ d. $2\log a + \log b + \frac{1}{7}\log c$
e. $5\ln w + 5\ln x - 5\ln y - 5\ln z$ f. $2\log_5 a + \frac{1}{3}\log_5 y - \log_5 w - \log_5 x$

9. a. $\ln \frac{x}{7}$ b. $\log(12y)$ c. $\ln(2\sqrt[3]{4})$ d. $\log \sqrt[5]{t}$

e. $\ln(7x^3)$ f. $\log \frac{xz}{y}$ g. $\ln \frac{a}{bc^3}$ h. $\log \frac{a^2}{\sqrt{b}\,\sqrt[3]{c^2}}$

10. a. 100 b. 7 c. 1 d. 8 e. 1
f. 0 g. 3 h. 4 i. 2

11. a. T b. F c. F d. T e. F f. F g. F h. F i. T
j. F k. F l. T m. T n. T o. T p. F q. F (tricky!)
r. T s. T t. F

12. Hint: Factor on the left side, or use the Third Law of Logs on the right side.

13. –2

14. a. N b. 56 c. $a - b$ d. 25 e. As is f. Undefined

15. x^2 **16.** 0 **17.** $\log a + \log b - 1$

18. $3\ln x - \ln y - \frac{1}{2}\ln z$ **19.** $\ln\left(\frac{xy}{z}\right)$

20. $\log\left[\frac{x^3\,\sqrt[3]{y^2}}{\sqrt{z}}\right]$ **21.** They're all false.

22. a. T b. F c. T d. F e. T f. F g. F h. T
i. T j. T k. F l. T m. F n. T o. T

Nothing can stop the man with the right mental attitude from achieving his goal; nothing on earth can help the man with the wrong mental attitude.

Thomas Jefferson