CH 4 – EQUATIONS AND INEQUALITIES: A GRAPHICAL APPROACH

❑ INTRODUCTION

Some equations, like $3x - 4 = 7x + 9$, are not too hard to solve. Others can be very difficult, if not impossible, to solve using algebra. For example, I have no idea how to solve the simple-looking equation

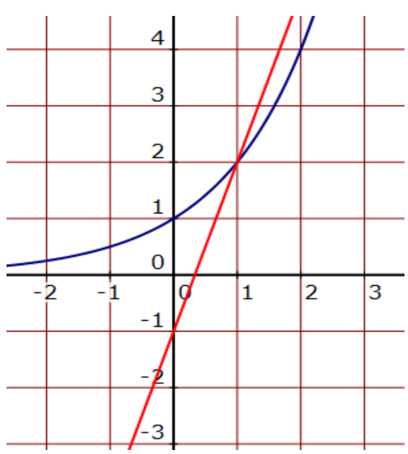
$$2^x = 3x - 1$$

using algebra. But we can find good (if not perfect) solutions using *GRAPHING*, as you'll see in the next example.

❑ OBTAINING SOLUTIONS FROM GRAPHS

EXAMPLE 1: **Solve the equation $2^x = 3x - 1$ graphically.**

Solution: I'm going to give you the graph of each side of the equation on the same grid:



From your knowledge of graphing lines, you should see that the red straight line is the graph of $y = 3x - 1$, the right side of the equation. And so the curvy blue graph is the graph of $y = 2^x$, the left side of the equation.

Now, the equation $2^x = 3x - 1$ is a statement of equality — we want to know what values of x will make each side of the equation result in the same number. To do that, we find any ***points of intersection*** of the two graphs. Looking at the graph, a point of intersection seems to be (1, 2). That means that when $x = 1$, both graphs have a y-value of 2, which also implies that when $x = 1$, both sides of the equation are equal (they're both equal to 2). In short, the solution of the equation $2^x = 3x - 1$ (as far as the graph is concerned) is

$$\boxed{x = 1}$$

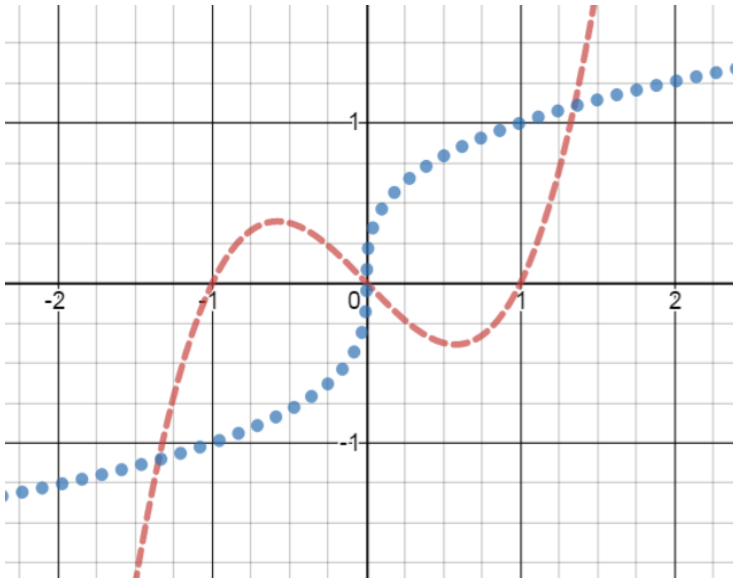
Two Caveats (things to watch out for):

1. How do we know that there's only one solution? Perhaps the two graphs cross somewhere higher up in Quadrant I, or maybe down in Quadrant III — using the graph we have, we can never know. [By simple substitution, you can show that $x = 3$ is another solution of the equation. But are there others?]

2. It may appear that the point of intersection is (1, 2), but a graph is just a rough picture — no graph can be perfectly accurate. So maybe the true value of x at the point of intersection is 0.999997, or maybe 1.0000203. If it IS one of those, the graph will NEVER tell us that. So the graphing method is not an exact method, but if the equation can't be solved using algebra, a graph may be the best we can get — and may be quite useful for applications both inside and outside of mathematics.

EXAMPLE 2: **Solve the equation $x^3 - x = \sqrt[3]{x}$ graphically.**

Solution: For this problem, I'm not going to even bother telling you which graph goes with which side of the equation. Just know that each side of the equation has been graphed. Your job is to pick out any points of intersection (which are, remember, just approximations). The x-values of those points of intersection will be the solutions of the equation.



First, do you see three points of intersection? There's one in Quadrant I, there's one in Quadrant III, and it also appears that the origin, (0, 0), is a point of intersection. Second comes the guessing part: The x-values of the three points of intersection are approximately -1.3, 0, and 1.3. So these are our best shots at the solutions of the equation:

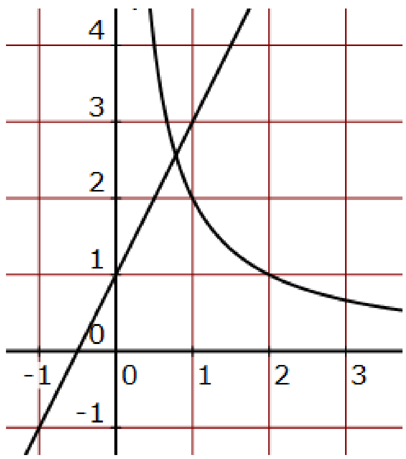
$x = -1.3$	$x = 0$	$x = 1.3$

For the rest of this chapter (after the following homework section), you must graph the given equation yourself before you find the point(s) of intersection.

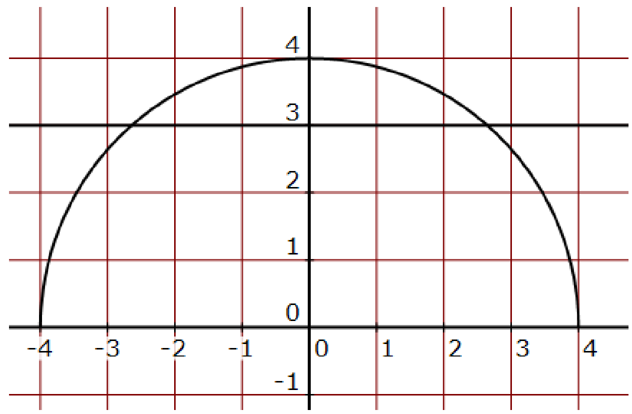
Homework

Solve each equation by using the associated graphs:

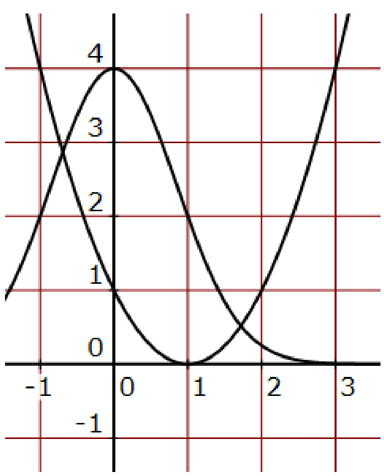
1. $\frac{1}{x} = 2x - 1$



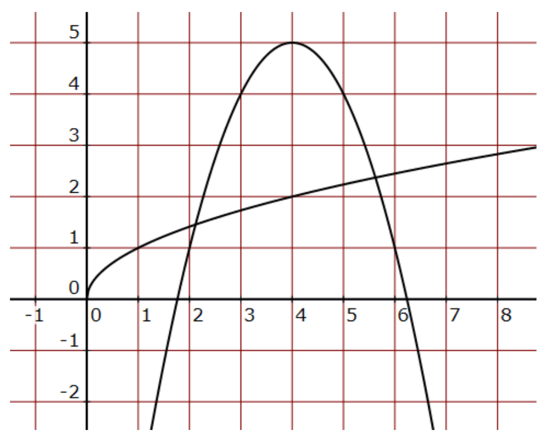
2. $\sqrt{16 - x^2} = 3$



3. $4^{-x^2} = (x-2)^2$



4. $-(x-4)^2 + 5 = \sqrt{x}$



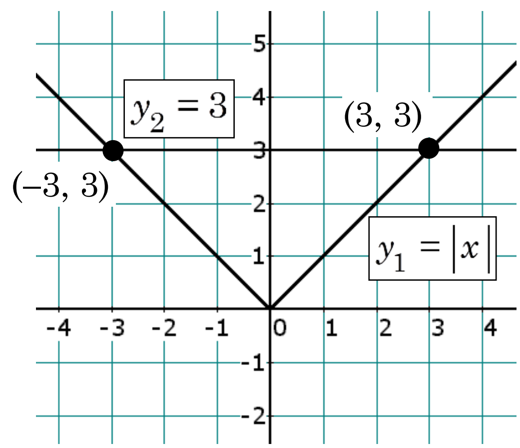
❑ ABSOLUTE VALUE EQUATIONS

EXAMPLE 3: **Solve the equation $|x| = 3$ graphically.**

Solution: We graph each side of the equation separately and look at any points of intersection we might find. To help us keep track, we'll call the left side of the equation y_1 and the right side y_2:

$$y_1 = |x| \qquad y_2 = 3$$

Now graph each equation on the same grid. It appears we have two points of intersection: (–3, 3) and (3, 3). Thus, if $x = -3$ or $x = 3$, the two sides of the equation balance, meaning we have two solutions to our equation:



$$\boxed{x = 3 \text{ or } x = -3}$$

[We could also write $x = \pm 3$]

Check: $x = 3$:

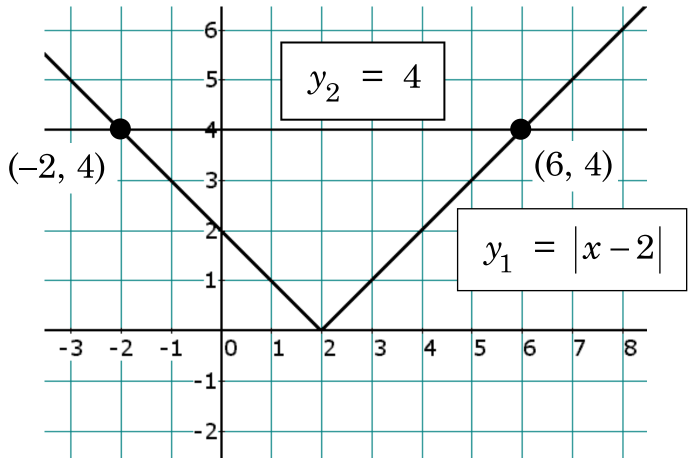
$\lvert x \rvert$	3
$\lvert 3 \rvert$	
3	✓

$x = -3$:

$\lvert x \rvert$	3
$\lvert -3 \rvert$	
3	✓

EXAMPLE 4: **Solve the equation $|x-2| = 4$ graphically.**

Solution: Let $y_1 = |x-2|$ and $y_2 = 4$. Graphing y_1 and y_2 on the same grid gives the following graphs:



It looks like y_1 and y_2 intersect at the points (–2, 4) and (6, 4). Therefore, the two solutions of the equation $|x-2| = 4$ are

$$x = -2 \text{ or } x = 6$$

Check:

$x = -2$:

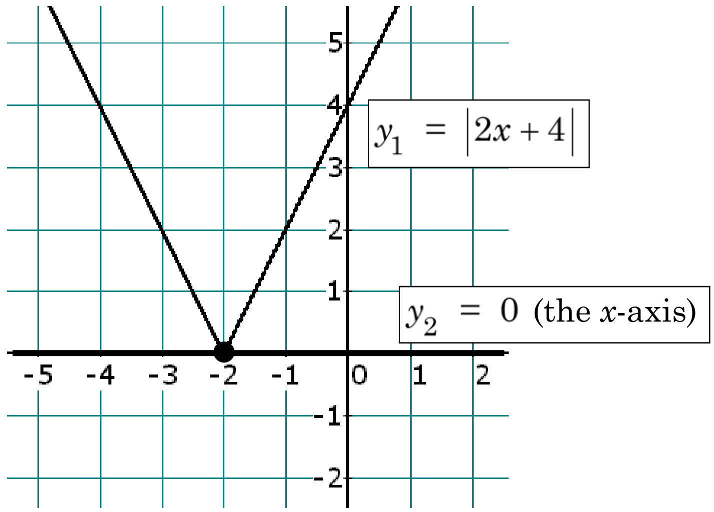
$\lvert x-2\rvert$	4
$\lvert -2-2\rvert$	
$\lvert -4\rvert$	
4	✓

$x = 6$:

$\lvert x-2\rvert$	4
$\lvert 6-2\rvert$	
$\lvert 4\rvert$	
4	✓

<u>EXAMPLE 5:</u> **Solve the equation $|2x+4| = 0$ graphically.**

<u>Solution:</u> Let $y_1 = |2x+4|$ and $y_2 = 0$. Noting that the graph of $y_2 = 0$ is just the x-axis, we graph y_1 and y_2 on the same grid:



The two graphs intersect at one point, (–2, 0). Thus, each formula has the same value (0) when $x = -2$. Thus, the only solution of the given equation is

$$\boxed{x = -2}$$

Check:

$\|2x+4\|$	0
$\|2(-2)+4\|$	
$\|-4+4\|$	
$\|0\|$	
0	✓

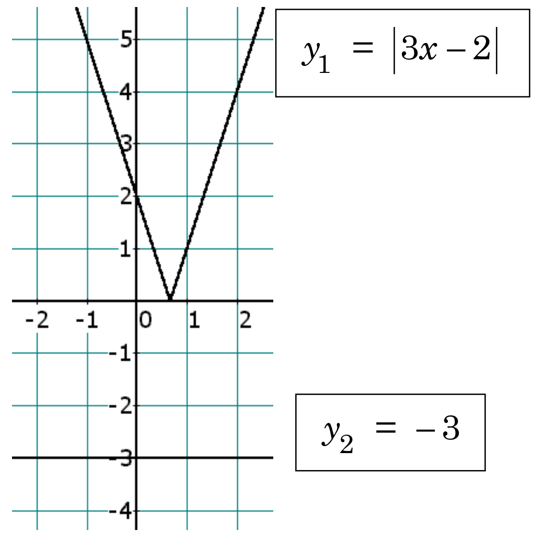
<u>EXAMPLE 6:</u> **Solve the equation $|3x-2| = -3$ graphically.**

<u>Solution:</u> Let's graph $y_1 = 3x - 2$ and $y_2 = -3$:

Look at the two graphs. Notice that y_1 and y_2 have <u>no</u> point of intersection. This means that y_1 can <u>never</u> be equal to y_2.

What's our conclusion? The given equation has

No solution



Homework

5. Solve each absolute-value equation graphically:

a. $|x| = 2$ b. $|x| = -3$ c. $|x+1| = 3$

d. $|x-3| = 0$ e. $|2x-2| = 4$ f. $|2x+4| = -1$

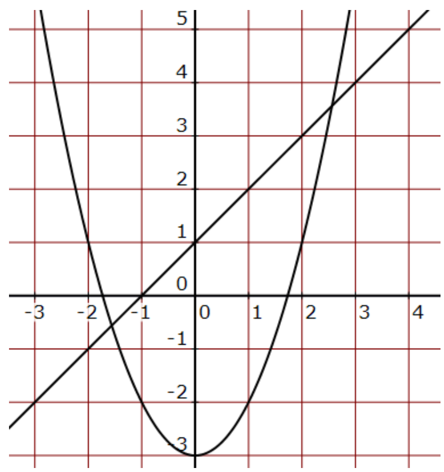
❑ LINES AND CURVES

EXAMPLE 7: **Solve the equation $x^2 - 3 = x + 1$ graphically.**

Solution: Let's graph $y_1 = x^2 - 3$ and $y_2 = x + 1$ on the same grid, and then check out if there are any points of intersection.

The formula $y_2 = x + 1$ is just a straight line, covered in the previous chapter. But the formula $y_1 = x^2 - 3$ might require a little more work. Let's make a table of *x*-*y* values:

x	y_1
–3	6
–2	1
–1	–2
0	–3
1	–2
2	1
3	6



Now we try to determine where the line and the curve intersect. It appears (although there's no guarantee) that the points of intersection are **(–1.5, –0.5)** and **(2.5, 3.5)**. We can therefore estimate the two solutions of the original equation as

$$x = -1.5 \text{ or } x = 2.5$$

Note: The solutions we obtained in this problem are NOT the actual solutions — they are simply the best we can get from the picture. Later in the course, we'll have a couple of ways to find the <u>exact</u> solutions.

Homework

Solve each equation by graphing each side of the equation by hand:

6. $x^2 - 2 = x + 4$ 7. $x^2 - 1 = 2x - 2$ 8. $x^2 + 2 = x - 1$

Use DESMOS to prove the following assertions:

9. The quadratic equation $2x^2 - 8x + 41 = 0$ has NO solution in $\mathbb{R}$.

10. The quadratic equation $25x^2 - 26x + 6.76 = 0$ has ONE solution in $\mathbb{R}$.

11. The quadratic equation $5x^2 + 97x - 168 = 0$ has TWO solutions in $\mathbb{R}$.

12. The equation $e^x = \ln x$ has NO solution in $\mathbb{R}$.

❑ *Inequalities*

An inequality is quite different from an equation. Whereas an equation usually has one or a couple of solutions, inequalities tend to have infinitely many solutions. Here's an example:

Solve for x: $5x - 3 > 12$

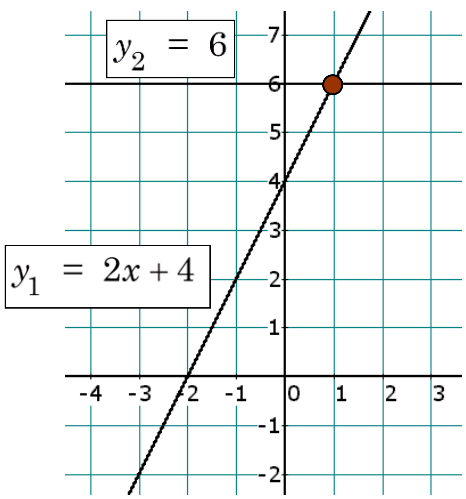
We're seeking any and all values of x which make the statement true. Suppose $x = 6$. Then $5(\mathbf{6}) - 3 = 30 - 3 = 27$, which is > 12. Thus $x = 6$ is a solution of the inequality; but it's "a" solution, not "the" solution. Why? Because there are others; $x = 5$ will work; $x = 3.7$ will work.

But will $x = 3$ work? Let's see: $5(\mathbf{3}) - 3 = 15 - 3 = 12$, which is not greater than 12. So $x = 3$ fails to satisfy the inequality. What about $x = 0$? $5(\mathbf{0}) - 3 = 0 - 3 = -3$, which is certainly not greater than 12.

Let's summarize: 0 and 3 failed, but 3.7 and 5 and 6 worked. Our best guess at this point is that any number greater than 3 will work. Our solution is therefore $\boldsymbol{x > 3}$.

EXAMPLE 8: **Solve the inequality $2x + 4 < 6$ graphically.**

Solution: We employ the same technique we used for all the previous equations in this chapter, except we will take into account the "less than" sign after we construct our graphs. Let $y_1 = 2x + 4$ and $y_2 = 6$, both of which are lines.



First we note that the two lines intersect at the point (1, 6). But remember, we're trying to solve the inequality $2x + 4 < 6$, so here's the question: For what values of x is the quantity $2x + 4$ smaller than 6? Equivalently — and here's the key question — where on the graph is y_1 below y_2? By looking at the graphs of the lines, we see that y_1 is below y_2 to the left of the point of intersection, (1, 6). In other words, whenever x is smaller than 1, it will follow that $2x + 4$ is less than 6. That is,

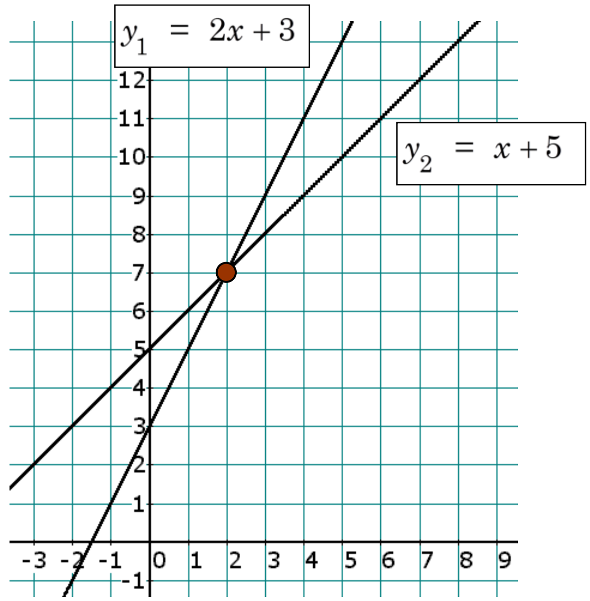
Whenever $x < 1$, we can conclude that $2x + 4 < 6$.

We have found the solution of the inequality $2x + 4 < 6$:

$\boxed{x < 1}$ which can be written $(-\infty, 1)$

<u>EXAMPLE 9:</u> **Solve the inequality $2x + 3 \ge x + 5$ graphically.**

<u>Solution:</u> Graph $y_1 = 2x + 3$ and $y_2 = x + 5$ on the same grid.



We're looking for where y_1 is greater than or equal to y_2. Clearly, y_1 and y_2 are equal to each other at the point of intersection, (2, 7). Also, $y_1 > y_2$ wherever the graph of y_1 is above the graph of y_2. This occurs to the <u>right</u> of the point (2, 7); that is, $y_1 > y_2$ whenever $x > 2$. In short, $y_1 \ge y_2$ whenever $x \ge 2$. Thus, the solution of the inequality $2x + 3 \ge x + 5$ is

$$\boxed{x \ge 2}$$ and in interval notation: $[2, \infty)$

Homework

13. Solve each inequality graphically:

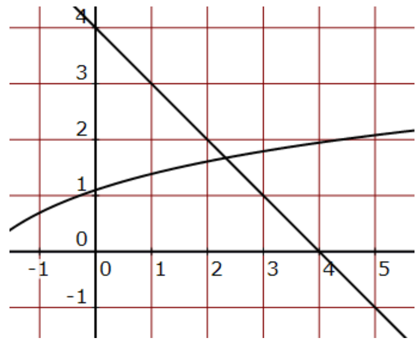
a. $x + 3 < 6$ b. $x - 1 \ge 5$

c. $2x + 1 \le 5$ d. $x + 5 \ge 3x - 1$

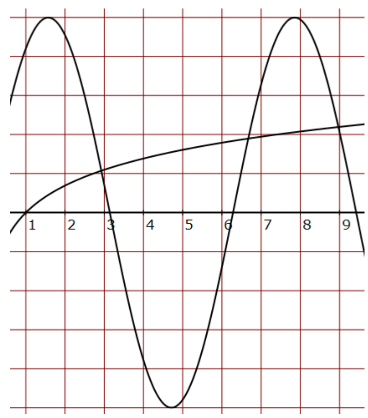
e. $2x - 1 < x + 2$ f. $3x + 1 \le 2x - 3$

Practice Problems

14. Solve the equation $\ln(x+3) = -x+4$ by using the associated graphs:



15. Solve the equation $5\sin x = \ln x$ by using the associated graphs:



16. Solve the equation $|2x-3| = 3$ graphically.

17. Solve the equation $|x+1|+3 = 3$ graphically.

18. Solve the equation $|x-1|+2 = 1$ graphically.

19. Solve the equation $x^2 - 4 = 2x - 3$ graphically.

20. Solve the inequality $2x - 2 \le 4$ graphically.

21. Solve the inequality $2x + 3 > x - 2$ graphically.

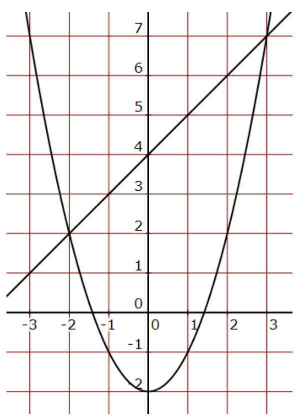
Solutions

Remember that these solutions are approximations; as long as your estimation is close to mine, you have the correct answer.

1. $x = 0.8$ **2.** $x = 2.6, -2.6$ **3.** $x = 1.7, -0.7$ **4.** $x = 2.1, 5.6$

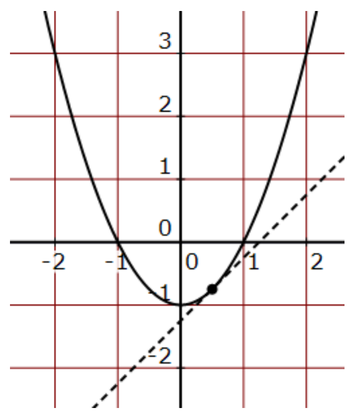
5. a. $x = 2$ or $x = -2$ b. No solution c. $x = 2$ or $x = -4$
d. $x = 3$ e. $x = 3$ or $x = -1$ f. No solution

6.



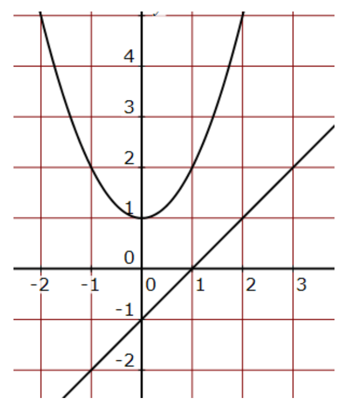
$x = -2, 3$

7.



$x = 0.5$

8.



No Solution

9. **10**. **11**. Graph each side of the equation, noting that $y_2 = 0$ is simply the x-axis. This means that the number of solutions is given by the number of points of intersection between the parabola on the left side of the equation and the x-axis on the right side.

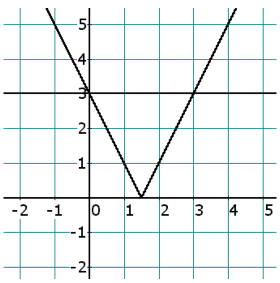
12. The graphs should prove to you that the equation has NO solution.

13. a. $x < 3$ b. $x \geq 6$ c. $x \leq 2$ d. $x \leq 3$
e. $x < 3$ f. $x \leq -4$

14. $x = 2.3$ (or anything reasonably close)

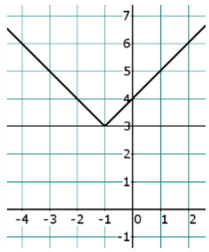
15. There are three points of intersection shown on the graph; the x-values are roughly 2.9, 6.7, and 9.

16.



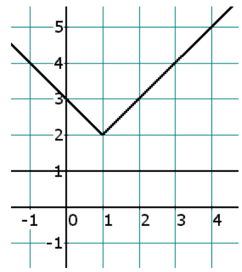
$x = 0, 3$

17.



$x = -1$

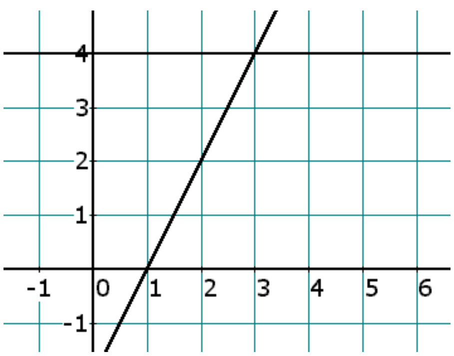
18.



No solution

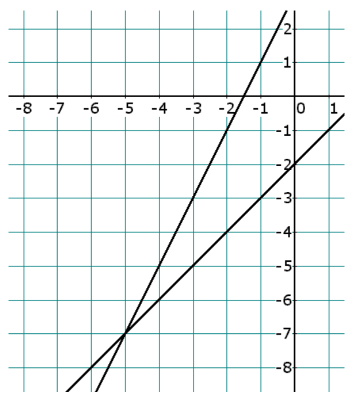
19. $x = -0.4$ or $x = 2.4$ (or anything reasonably close)

20.



$x \leq 3$

21.



$x > -5$

"The first step to getting the things you want out of life is this:

Decide what you want."

- Ben Stein