CH 13 – BREAK-EVEN POINT, QUADRATIC FUNCTIONS

❑ *INTRODUCTION*

Back in Chapter 7, we discussed the notions of revenue (R), cost (C), profit (P), and the break-even point:

$$\text{Profit:} \quad P = R - C$$

$$\text{Break-even:} \quad R = C \quad \text{or} \quad P = 0$$

Now that we're getting proficient at factoring, we can solve some more break-even business problems; these problems will result in a **quadratic equation**, which can be roughly defined as an equation where the variable is squared. An example of a quadratic equation is $x^2 + 5x + 6 = 0$.

❑ *CONFIRMING THE SOLUTIONS OF A QUADRATIC EQUATION*

Let's look at the solutions of the quadratic equation

$$x^2 - 10x + 16 = 0$$

First, let's verify that $\underline{x = 2}$ is a solution of this equation (don't worry about where the 2 came from):

$$x^2 - 10x + 16 = 0$$

$$\mathbf{2}^2 - 10(\mathbf{2}) + 16 \stackrel{?}{=} 0$$

$$4 - 20 + 16 \stackrel{?}{=} 0$$

$$-16 + 16 \stackrel{?}{=} 0$$

$$0 = 0 \quad ✓$$

Fine, we have a solution. Here comes the (possibly) surprising fact: This equation has another solution, namely <u>$x = 8$</u>. Watch this:

$$x^2 - 10x + 16 = 0$$

$$\mathbf{8}^2 - 10(\mathbf{8}) + 16 \stackrel{?}{=} 0$$

$$64 - 80 + 16 \stackrel{?}{=} 0$$

$$-16 + 16 \stackrel{?}{=} 0$$

$$0 = 0 \quad ✓$$

One equation with <u>two</u> solutions? Yep, that's what we have. This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: we call it a ***quadratic equation***.

Homework

1. For each quadratic equation, verify that the two given solutions are really solutions:

   a. $x^2 + 3x - 10 = 0$ $\qquad x = -5;\; x = 2$

   b. $n^2 - 25 = 0$ $\qquad n = 5;\; n = -5$

   c. $a^2 + 7a = -12$ $\qquad a = -3;\; a = -4$

   d. $w^2 = 7w + 18$ $\qquad w = 9;\; w = -2$

   e. $2y^2 + 8y = 0$ $\qquad y = -4;\; y = 0$

❑ SOLVING QUADRATIC EQUATIONS

We know how to check that a number is indeed a solution to a quadratic equation, and we've learned that a quadratic equation can have two different solutions. It's time to begin the discussion of *solving* such equations, and we'll begin with a quadratic equation given to us in factored form.

Consider the quadratic equation

$$(x+3)(x-7) = 0$$

Can you see why this is a quadratic equation?

There are two solutions to this equation — what are they? Before we present the formal process, here's what you should note: What would happen if we let $x = -3$ in the equation $(x+3)(x-7) = 0$? We'd get

$$(\mathbf{-3}+3)(\mathbf{-3}-7) = (0)(-10) = 0 \quad ✓$$

Look at that! We've stumbled upon a solution of the equation $(x+3)(x-7) = 0$. Let's "stumble" one more time and choose $x = 7$:

$$(\mathbf{7}+3)(\mathbf{7}-7) = (10)(0) = 0 \quad ✓$$

Can you see how we stumbled across these two solutions, –3 and 7? Each solution was chosen so that one of the two factors would turn into zero. That way, the product of that zero factor with the other factor (no matter what it is) would <u>have</u> to be zero.

> If $a \times b = 0$, then $a = 0$ or $b = 0$.

Now let's solve the quadratic equation $(x-9)(x+17) = 0$ without stumbling upon the solutions. Here's our reasoning: Since we have two factors whose product is 0, we know that either of the two factors could be 0. Setting each factor to 0 gives two possibilities:

$$x - 9 = 0 \Rightarrow \boldsymbol{x = 9}$$

$$x + 17 = 0 \Rightarrow \boldsymbol{x = -17}$$, and we have our two solutions.

A quadratic equation is said to be in **standard form** when the order of the terms in the equation is the squared term first, followed by the linear term, followed by the constant, followed by the equals sign, followed by a zero. Our next example, $3x^2 - 7x - 40 = 0$, is already in standard form. If a quadratic equation is not given to us in standard form, a little algebra can always convert it to standard form.

<u>EXAMPLE 1:</u> **Solve for x: $3x^2 - 7x - 40 = 0$**

<u>Solution:</u> This equation is in standard quadratic form, so it's all set to factor:

	$3x^2 - 7x - 40 = 0$	(the original equation)
$\Rightarrow$	$(3x+8)(x-5) = 0$	(factor the left side)
$\Rightarrow$	$3x + 8 = 0$ or $x - 5 = 0$	(set each factor to 0)
$\Rightarrow$	$x = \frac{-8}{3} = -\frac{8}{3}$ or $x = 5$	(solve each equation)

Thus, the final solutions to the quadratic equation are

$$\boxed{5, -\frac{8}{3}}$$

<u>EXAMPLE 2:</u> **Solve for y: $9y^2 - 16 = 0$**

<u>Solution:</u> It's a good-looking quadratic (even though the middle term is missing), so let's factor and set the factors to zero:

	$9y^2 - 16 = 0$	(the original equation)
$\Rightarrow$	$(3y+4)(3y-4) = 0$	(factor the left side)
$\Rightarrow$	$3y + 4 = 0$ or $3y - 4 = 0$	(set each factor to 0)
$\Rightarrow$	$y = -\frac{4}{3}$ or $y = \frac{4}{3}$	(solve each equation)

Therefore, the solutions of the equation are

$\boxed{\frac{4}{3}, -\frac{4}{3}}$ which can also be written $\pm\frac{4}{3}$.

EXAMPLE 3: **Solve for u: $9u^2 = 42u - 49$**

Solution: This quadratic equation is not in standard form, so the first steps will be to transform it into standard form:

	$9u^2 = 42u - 49$	(the original equation)
$\Rightarrow$	$9u^2 - 42u = -49$	(subtract $42u$)
$\Rightarrow$	$9u^2 - 42u + 49 = 0$	(add 49)
$\Rightarrow$	$(3u - 7)(3u - 7) = 0$	(factor)
$\Rightarrow$	$3u - 7 = 0$ or $3u - 7 = 0$	(set each factor to 0)
$\Rightarrow$	$u = \frac{7}{3}$ or $u = \frac{7}{3}$	(solve each equation)

We obtained two solutions, but they're the same, so we really have just one solution:

$$\boxed{\frac{7}{3}}$$

Homework

2. Solve each quadratic equation:

a. $x^2 + 5x - 14 = 0$

b. $25y^2 = 4$

c. $z^2 + 1 = -2z$

d. $4a^2 = 3 - 4a$

e. $6u^2 = 47u + 8$

f. $0 = 4t^2 + 8t + 3$

g. $-n^2 + n + 56 = 0$ [Hint: multiply each side by -1]

h. $-2x^2 - 7x + 15 = 0$

i. $144n^2 - 49 = 0$

j. $81q^2 = 126q - 49$

k. $-30a^2 + 13a + 3 = 0$

❑ BREAK-EVEN

EXAMPLE 4: **Find the break-even point(s) if the profit formula is given by $P = 2w^2 - 31w + 84$.**

Solution: We find the break-even points by setting the profit formula to zero:

$$2w^2 - 31w + 84 = 0 \qquad \text{(set profit to 0)}$$

$$\Rightarrow \quad (2w - 7)(w - 12) = 0 \qquad \text{(factor)}$$

$$\Rightarrow \quad 2w - 7 = 0 \quad \text{or} \quad w - 12 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad 2w = 7 \quad \text{or} \quad w = 12 \qquad \text{(solve each equation)}$$

$$\Rightarrow \quad w = 3\tfrac{1}{2} \quad \text{or} \quad w = 12$$

Thus, the break-even points are

$$\boxed{3\tfrac{1}{2} \text{ widgets and 12 widgets}}$$

Of course, $3\frac{1}{2}$ widgets can't really exist, but it's good enough for this problem. Indeed, if w stood for wages, for instance, then $3\frac{1}{2}$ would make sense, since that number represents \$3.50, a perfectly legit answer.

EXAMPLE 5: **Find the break-even point(s) if revenue and cost are given by the formulas**

$$\boldsymbol{R = 3w^2 - 3w - 8}$$
$$\boldsymbol{C = 2w^2 + 30w - 268}$$

Solution: Recall that one of the two ways to describe the **break-even points** is by equating revenue and cost. Notice that we put the resulting equation in standard form by bringing all the terms on the right side to the left side so that the right side will be zero.

	$R = C$	(to find break-even)
$\Rightarrow$	$3w^2 - 3w - 8 = 2w^2 + 30w - 268$	(use the given formulas)
$\Rightarrow$	$w^2 - 3w - 8 = 30w - 268$	(subtract $2w^2$)
$\Rightarrow$	$w^2 - 33w - 8 = -268$	(subtract $30w$)
$\Rightarrow$	$w^2 - 33w + 260 = 0$	(add 268 $\Rightarrow$ standard form)
$\Rightarrow$	$(w - 20)(w - 13) = 0$	(factor)
$\Rightarrow$	$w - 20 = 0$ or $w - 13 = 0$	(set each factor to 0)
$\Rightarrow$	$w = 20$ or $w = 13$	(solve each equation)

And so the two break-even points are

20 widgets and 13 widgets

Homework

3. Find the **break-even points** for the given profit formula:

a. $P = w^2 - 12w + 35$ b. $P = 2w^2 - 13w + 15$

c. $P = w^2 - 25w + 150$ d. $P = 6w^2 - 31w + 40$

4. Find the **break-even points** given the revenue and expense formulas:

   a. $R = 2w^2 + 8w - 20$ $\qquad C = w^2 + 24w - 75$

   b. $R = 2w^2 - 3w + 1$ $\qquad C = -w^2 + 16w - 29$

   c. $R = 5w^2 - 26w + 80$ $\qquad C = 3w^2 + w + 10$

❑ *COMPLETE FACTORING AND MORE QUADRATICS*

Just as factoring 12 as 3 × 4 isn't complete, (the complete factorization is 12 = 2 × 2 × 3), factoring an algebraic expression may require more than one step.

EXAMPLE 6: **Factor completely: $10x^2 + 50x + 60$**

Solution: Look at the 10. Its factor pairs are 1 and 10, or 2 and 5. Now take a gander at the 60. It's downright scary to consider all the pairs of factors of that number. But watch what happens if we deal with the greatest common factor first, and then worry about the rest later.

The variable x is not common to all three terms, so we'll ignore it. But each of the three terms does contain a factor of 10. Thus,

$$\begin{aligned} & 10x^2 + 50x + 60 && \text{(the given expression)} \\ = \ & 10(x^2 + 5x + 6) && \text{(pull out the GCF of 10)} \\ = \ & 10(x + 3)(x + 2) && \text{(factor the quadratic)} \end{aligned}$$

Not so difficult, after all. Therefore, the complete factorization of $10x^2 + 50x + 60$ is

$$\boxed{10(x + 3)(x + 2)}$$

**** The key to <u>complete</u> factoring is to FIRST pull out the GCF! ****

Homework

5. Factor each expression <u>completely</u>:

a. $4a^2 + 8b^2$ b. $6x^2 - 9x$ c. $15y^2 - 5y$

d. $30z^2 + 20z$ e. $7x - 10y$ f. $9x^2 + 10x$

6. Factor each expression <u>completely</u>:

a. $7x^2 - 35x + 42$ b. $10n^2 - 10$

c. $5a^2 - 30a + 45$ d. $50u^2 - 25u - 25$

e. $7w^2 - 700$ f. $9n^2 + 9$

g. $5y^2 - 125$ h. $3x^2 + 15x + 12$

i. $14x^2 - 7x - 7$ j. $13t^2 + 117$

k. $48z^2 - 28z + 4$ l. $24a^2 - 120a + 150$

Additional Quadratic Equations

Now we'll combine the GCF method of factoring with the methods of this chapter to solve more quadratic equations. The following example should convince you that factoring out a simple number first makes the rest of the factoring, and thus the solving of the equation, vastly easier.

<u>EXAMPLE 7</u>: **Solve for k: $16k^2 = 40k + 24$**

<u>Solution</u>: Solving a quadratic equation requires that we make one side of the equation zero. To this end, we will first bring the

$40k$ and the 24 to the left side, factor in two steps, divide each side by the greatest common factor, set each factor to 0, and then solve each resulting equation.

	$16k^2 = 40k + 24$	(the original equation)
$\Rightarrow$	$16k^2 - 40k - 24 = 0$	(subtract $40k$ and 24)
$\Rightarrow$	$8(2k^2 - 5k - 3) = 0$	(factor out 8, the GCF)
$\Rightarrow$	$\dfrac{\cancel{8}(2k^2 - 5k - 3)}{\cancel{8}} = \dfrac{0}{8}$	(divide by 8, the GCF)
$\Rightarrow$	$2k^2 - 5k - 3 = 0$	(simplify)
$\Rightarrow$	$(2k + 1)(k - 3) = 0$	(factor)
$\Rightarrow$	$2k + 1 = 0 \quad \text{or} \quad k - 3 = 0$	(set each factor to 0)
$\Rightarrow$	$\boxed{k = -\dfrac{1}{2} \quad \text{or} \quad k = 3}$	(solve each equation)

Warning!!

Do you see the step in the preceding example where we divided both sides of the equation by 8? This was legal because we did the same thing to both sides of the equation, and we did not divide by zero. Do not ever fall into the trap of dividing each side of an equation by an expression with the variable in it; that expression might be equal to zero. The upshot is that you may lose a solution to the equation.

For example, the correct way to solve the quadratic equation $x^2 + x = 0$ is as follows:

$$x^2 + x = 0$$

$$\Rightarrow \quad x(x + 1) = 0$$

$$\Rightarrow \quad x = 0 \ \text{ or } \ x + 1 = 0$$

$$\Rightarrow \quad x = 0 \ \text{ or } \ x = -1$$

That is, we have <u>two</u> solutions: **0** and **–1**.

<u>Check:</u> $\mathbf{0}^2 + \mathbf{0} = 0 + 0 = 0$ ✓

$(\mathbf{-1})^2 + (\mathbf{-1}) = 1 + (-1) = 1 - 1 = 0$ ✓

Now let's do it the <u>wrong</u> way:

$$x^2 + x = 0 \Rightarrow x(x+1) = 0 \Rightarrow \frac{\cancel{x}(x+1)}{\cancel{x}} = \frac{0}{x}$$

$$\Rightarrow x + 1 = 0 \Rightarrow x = -1,$$

which is merely one of the two solutions. That is, we lost a solution when we divided by the variable. Since the purpose of algebra is to obtain solutions — not throw them away — we see that dividing by the variable was a really bad idea.

Homework

7. Solve each quadratic equation:

a. $7x^2 - 35x + 42 = 0$

b. $10n^2 = 10$

c. $5a^2 + 45 = 30a$

d. $50u^2 = 25u + 25$

e. $7w^2 - 700 = 0$

f. $180z^2 - 30z - 60 = 0$

g. $4x^2 + 4x - 24 = 0$

h. $16x^2 = 6 - 4x$

i. $10x^2 - 490 = 0$

j. $75w^2 + 48 = 120w$

8. In a certain right triangle the longer leg is 6 more than the shorter leg, while the hypotenuse is 6 more than the longer leg. Find all three sides.

9. The three sides of a right triangle form three consecutive EVEN numbers. Find the lengths of the three sides.

10. The hypotenuse of a right triangle is 2 more than the longer leg, while the longer leg is itself 14 more than the shorter leg. Find the length of the hypotenuse.

11. One leg of a right triangle is 2 less than the other leg. The hypotenuse is 2 less than 2 times the shorter leg. What is the length of the hypotenuse?

12. In a certain right triangle the longer leg is 3 more than the shorter leg, while the hypotenuse is 3 more than the longer leg. Find all three sides.

13. One leg of a right triangle is 17 more than the other leg. The hypotenuse is 4 more than 3 times the shorter leg. Find the length of the shorter leg.

Practice Problems

14. Solve for x: $40x^2 + x - 6 = 0$

15. Solve for y: $100y^2 = 49$

16. Solve for n: $25n^2 + 9 = -30n$

17. If the profit formula is given by $P = w^2 - 57w + 350$, find the two break-even points.

18. If the revenue and cost are given by the formulas $R = 5w^2 - 30w + 100$ and $C = 4w^2 + 4w - 180$, find the break-even points.

19. Solve by factoring: $30q^2 + 68q = -30$

20. Solve by factoring: $15x^2 = 95x + 70$

21. Solve by factoring: $16x^2 + 80x + 100 = 0$

22. The hypotenuse of a right triangle is 8 more than the longer leg, while the longer leg is itself 1 more than the shorter leg. Find the length of the hypotenuse.

23. One leg of a right triangle is 4 more than the other leg. The hypotenuse is 4 less than 2 times the shorter leg. Find the length of the shorter leg.

24. One leg of a right triangle is 7 more than the other leg. The hypotenuse is 7 less than 4 times the shorter leg. What is the length of the hypotenuse?

Solutions

1. For each quadratic equation, substitute each "solution" (separately) into the original equation. Then work the arithmetic on each side of the equation separately. [Do NOT swap things back and forth across the equals sign.] In all cases, the two sides should balance at the end of the calculations.

2. a. $x = -7, 2$ b. $y = \pm\frac{2}{5}$ c. $z = -1$

d. $x = \frac{1}{2}, -\frac{3}{2}$ e. $u = -\frac{1}{6}, 8$ f. $t = -\frac{1}{2}, -\frac{3}{2}$

g. $n = 8, -7$ h. $x = -5, \frac{3}{2}$ i. $n = \pm\frac{7}{12}$

j. $q = \frac{7}{9}$ k. $a = \frac{3}{5}, -\frac{1}{6}$

3. a. $w = 5, 7$ b. $w = \frac{3}{2}, 5$ c. $w = 10, 15$ d. $w = \frac{8}{3}, \frac{5}{2}$

4. a. $w = 5, 11$ b. $w = \frac{10}{3}, 3$ c. $w = \frac{7}{2}, 10$

5. a. $4(a^2 + 2b^2)$ b. $3x(2x - 3)$ c. $5y(3y - 1)$
d. $10z(3z + 2)$ e. Not factorable f. $x(9x + 10)$

6. a. $7(x - 3)(x - 2)$ b. $10(n + 1)(n - 1)$ c. $5(a - 3)^2$
d. $25(2u + 1)(u - 1)$ e. $7(w + 10)(w - 10)$ f. $9(n^2 + 1)$
g. $5(y + 5)(y - 5)$ h. $3(x + 1)(x + 4)$ i. $7(2x + 1)(x - 1)$
j. $13(t^2 + 9)$ k. $4(4z - 1)(3z - 1)$ l. $6(2a - 5)^2$

7. a. 2, 3 b. ±1 c. 3 d. $1, -\frac{1}{2}$
e. ±10 f. $\frac{2}{3}, -\frac{1}{2}$ g. 2, –3 h. $\frac{1}{2}, -\frac{3}{4}$
i. ±7 j. $\frac{4}{5}$

8. Outline: Let x = the shorter leg
Then $x + 6$ = the longer leg
And $x + 12$ = the hypotenuse
From the Pythagorean Theorem, $x^2 + (x + 6)^2 = (x + 12)^2$
Final answer: 18, 24, and 30

9. The three sides of the right triangle can be written a, $a + 2$, and $a + 4$.
Final answer: 6, 8, and 10

10. 26 **11.** 10 **12.** 9, 12, and 15 **13.** 7

14. $x = \frac{3}{8}, -\frac{2}{5}$ **15.** $y = \pm\frac{7}{10}$ **16.** $n = -\frac{3}{5}$

17. $w = 7$ and $w = 50$ **18.** $w = 14$ and $w = 20$

19. $-\frac{3}{5}, -\frac{5}{3}$ **20.** $7, -\frac{2}{3}$ **21.** $x = -\frac{5}{2}$

22. 29 **23.** 12 **24.** 13

"There is one purpose to life and one only: to bear witness to and understand as much as possible of the complexity of the world – its beauty, its mysteries, its riddles. The more you understand, the more you look – the greater is your enjoyment of life and your sense of peace. That's all there is to it. If an activity is not grounded in '***to love***' or '***to learn,***' it does not have value."

– Anne Rice, American Author