CH 19 – FACTORING, PART II

❑ INTRODUCTION

We can now factor lots of quadratic binomials and trinomials. Sorry to have to tell you this, but we're not done with factoring just yet. In this chapter, we learn how to factor expressions with an exponent of 4 in them, expressions containing four terms, expressions containing GCFs you might never have seen before, and expressions that are the sum or difference of cubes.

❑ FACTORING QUARTICS

EXAMPLE 1: **Factor each quartic (4th degree) polynomial:**

A. $\boldsymbol{c^4 - 256}$

= $(c^2 + 16)(c^2 - 16)$ (difference of squares)

= $(c^2 + 16)(c + 4)(c - 4)$ (difference of squares again)

Note: $c^2 + 16$ can<u>not</u> be factored any further.

B. $\boldsymbol{9a^4 - 37a^2 + 4}$

= $(9a^2 - 1)(a^2 - 4)$ (factor trinomial)

Now we notice that each factor is quadratic and is the difference of two squares. Therefore, each factor can be factored further to get a final answer of

$(3a + 1)(3a - 1)(a + 2)(a - 2)$

Homework

1. Factor each quartic polynomial:

a. $x^4 - 1$

b. $x^4 - x^2 - 6$

c. $n^4 - 10n^2 + 9$

d. $a^4 - 81$

e. $36w^4 - 25w^2 + 4$

f. $9x^4 - 34x^2 + 25$

g. $c^4 - 16$

h. $x^4 - 8x^2 - 9$

i. $x^4 - 3x^2 - 10$

j. $g^4 - 256$

k. $36u^4 - 85u^2 + 9$

l. $y^4 + 81$

❑ *THE GCF REVISITED*

EXAMPLE 2: **Factor: $(a + b)^2 + 4(a + b)$**

Solution: There are two terms in this expression: $(a + b)^2$ and $4(a + b)$. Notice that each of these two terms contains the same factor, namely $a + b$. In other words, the GCF of the two terms is $a + b$. Factoring out this GCF gives us the final factored form, a single term:

$$(a + b)(a + b + 4)$$

The thing not to do in this kind of problem is to distribute the original expression; if you do, you'll be going in the wrong direction. Check it out:

$$(a + b)^2 + 4(a + b) = a^2 + 2ab + b^2 + 4a + 4b$$

Do you really want to factor that last expression?

So, when you see an expression, like $a + b$ in this problem, occurring multiple times in an expression, it's usually best to leave it intact. Also notice that we have converted a 2-termed expression into 1 term – we have factored.

Alternate Method: Let's try a substitution method. We might better see the essence of the problem if we replace $a + b$ with a simpler symbol — for example, x will represent $a + b$. Then the original expression

$$(a + b)^2 + 4(a + b)$$

is transformed into

$$x^2 + 4x$$

The GCF in this form is clearly x, so we pull it out in front:

$$x(x + 4)$$

Now substitute in the reverse direction, to get $a + b$ back in the problem:

$$(a + b)(a + b + 4)$$ (the same answer as before)

EXAMPLE 3: **Factor:** $\boldsymbol{x^2(u - w) - 100(u - w)}$

Solution: The two given terms have a GCF of $u - w$. Factoring this GCF out gives

$$(u - w)(x^2 - 100)$$

But we're not done yet. The second factor is a difference of squares. Factoring that part gives us our final factorization:

$$\boxed{(u - w)(x + 10)(x - 10)}$$

<u>EXAMPLE 4:</u> **Factor:** $w^2(x + z) - 4w(x + z) + 3(x + z)$

<u>Solution:</u> Let's use substitution to make this expression appear a little less intimidating; we'll convert every occurrence of $x + z$ to the symbol A:

$$w^2A - 4wA + 3A$$

Pulling out the GCF of A, we get

$$A(w^2 - 4w + 3)$$

Factor the trinomial in the usual way:

$$A(w - 3)(w - 1)$$

Last, replace the A with its original definition of $x + z$:

$$\boxed{(x + z)(w - 3)(w - 1)}$$

Homework

2. Factor each expression:

a. $(x + y)^2 + 7(x + y)$

b. $(a - b)^2 - c(a - b)$

c. $x^2(c + d) + 5(c + d)$

d. $n^2(a - b) - 9(a - b)$

e. $x^2(a + 4) + 5x(a + 4) + 6(a + 4)$

f. $y^2(m + n) + 7y(m + n)$

g. $2x^2(a + b) + 3x(a + b) - 5(a + b)$

h. $4x^2(w + z) - 9(w + z)$

i. $(u - w)^2 - 9(u - w)$

j. $n^2(a + b) - 9n(a + b)$

k. $(t + r)y^2 - 100(t + r)$

l. $3ax^2 - 20ax - 7a$

❑ *GROUPING WITH FOUR TERMS*

EXAMPLE 5: **Factor: $a^2 + ac + ab + bc$**

Solution: Group the first two terms and the last two terms:

$$(a^2 + ac) + (ab + bc)$$

Now factor each pair of grouped terms separately (using the GCF) :

$$a(a + c) + b(a + c)$$

Even though we've grouped and factored, we can't be done because there are still two terms, and we need one term in the final answer to a factoring question. So we continue — using our knowledge of the previous section — and factor out the GCF, which is $a + c$:

$$\boxed{(a + c)(a + b)}$$

By the commutative property of multiplication, the final answer could also be written $(a + b)(a + c)$. Also, to check our answer, just double distribute the answer and obtain the original expression.

EXAMPLE 6: **Factor: $x^3 - 7x^2 - 9x + 63$**

Solution: Group the first two terms and the last two terms:

$$(x^3 - 7x^2) + (-9x + 63)$$

Now factor the GCF in each pair of grouped terms. The first GCF is obvious: x^2. Choosing the GCF in the second grouping is a little trickier — should we choose 9 or –9? Ultimately, it's a trial-and-error process. Watch what happens if we choose –9 for the GCF:

$x^2(x-7) - 9(x-7)$ (check the signs carefully)

We now see two terms whose GCF is $x - 7$:

$(x-7)(x^2-9)$

All this, and we're still not done. The second factor is the difference of two squares — now we're done:

$$\boxed{(x-7)(x+3)(x-3)}$$

EXAMPLE 7: **Factor: $ab + cd + ad + bc$**

Solution: Group the first two terms and the last two terms (after all, this technique worked quite well in the previous two examples):

$(ab+cd)+(ad+bc)$

We're stuck; there's no way to factor either pair of terms (the GCF = 1 in each case), so let's swap the two middle terms of the original problem and again group in pairs:

$(ab+ad)+(cd+bc)$

Pull out the GCF from each set of parentheses:

$a(b+d)+c(d+b)$

Do we have a common factor in these two terms? Well, does $b + d = d + b$? Since addition is commutative, of course they are equal. So the GCF is $b + d$, and when we pull it out in front, we're done:

$$\boxed{(b+d)(a+c)}$$

EXAMPLE 8: **Factor: $2ax - bx - 2ay + by$**

Solution: Group in pairs, as usual:

$(2ax - bx) + (-2ay + by)$

Pull out the GCF in each grouping:

$x(2a - b) + y(-2a + b)$

Problem: There's no common factor; however, the factors $2a - b$ and $-2a + b$ are opposites of each other, and that gives us a clue. Let's go back to our first step and factor out $-y$ rather than y:

$x(2a - b) - y(2a - b)$ (distribute to make sure we're right)

Now we see a good GCF, so we pull it out in front, and we're done:

$$\boxed{(2a - b)(x - y)}$$

Homework

3. Factor each expression:

a. $xw + xz + wy + yz$

b. $a^2 + ac + ab + bc$

c. $x^3 - 4x^2 + 3x - 12$

d. $n^3 - n^2 - 5n + 5$

e. $x^3 + x^2 - 9x - 9$

f. $ac - bd + bc - ad$

g. $xw + yz - xz - wy$

h. $2ac - 2ad + bc - bd$

i. $6xw - yz + 3xz - 2wy$

j. $hj - j^2 - hk + jk$

k. $ax + ay - bx - by$

l. $x^3 - 2x^2 - 25x + 50$

m. $xw + 2wy - xz - 2yz$

n. $a^3 - a^2 - 5a + 5$

o. $4tw - 2tx + 2w^2 - wx$

p. $6x^3 + 2x^2 - 9x - 3$

q. Not factorable

r. $6a^3 - 15a^2 + 10a - 25$

❑ MORE GROUPING AND SUBSTITUTION PROBLEMS

EXAMPLE 9: **Factor:** $(w + z)^2 - a^2$

Solution: After some practice, you might not need a substitution for this kind of problem, but we'll use one for this problem. Let $n = w + z$. The starting problem then becomes

$$n^2 - a^2$$

This is just a standard difference of squares:

$$(n + a)(n - a)$$

Now substitute in the other direction:

$$\boxed{(w + z + a)(w + z - a)}$$

EXAMPLE 10: **Factor:** $x^2 + 6x + 9 - y^2$

Solution: Grouping in pairs has worked quite well so far, so let's try it again:

$$(x^2 + 6x) + (9 - y^2)$$

We see that the first pair of terms has a nice GCF of x, and the second is the difference of squares:

$$x(x + 6) + (3 + y)(3 - y)$$

Good try, but there's no common factor in these two terms. In fact, no grouping into pairs will result in a common factor — a dead end. Let's go back to the original problem and regroup so that the first three terms are together:

$$(x^2 + 6x + 9) - y^2$$

The first set of three terms is a perfect square trinomial, and factors into the square of a binomial:

$$(x + 3)^2 - y^2$$

leaving us with another difference of squares (just like the previous example), which factors to

$$\boxed{(x + 3 + y)(x + 3 - y)}$$

Homework

4. Factor each expression:

a. $(x + y)^2 - z^2$

b. $(a - b)^2 - c^2$

c. $x^2 + 4x + 4 - y^2$

d. $n^2 - 6n + 9 - Q^2$

e. $(u + w)^2 - T^2$

f. $y^2 + 10y + 25 - x^2$

g. $a^2 + 2ab + b^2 - c^2$

h. $w^2 - 2wy + y^2 - 49$

i. $4x^2 + 4x + 1 - t^2$

j. $9x^2 - 12x + 4 - y^2$

❑ *FACTORING CUBICS USING THE GCF*

EXAMPLE 11: **Factor each cubic (3rd degree) polynomial:**

A. $\mathbf{5q^3 + 10q^2 + 5q}$

This is not as bad as it looks, if we remember to start with the GCF:

$5q^3 + 10q^2 + 5q$ (the polynomial to factor)

$= 5q(q^2 + 2q + 1)$ (factor out $5q$, the GCF)

$= \quad 5q(q + 1)(q + 1)$ (factor the trinomial)

$= \quad \boxed{5q(q + 1)^2}$ (write it more simply)

B. $\mathbf{4x^3 - x}$

$= \quad x(4x^2 - 1)$ (factor out x, the GCF)

$= \quad \boxed{x(2x + 1)(2x - 1)}$ (difference of squares)

Homework

5. Factor each cubic polynomial:

a. $x^3 - x$

b. $2n^3 + 6n^2 + 4n$

c. $10a^3 - 5a^2 - 5a$

d. $7y^3 + 70y^2 + 175y$

e. $36w^3 - 9w$

f. $24z^3 - 20z^2 - 24z$

❑ *FACTORING THE SUM AND DIFFERENCE OF CUBES*

We've learned that we can factor the *difference of squares* $x^2 - y^2$ into $(x + y)(x - y)$. We've also determined that the *sum of squares* $x^2 + y^2$ cannot be factored. Now we're about to show that the ***difference of cubes*** $x^3 - y^3$ can also be factored — and perhaps surprisingly — even the ***sum of cubes*** $x^3 + y^3$ can be factored. We begin with a discussion of division, remainders, and factors.

Is 3 a factor of 161? No — divide 161 by 3 and you'll get 53 remainder 2. Since the remainder is not zero, 3 is not a factor of 161. In other words, 3 does not go into 161 "evenly."

Is 7 a factor of 161? Yes — divide 161 by 7 and you'll get 23, remainder 0. Thus, 7 divides into 161 exactly 23 times. And therefore, $161 = 7 \times 23$. We have factored 161 into 7×23 by showing that the factor 7 divides into 161 without remainder. These observations are the key to factoring the sum and difference of cubes.

Perfect Cubes

We know that $2^3 = 8$. Since the cube of 2 is 8, we say that 8 is a ***perfect cube***. Here are some more examples of perfect cubes:

125 is a perfect cube because it's the cube of 5.

1 is a perfect cube because it's the cube of 1.

x^3 is a perfect cube because it's the cube of x.

$27y^3$ is a perfect cube because it's the cube of $3y$.

$8n^6$ is a perfect cube because it's the cube of $2n^2$.

$64z^{12}$ is a perfect cube because it's the cube of $4z^4$.

Homework

6. a. $64m^3$ is a perfect cube because it's the cube of _____.

b. $216n^3$ is a perfect cube because it's the cube of _____.

c. $27A^6$ is a perfect cube because it's the cube of _____.

d. _____ is a perfect cube because it's the cube of $7z^2$.

e. _____ is a perfect cube because it's the cube of $-3a^3$.

Factoring a *Sum* of Cubes

We're now ready to try to factor a sum of cubes; for example, what is the factorization of $x^3 + 8$? To answer this question, we should try to divide $x^3 + 8$ by something that goes into it evenly; that is, divide $x^3 + 8$ by something that will leave a remainder of zero. But what should we divide by? Since both terms of $x^3 + 8$ are perfect cubes, let's divide it by the binomial $x + 2$, since these two terms are the cube roots of x^3 and 8. Maybe this will work and maybe it won't, but we've got to try something.

$$\begin{array}{r|l} & \;x^2 - 2x + 4 \\ \hline x+2 & x^3 + 0x^2 + 0x + 8 \\ & \underline{x^3 + 2x^2} \\ & \quad -2x^2 + 0x \\ & \quad \underline{-2x^2 - 4x} \\ & \qquad\quad 4x + 8 \\ & \qquad\quad \underline{4x + 8} \\ & \qquad\qquad 0 \end{array}$$

Here's the division of $x^3 + 8$ by $x + 2$. Note that the dividend has two zeros placed in it to account for the missing terms.

Also note that the remainder is 0. This means that $x + 2$ is a factor of $x^3 + 8$ and therefore, that $x^2 - 2x + 4$ is the other factor.

Now we write the results of our long division in the form of a multiplication problem, giving us the factorization of $x^3 + 8$:

$$\boxed{x^3 + 8 = (x + 2)(x^2 - 2x + 4)}$$

Factoring a *Difference* of Cubes

For our difference of cubes, let's try to factor $n^3 - 27$. What do you think one of the factors will be? Consider the binomial consisting of the individual cube roots of n^3 and -27, namely $n - 3$. This time it's your turn to carry out the long division. Here's what you should end up with:

$$n - 3 \overline{\smash{)}\, n^3 + 0n^2 + 0n - 27}^{\;n^2 + 3n + 9}$$

We now have our factorization:

$$\boxed{n^3 - 27 = (n-3)(n^2 + 3n + 9)}$$

EXAMPLE 12:

A. Factor: $N^3 - 1$. Divide $N^3 - 1$ by $N - 1$ and you should get the factorization $\mathbf{N^3 - 1 = (N - 1)(N^2 + N + 1)}$.

B. Factor: $8p^3 + 27$. Divide $8p^3 + 27$ by $2p + 3$. It should divide evenly, thus giving $\mathbf{8p^3 + 27 = (2p + 3)(4p^2 - 6p + 9)}$.

C. Factor: $(a + b)^3 - 125$. This is tricky, and it will be much easier to perform the long division if we make a substitution first. If we let $x = a + b$, then the expression to factor becomes $x^3 - 125$. The quantity to divide this by would be $x - 5$. When the long division is finished, the quotient is $x^2 + 5x + 25$ with remainder 0. We therefore get the factorization

$$x^3 - 125 = (x - 5)(x^2 + 5x + 25)$$

But the original problem didn't have any x's in it. So we need to substitute *back the other way* — converting each x back into $a + b$, we get the factorization

$$(a + b)^3 - 125 = \big((a + b) - 5\big)\big((a + b)^2 + 5(a + b) + 25\big),$$

which can be simplified to the final answer of

$$\mathbf{(a + b)^3 - 125 = (a + b - 5)(a^2 + 2ab + b^2 + 5a + 5b + 25)}.$$

Homework

7. In the discussion above, we arrived at the following factorizations:

   a. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

   b. $n^3 - 27 = (n - 3)(n^2 + 3n + 9)$

   Verify each result by simplifying the right side of the statement so that it becomes the left side.

8. Factor each expression:

   a. $x^3 - 8$ b. $n^3 + 27$ c. $z^3 + 1$

   d. $8x^3 - 27$ e. $27y^3 + 125$ f. $64a^3 - 1$

9. Factor each expression:

   a. $(x + y)^3 + 8$ b. $(a - b)^3 - 27$ c. $(p + q)^3 + 1$

10. Factor $x^5 + 1$. Hint: Divide by $x + 1$.

11. Factor $A^5 - 32$. Hint: Divide by $A - 2$.

12. Factor each expression:

   a. $w^5 + 1$ b. $c^5 - 1$ c. $y^5 - 32$

   d. $z^5 + 32$ e. $n^5 + 243$ f. $m^5 - 243$

13. Factor each expression:

   a. $x^7 - 1$ b. $y^7 + 1$

   c. $u^7 - 128$ d. $z^7 + 128$

Practice Problems

14. Factor each expression:

a. $10ax^4 - 160a$

b. $Z^2(P - Q) - 144(P - Q)$

c. $50x^3 - 75x^2 - 2x + 3$

d. $12ac - 10bd + 8bc - 15ad$

e. $a^2 - 2ab + b^2 - c^2$

f. $x^2 + 2xy + y^2 - 144$

g. $x^4 - 34x^2 + 225$

h. $x^4 - 8x^2 - 9$

i. $x^3 - 7x^2 + 9x - 63$

j. $n^3 + 3n^2 - 16n - 48$

k. $(a + b)^2 - 5(a + b) + 6$

l. $(x - y)^2 + 7(x - y) + 6$

m. $(a - b)^2 + 6(a - b) - 16$

n. $hm - hn + km - kn$

15. Factor each expression:

a. $n^3 + 64$

b. $a^3 - 125$

c. $8T^3 - 27$

d. $27x^3 + 1$

Solutions

1. a. $(x^2 + 1)(x + 1)(x - 1)$

b. $(x^2 + 2)(x^2 - 3)$

c. $(n + 1)(n - 1)(n + 3)(n - 3)$

d. $(a^2 + 9)(a + 3)(a - 3)$

e. $(2w + 1)(2w - 1)(3w + 2)(3w - 2)$

f. $(x + 1)(x - 1)(3x + 5)(3x - 5)$

g. $(c^2 + 4)(c + 2)(c - 2)$

h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 2)(x^2 - 5)$

j. $(g^2 + 16)(g + 4)(g - 4)$

k. $(2u + 3)(2u - 3)(3u + 1)(3u - 1)$

l. Not factorable

2. a. $(x + y)(x + y + 7)$ b. $(a - b)(a - b - c)$
c. $(c + d)(x^2 + 5)$ d. $(a - b)(n + 3)(n - 3)$
e. $(a + 4)(x + 3)(x + 2)$ f. $y(m + n)(y + 7)$
g. $(a + b)(2x + 5)(x - 1)$ h. $(w + z)(2x + 3)(2x - 3)$
i. $(u - w)(u - w - 9)$ j. $n(a + b)(n - 9)$
k. $(t + r)(y + 10)(y - 10)$ l. $a(3x + 1)(x - 7)$

3. a. $(x + y)(w + z)$ b. $(a + b)(a + c)$ c. $(x^2 + 3)(x - 4)$
d. $(n^2 - 5)(n - 1)$ e. $(x + 1)(x + 3)(x - 3)$ f. $(a + b)(c - d)$
g. $(x - y)(w - z)$ h. $(2a + b)(c - d)$ i. $(3x - y)(2w + z)$
j. $(h - j)(j - k)$ k. $(a - b)(x + y)$ l. $(x - 2)(x + 5)(x - 5)$
m. $(x + 2y)(w - z)$ n. $(a^2 - 5)(a - 1)$ o. $(2t + w)(2w - x)$
p. $(2x^2 - 3)(3x + 1)$ q. Not factorable r. $(3a^2 + 5)(2a - 5)$

4. a. $(x + y + z)(x + y - z)$ b. $(a - b + c)(a - b - c)$
c. $(x + 2 + y)(x + 2 - y)$ d. $(n - 3 + Q)(n - 3 - Q)$
e. $(u + w + T)(u + w - T)$ f. $(y + 5 + x)(y + 5 - x)$
g. $(a + b + c)(a + b - c)$ h. $(w - y + 7)(w - y - 7)$
i. $(2x + 1 + t)\ (2x + 1 - t)$ j. $(3x - 2 + y)(3x - 2 - y)$

5. a. $x(x + 1)(x - 1)$ b. $2n(n + 1)(n + 2)$
c. $5a(2a + 1)(a - 1)$ d. $7y(y + 5)^2$
e. $9w(2w + 1)(2w - 1)$ f. $4z(3z + 2)(2z - 3)$

6. a. $4m$ b. $6n$ c. $3A^2$ d. $343z^6$ e. $-27a^9$

7. a. $(x + 2)(x^2 - 2x + 4) = x^3 - 2x^2 + 4x + 2x^2 - 4x + 8 = x^3 + 8$ ✓
b. You try it.

8. a. $(x-2)(x^2+2x+4)$

b. $(n+3)(n^2-3n+9)$

c. $(z+1)(z^2-z+1)$

d. $(2x-3)(4x^2+6x+9)$

e. $(3y+5)(9y^2-15y+25)$

f. $(4a-1)(16a^2+4a+1)$

9. a. $(x+y+2)(x^2+2xy+y^2-2x-2y+4)$

b. $(a-b-3)(a^2-2ab+b^2+3a-3b+9)$

c. $(p+q+1)(p^2+2pq+q^2-p-q+1)$

10. $(x+1)(x^4-x^3+x^2-x+1)$

11. $(A-2)(A^4+2A^3+4A^2+8A+16)$

12. a. $(w+1)(w^4-w^3+w^2-w+1)$

b. $(c-1)(c^4+c^3+c^2+c+1)$

c. $(y-2)(y^4+2y^3+4y^2+8y+16)$

d. $(z+2)(z^4-2z^3+4z^2-8z+16)$

e. $(n+3)(n^4-3n^3+9n^2-27n+81)$

f. $(m-3)(m^4+3m^3+9m^2+27m+81)$

13. a. $(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

b. $(y+1)(y^6-y^5+y^4-y^3+y^2-y+1)$

c. $(u-2)(u^6+2u^5+4u^4+8u^3+16u^2+32u+64)$

d. $(z+2)(z^6-2z^5+4z^4-8z^3+16z^2-32z+64)$

14. a. $10a(x^2 + 4)(x + 2)(x - 2)$ b. $(P - Q)(Z + 12)(Z - 12)$

c. $(2x - 3)(5x + 1)(5x - 1)$ d. $(3a + 2b)(4c - 5d)$

e. $(a - b + c)(a - b - c)$ f. $(x + y + 12)(x + y - 12)$

g. $(x + 5)(x - 5)(x + 3)(x - 3)$ h. $(x^2 + 1)(x + 3)(x - 3)$

i. $(x^2 + 9)(x - 7)$ j. $(n + 4)(n - 4)(n + 3)$

k. $(a + b - 3)(a + b - 2)$ l. $(x - y + 6)(x - y + 1)$

m. $(a - b + 8)(a - b - 2)$ n. $(m - n)(h + k)$

15. a. $(n + 4)(n^2 - 4n + 16)$ b. $(a - 5)(a^2 + 5a + 25)$

c. $(2T - 3)(4T^2 + 6T + 9)$ d. $(3x + 1)(9x^2 - 3x + 1)$

"A college degree is not a sign that one is a finished product, but an indication a person is prepared for life."

Reverend Edward A. Malloy, *Monk's Reflections*