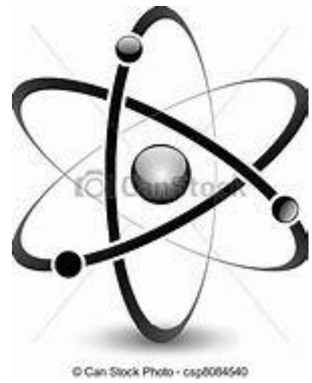
CH 21 – NEGATIVE EXPONENTS

❑ INTRODUCTION



The mass of a proton (the positively charged particle in the nucleus of an atom) is written in "scientific notation" as

1.67×10^{-27} kilograms.

How do we read that? Is that a really big or a really small number? In this chapter we will learn exactly what is meant by a negative exponent.

If n is a natural number (n = 1, 2, 3, . . .), then we know that the meaning of x^n is based upon repeated multiplication:

$$x^n = \underbrace{(x)(x)(x)\cdots(x)}_{n \text{ factors of } x}$$

For instance, we know that x^3 means $x \cdot x \cdot x$. We also learned that $x^0 = 1$ (as long as x itself isn't 0). But what does something like x^{-3} mean?

❑ REVIEW OF THE FIVE LAWS OF EXPONENTS

Before we begin the discussion of negative exponents, it would be beneficial for us to review the five laws of exponents.

$$x^a x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

<u>EXAMPLE 1</u>: **Simplify each expression:**

A. $x^3x^4 = x^7$

When multiplying powers of the same base, add the exponents.

B. $\frac{a^{10}}{a^2} = a^8$

When dividing powers of the same base, subtract the exponents.

C. $(z^3)^4 = z^{12}$

When raising a power to a power, multiply the exponents.

D. $(mn)^4 = m^4n^4$

When raising a product to a power, apply the power to all the factors of the product.

E. $\left(\frac{x}{y}\right)^7 = \frac{x^7}{y^7}$

When raising a quotient to a power, apply the power to both the numerator and denominator.

<u>EXAMPLE 2</u>: **Simplify each expression:**

A. r^4t^5 = As is

Don't do anything. You cannot multiply powers of <u>different</u> bases by adding the exponents.

B. $a^2 + a^4$ = As is

Again, don't try combining these terms. First, the exponent rule regarding adding the exponents applies only when the operation is multiplication, not addition. Second, they can't be added either, since they're not like terms.

C. $w^3 + w^3 = 2w^3$

This sum can be simplified, but not by the first law of exponents, since the powers of w are not being multiplied. But the two terms are like terms, which means we simply add them together to get $2w^3$.

D. $(a + b)^{23}$ = As is (for now)

It does not equal $a^{23} + b^{23}$. The third law of exponents, $(xy)^n = x^n y^n$, does not apply because xy is a single term, but $a + b$ consists of two terms.

Homework

1. Simplify each expression:

a. $n^4 n^5$ b. $\dfrac{b^{18}}{b^6}$ c. $(z^4)^5$ d. $(abc)^4$

e. $\left(\dfrac{k}{n}\right)^4$ f. $a^3 b^7$ g. $x^3 + x^2$ h. $p^4 + p^4$

i. $u^3 - u^3$ j. $(a + b)^2$ k. $(x - y)^3$ l. $(g + h)^{56}$

m. $\dfrac{d^{20}}{d^{30}}$ n. $x^3 x^5 + x^6 x^2$

❑ DEVELOPING THE MEANING OF A NEGATIVE EXPONENT

Consider the following expression containing a negative exponent:

$$x^{-3}$$

What could it mean? To find out, let's multiply it by x^3 and see what happens:

$$x^{-3} \cdot x^{3} = x^{-3+3} = x^{0} = 1$$

which implies that

$$x^{-3} \cdot x^{3} = 1 \qquad \text{(using rules we already have)}$$

Now, x^{-3} is what we're analyzing, so let's "solve" for it (or isolate it) by dividing each side of the equation by x^3:

$$\frac{x^{-3} \cdot x^{3}}{x^{3}} = \frac{1}{x^{3}}$$

$$\Rightarrow \quad \boxed{x^{-3} = \frac{1}{x^{3}}}$$

You could do the same thing to n^{-12}:

$$n^{-12} \cdot n^{12} = n^{0} = 1$$

$$\Rightarrow \quad n^{-12} \cdot n^{12} = 1$$

$$\Rightarrow \quad \frac{n^{-12} \cdot n^{12}}{n^{12}} = \frac{1}{n^{12}}$$

$$\Rightarrow \quad \boxed{n^{-12} = \frac{1}{n^{12}}}$$

Homework

2. Use the above logic to rewrite the expression with NO negative exponents:

   a. w^{-4} b. y^{-19} c. z^{-50} d. 2^{-10}

❑ WORKING WITH NEGATIVE EXPONENTS

The previous section should convince you that a negative exponent means reciprocal. In fact, the exponent doesn't have to be a negative whole number. [Even things like $w^{-5/4}$ and $h^{-\sqrt{2\pi}}$ represent reciprocals.]

In general, as long as $x \neq 0$,

$$x^{-n} = \frac{1}{x^n}$$ for ANY number n.

This fact, together with our knowledge of fractions and the laws of exponents, is all we need to understand all the examples in this chapter.

EXAMPLE 2:

A. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B. $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

C. $-3^{-4} = -\frac{1}{3^4} = -\frac{1}{81}$

See the difference?

D. $(-3)^{-4} = \frac{1}{(-3)^4} = \frac{1}{81}$

E. $\frac{1}{2^{-5}} = \frac{1}{\frac{1}{2^5}} = \frac{1}{\frac{1}{32}} = 32$

F. $\frac{1}{10^{-3}} = \frac{1}{\frac{1}{10^3}} = 10^3 = 1000$

G. $\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^3} = \frac{1}{\frac{8}{27}} = \frac{27}{8}$

H. $0^{-5} = \frac{1}{0^5} = \frac{1}{0} = \text{Undefined}$

Homework

3. Evaluate each expression:

a. 3^{-2} b. 2^{-5} c. 10^{-3} d. 5^{-4} e. 1^{-10}

f. -2^{-3} g. $(-3)^{-3}$ h. -10^{-4} i. $(-5)^{-1}$ j. -6^{-2}

k. $\frac{1}{3^{-2}}$ l. $\frac{1}{10^{-4}}$ m. $\frac{1}{5^{-3}}$ n. $\left(\frac{3}{4}\right)^{-2}$ o. $\left(\frac{1}{6}\right)^{-3}$

4. Explain why $x = 0$ is NOT allowed in the expression x^{-8}.

EXAMPLE 3:

A. $n^{-1} = \frac{1}{n}$

B. $a^{-12} = \frac{1}{a^{12}}$

C. $\frac{x}{y^{-3}} = \frac{x}{\frac{1}{y^3}} = x \cdot \frac{y^3}{1} = xy^3$

D. $\frac{x^{-7}}{a^{12}} = \frac{\frac{1}{x^7}}{a^{12}} = \frac{1}{x^7} \cdot \frac{1}{a^{12}} = \frac{1}{a^{12}x^7}$

E. $a^2b^{-3} = a^2 \cdot \frac{1}{b^3} = \frac{a^2}{b^3}$

F. $R^{-1}T^{-7} = \frac{1}{R} \cdot \frac{1}{T^7} = \frac{1}{RT^7}$

EXAMPLE 4:

A. $x + y^{-5} = x + \frac{1}{y^5} = \left[\frac{y^5}{y^5}\right]\frac{x}{1} + \frac{1}{y^5} = \frac{xy^5}{y^5} + \frac{1}{y^5} = \frac{xy^5 + 1}{y^5}$

B. $a^{-2} + b^{-7} = \frac{1}{a^2} + \frac{1}{b^7} = \frac{1}{a^2}\left[\frac{b^7}{b^7}\right] + \frac{1}{b^7}\left[\frac{a^2}{a^2}\right] = \frac{b^7 + a^2}{a^2b^7}$

Homework

5. Simplify each expression (no negative exponents in the answer):

a. x^{-9} b. a^{-30} c. $\frac{a}{b^{-2}}$ d. $\frac{x^{-4}}{y^{2}}$

e. $\frac{w^{3}}{x^{-3}}$ f. $\frac{m^{-3}}{n^{-4}}$ g. $a^{-4}b^{-5}$ h. $k^{3}p^{-5}$

i. $h^{-5}z^{7}$ j. $\frac{w^{-5}}{w^{-5}}$ k. $x+y^{-1}$ l. $a^{-3}+b^{2}$

m. $p^{-1}+r^{-1}$ n. $w^{-3}-z^{-2}$ o. $a^{-3}-a^{-3}$ p. $a^{-1}b^{-2}c^{-3}$

<u>EXAMPLE 5:</u> **Using the Five Laws of Exponents:**

A. $x^{-6}x^{-8} = x^{-14} = \mathbf{\frac{1}{x^{14}}}$

Add the exponents

B. $\left(N^{-5}\right)^{-7} = \mathbf{N^{35}}$

Multiply the exponents

C. $(abc)^{-4} = a^{-4}b^{-4}c^{-4} = \frac{1}{a^4}\cdot\frac{1}{b^4}\cdot\frac{1}{c^4} = \mathbf{\frac{1}{a^4b^4c^4}}$

Apply exponent to each factor

D. $\frac{w^5}{w^{15}} = w^{5-15} = w^{-10} = \mathbf{\frac{1}{w^{10}}}$

Subtract the exponents

E. $$\frac{x^{-17}}{x^{-12}} = x^{-17-(-12)} = x^{-17+12} = x^{-5} = \frac{\mathbf{1}}{\boldsymbol{x^5}}$$

Subtract the exponents

F. $$\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}} = \frac{\frac{1}{a^3}}{\frac{1}{b^3}} = \frac{1}{a^3}\cdot\frac{b^3}{1} = \frac{\boldsymbol{b^3}}{\boldsymbol{a^3}}$$

Apply exponent to
top and bottom

Another way: $$\left(\frac{a}{b}\right)^{-3} = \frac{1}{\left(\frac{a}{b}\right)^3} = \frac{1}{\frac{a^3}{b^3}} = \frac{b^3}{a^3}$$

G. $(x+y)^{-2}$

Be careful; no law of exponents applies here since the quantity $x + y$ consists of <u>two</u> terms — it is not a product — so do NOT apply the exponent to each term of the sum. Begin the problem by dealing with the meaning of the negative exponent.

$$(x+y)^{-2} = \frac{1}{(x+y)^2} = \frac{1}{(x+y)(x+y)} = \frac{\mathbf{1}}{\boldsymbol{x^2+2xy+y^2}}$$

Homework

6. Simplify each expression (no negative exponents in the answer):

a. $a^{-3}a^7$ b. $x^{-5}x^{-10}$ c. $k^{-10}k^{10}$ d. $m^{-5}w^{-3}$

e. $(x^2)^{-5}$ f. $(a^{-5})^3$ g. $(c^{-3})^{-4}$ h. $(b^2)^{-1}$

i. $(ax)^{-5}$ j. $(xyz)^{-3}$ k. $(x+y)^{-1}$ l. $(a-b)^{-2}$

m. $\dfrac{x^{-5}}{x^3}$ n. $\dfrac{z^5}{z^{-4}}$ o. $\dfrac{a^{-5}}{a^{-8}}$ p. $\dfrac{y^{-12}}{y^{-7}}$

q. $\left(\dfrac{a}{b}\right)^{-4}$ r. $\left(\dfrac{y}{x}\right)^{-1}$ s. $\left(\dfrac{a}{b+c}\right)^{-2}$ t. $\left(\dfrac{a+b}{xy+wz}\right)^{0}$

<u>EXAMPLE 6:</u> **Simplify each expression:**

A. $(2x^{-8})(-5x^6) = (2)(-5)x^{-8}x^6 = -10x^{-2} = -10\left(\dfrac{1}{x^2}\right) = -\boldsymbol{\dfrac{10}{x^2}}$

B. $\left(x^2y^{-3}\right)^4 = \left(x^2\right)^4\left(y^{-3}\right)^4 = x^8y^{-12} = x^8\left(\dfrac{1}{y^{12}}\right) = \boldsymbol{\dfrac{x^8}{y^{12}}}$

C. $\left(\dfrac{a^{-3}}{b^4}\right)^{-5} = \dfrac{\left(a^{-3}\right)^{-5}}{\left(b^4\right)^{-5}} = \dfrac{a^{15}}{b^{-20}} = \dfrac{a^{15}}{\dfrac{1}{b^{20}}} = \dfrac{a^{15}}{1}\cdot\dfrac{b^{20}}{1} = \boldsymbol{a^{15}b^{20}}$

Final Note: In the Introduction we learned about the mass of a proton. We're now able to see just what that number is:

$$1.67\times10^{-27} = 1.67\times\frac{1}{10^{27}} = \frac{1.67}{1}\times\frac{1}{10^{27}} = \frac{1.67}{10^{27}}$$

$$= \frac{1.67}{1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000}$$

= 0.00000000000000000000000000167 kilograms

A proton certainly doesn't weigh very much! In addition, I hope you can appreciate why we use scientific notation (1.67×10^{-27}) rather than the number in the box.

Homework

7. Simplify each expression (no negative exponents in the answer):

a. $(2x^{-4})(3x^7)$ b. $(-3y^{-3})(2y^{-5})$ c. $(-4a^3)(a^{-7})$

d. $\left(u^3u^{-5}\right)^{-2}$ e. $\left(w^{-5}w^{-1}\right)^7$ f. $\left(a^{-2}b^{-3}\right)^{-5}$

g. $\left(t^3u^{-4}\right)^3$ h. $\left(\dfrac{x^2}{x^5}\right)^{-2}$ i. $\left(\dfrac{y^{-3}}{y^{-4}}\right)^5$

j. $\left(\dfrac{a^{-2}}{b^5}\right)^{-3}$ k. $\left(\dfrac{c^3}{d^{-4}}\right)^{-5}$ l. $\left(\dfrac{a^3b^{-4}}{x^{-12}z^{-2}}\right)^0$

8. Express each scientific notation number as a regular number:

a. 2.3×10^7 b. 7.11×10^{-5} c. 5.09×10^{-10}

Practice Problems

9. Evaluate: a. -2^{-4} b. $(-2)^{-4}$

10. Evaluate: a. $\left(\dfrac{4}{5}\right)^{-3}$ b. $\left(\dfrac{1}{9}\right)^{-1}$

11. Simplify: a. $\dfrac{a^{-2}}{b^{-3}}$ b. $\dfrac{x^{-9}}{x^{9}}$

12. Simplify: a. $a^{-2}a^{-3}$ b. $a^{-2} - a^{-3}$

13. Simplify: a. $(x^{-10})^5$ b. $(z^{-5})^{-5}$

14. Simplify: $(3x^{-3}y^{-2})^{-1}(-4x^3y^{10})^{-2}$

15. Simplify: a. $\left(\dfrac{x^{-4}}{x^{-6}}\right)^{-2}$ b. $\left(\dfrac{a^{3}}{b^{-4}}\right)^{5}$

16. Simplify: a. $(uw)^{-2}$ b. $(u + w)^{-2}$

17. Explain why $x = 0$ is NOT allowed in the expression x^{-2}.

18. Express as a regular number: 4.9×10^{-13}

Solutions

1. a. n^9 b. b^{12} c. z^{20} d. $a^4b^4c^4$ e. $\dfrac{k^4}{n^4}$ f. As is
g. As is h. $2p^4$ i. 0 j. $a^2 + 2ab + b^2$
k. $x^3 - 3x^2y + 3xy^2 - y^3$ l. As is (for now) m. $\dfrac{1}{d^{10}}$ n. $2x^8$

2. a. $\dfrac{1}{w^4}$ b. $\dfrac{1}{y^{19}}$ c. $\dfrac{1}{z^{50}}$ d. $\dfrac{1}{2^{10}} = \dfrac{1}{1{,}024}$

3. a. $\dfrac{1}{9}$ b. $\dfrac{1}{32}$ c. $\dfrac{1}{1000}$ d. $\dfrac{1}{625}$ e. 1 f. $-\dfrac{1}{8}$

g. $-\frac{1}{27}$ h. $-\frac{1}{10{,}000}$ i. $-\frac{1}{5}$ j. $-\frac{1}{36}$ k. 9 l. 10,000

m. 125 n. $\frac{16}{9}$ o. 216

4. If x were 0 in the expression x^{-8}, we would have

$x^{-8} = 0^{-8} = \frac{1}{0^8} = \frac{1}{0}$, which is Undefined.

5. a. $\frac{1}{x^9}$ b. $\frac{1}{a^{30}}$ c. ab^2 d. $\frac{1}{x^4y^2}$ e. x^3w^3 f. $\frac{n^4}{m^3}$

g. $\frac{1}{a^4b^5}$ h. $\frac{k^3}{p^5}$ i. $\frac{z^7}{h^5}$ j. 1 k. $\frac{xy+1}{y}$ l. $\frac{1+a^3b^2}{a^3}$

m. $\frac{r+p}{pr}$ n. $\frac{z^2-w^3}{w^3z^2}$ o. 0 p. $\frac{1}{ab^2c^3}$

6. a. a^4 b. $\frac{1}{x^{15}}$ c. 1 d. $\frac{1}{m^5w^3}$ e. $\frac{1}{x^{10}}$

f. $\frac{1}{a^{15}}$ g. c^{12} h. $\frac{1}{b^2}$ i. $\frac{1}{a^5x^5}$ j. $\frac{1}{x^3y^3z^3}$

k. $\frac{1}{x+y}$ l. $\frac{1}{a^2-2ab+b^2}$ m. $\frac{1}{x^8}$ n. z^9

o. a^3 p. $\frac{1}{y^5}$ q. $\frac{b^4}{a^4}$ r. $\frac{x}{y}$ s. $\frac{b^2+2bc+c^2}{a^2}$

t. 1

7. a. $6x^3$ b. $-\frac{6}{y^8}$ c. $-\frac{4}{a^4}$ d. u^4 e. $\frac{1}{w^{42}}$ f. $a^{10}b^{15}$

g. $\frac{t^9}{u^{12}}$ h. x^6 i. y^5 j. a^6b^{15} k. $\frac{1}{c^{15}d^{20}}$ l. 1

8. a. 23,000,000 b. 0.0000711 c. 0.0000000000509

9. a. $-\frac{1}{16}$ b. $\frac{1}{16}$ **10.** a. $\frac{125}{64}$ b. 9

11. a. $\frac{b^3}{a^2}$ b. $\frac{1}{x^{18}}$ **12.** a. $\frac{1}{a^5}$ b. $\frac{a-1}{a^3}$

13. a. $\frac{1}{x^{50}}$ b. z^{25} **14**. $\frac{1}{48x^3y^{18}}$

15. a. $\frac{1}{x^4}$ b. $a^{15}b^{20}$ **16**. a. $\frac{1}{u^2w^2}$ b. $\frac{1}{u^2+2uw+w^2}$

17. Because $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ which is Undefined.

18. 0.00000000000049

"A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing."

– ***George Bernard Shaw***