CH 27 – RADICALS, PART II

❑ *PROVING THE THEOREMS FROM CHAPTER 24*

The chapter on fractional exponents showed us that $x^{1/n} = \sqrt[n]{x}$. This way of expressing radicals using exponents allows us to prove a pair of rules we took for granted in Chapter 24.

<u>THEOREM:</u> Assume that x and y represent non-negative numbers. Then

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

The root of a product
is the product of the roots.

<u>Proof:</u> Our technique will be to convert the left side of the formula to exponent form, use a law of exponents, and then convert back to radical form to arrive at the right side of the formula.

	$\sqrt[n]{xy}$	
=	$(xy)^{1/n}$	(definition of fractional exponent)
=	$x^{1/n}\,y^{1/n}$	(law of exponents)
=	$\sqrt[n]{x}\,\sqrt[n]{y}$	(definition of fractional exponent)

THEOREM: If x represents a non-negative number (that is, if $x \geq 0$), then

$$\sqrt[n]{x^n} = x$$

The *n*th root cancels the *n*th power.

Proof: Our method of proof is similar to the previous theorem.

$$\sqrt[n]{x^n} = (x^n)^{1/n} = x^{n \cdot \frac{1}{n}} = x^1 = x$$

❑ ADDING AND SUBTRACTING RADICALS

We have a simple radical rule that states that $\sqrt{a}\sqrt{b} = \sqrt{ab}$. How about a rule like $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$? Seems reasonable; let's perform an experiment. Let $a = 16$ and $b = 9$.

Left side: $\sqrt{a} + \sqrt{b} = \sqrt{16} + \sqrt{9} = 4 + 3 = \mathbf{7}$

Right side: $\sqrt{a+b} = \sqrt{16+9} = \sqrt{25} = \mathbf{5}$

We get two different result; we thus conclude that $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$.

So maybe we can't add $\sqrt{7}$ and $\sqrt{11}$ together, but we can add $2\sqrt{7}$ and $3\sqrt{7}$ together. This is simply the *combining of like terms*. After all, we may not be very comfortable with a number like $\sqrt{7}$, but certainly 2 of them plus 3 more of them should total 5 of them: $2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$.

Alternatively, we can apply the distributive property to the sum $2\sqrt{7} + 3\sqrt{7}$ by factoring out the GCF: $\sqrt{7}(2+3)$, giving $5\sqrt{7}$.

Thus, to add (or subtract) radicals, we can use either the concept of like terms or the idea of factoring.

<u>EXAMPLE 1:</u> **Simplify each radical expression:**

A. $3\sqrt{11}+5\sqrt{11} = 8\sqrt{11}$

They're like terms, so we can add them up, just like $3x + 5x = 8x$.
Or, factor out the GCF: $\sqrt{11}(3+5) = 8\sqrt{11}$

B. $18\sqrt[3]{10}-20\sqrt[3]{10} = -2\sqrt[3]{10}$

C. $\sqrt{3}+\sqrt{5}$ cannot be worked out. They're unlike terms, and no amount of simplification will turn them into like terms.

D. $3\sqrt{20}+10\sqrt{45}$ It might appear that these terms cannot be combined, but don't jump to conclusions — a little simplification first will yield an answer:

$$3\sqrt{20}+10\sqrt{45}$$
$$= 3\sqrt{4\cdot 5}+10\sqrt{9\cdot 5}$$
$$= 3\cdot 2\sqrt{5}+10\cdot 3\sqrt{5}$$
$$= 6\sqrt{5}+30\sqrt{5} \qquad \text{[Like Terms]}$$
$$= 36\sqrt{5}$$

E. $\sqrt{7}+\sqrt[3]{7}$ can't be simplified. Even though the radicands (the 7's) are the same, the roots are different, and therefore they are unlike terms and cannot be added together.

Homework

1. Simplify each radical expression:

a. $2\sqrt{x}+10\sqrt{x}$ b. $3\sqrt{7}-2\sqrt{2}$ c. $10\sqrt[3]{n}+3\sqrt[3]{n}$

d. $6\sqrt{x}-2\sqrt[3]{x}$ e. $9\sqrt{8}+10\sqrt{50}$ f. $\sqrt[3]{24}+\sqrt[3]{81}$

g. $3\sqrt{80}-5\sqrt{45}$ h. $10\sqrt[3]{189}+\sqrt[3]{56}$ i. $\sqrt[3]{z^2}+2\sqrt[3]{z^2}$

j. $\sqrt{12}+2\sqrt{75}$ k. $2\sqrt{20}+2\sqrt{45}$ l. $2\sqrt{275}-3\sqrt{99}$

❑ *MULTIPLYING RADICALS*

Using the formula $\sqrt[n]{x}\,\sqrt[n]{y} = \sqrt[n]{xy}$, we can multiply "like roots."

<u>EXAMPLE 2:</u> **Simplify each radical expression:**

A. $\sqrt[5]{3}\cdot\sqrt[5]{7} = \mathbf{\sqrt[5]{21}}$

B. $\sqrt{7}\sqrt{7} = \sqrt{49} = \mathbf{7}$

C. $\sqrt[3]{17}\times\sqrt[3]{3} = \mathbf{\sqrt[3]{51}}$

D. $\sqrt{2}\sqrt{8} = \sqrt{16} = \mathbf{4}$

E. $\sqrt{14}\sqrt{2} = \sqrt{28} = \sqrt{4\cdot 7} = \sqrt{4}\sqrt{7} = \mathbf{2\sqrt{7}}$

F. $\sqrt{2}\left(1+\sqrt{2}\right) = \sqrt{2}+\sqrt{2}\sqrt{2} = \mathbf{\sqrt{2}+2}$

G. $\sqrt{6}\left(\sqrt{10}-\sqrt{6}\right) = \sqrt{6}\sqrt{10}-\sqrt{6}\sqrt{6} = \sqrt{60}-6 = \mathbf{2\sqrt{15}-6}$

H. $\left(\sqrt{3}+1\right)\left(\sqrt{3}-5\right) = \sqrt{3}\sqrt{3}-5\sqrt{3}+1\sqrt{3}-5$

$= 3-4\sqrt{3}-5 = \mathbf{-2-4\sqrt{3}}$

I. $\left(1+\sqrt{7}\right)^2 = \left(1+\sqrt{7}\right)\left(1+\sqrt{7}\right) = 1+\sqrt{7}+\sqrt{7}+7 = \mathbf{8+2\sqrt{7}}$

J. $\left(3+\sqrt{10}\right)\left(3-\sqrt{10}\right) = 9-3\sqrt{10}+3\sqrt{10}-\sqrt{10}\sqrt{10}$

$= 9-10 = \mathbf{-1}$

Homework

2. Find the product:

a. $\sqrt{14}\sqrt{6}$ b. $\sqrt[3]{3}\times\sqrt[3]{9}$ c. $\sqrt{2}\sqrt{7}$

d. $\sqrt{171}\sqrt{171}$ e. $\sqrt{14}\sqrt{2}$ f. $\sqrt[3]{10}\,\sqrt[3]{12.5}$

g. $\sqrt[4]{2}\,\sqrt[4]{8}$ h. $\sqrt{x}\sqrt{yz}$ i. $\sqrt[3]{2}\,\sqrt[3]{4}$

3. Find the product:

a. $\sqrt{n}\left(1+\sqrt{n}\right)$ b. $\left(x+\sqrt{y}\right)\left(x-\sqrt{y}\right)$ c. $\left(2+\sqrt{3}\right)^2$

d. $\left(\sqrt{p}-q\right)^2$ e. $\left(\sqrt{7}-2\right)\left(\sqrt{7}+2\right)$ f. $\left(x-\sqrt{n}\right)^2$

g. $\left(\sqrt{a}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{c}\right)$ h. $2\left(\sqrt{3}+\sqrt{4}\right)$ i. $\left(\sqrt{x}+\sqrt{x}\right)^2$

❑ *DIVIDING RADICALS*

Just as the square root of a product is the product of the square roots ($\sqrt{ab} = \sqrt{a}\sqrt{b}$), the square root of a quotient is the quotient of the square roots.

<u>THEOREM:</u> Assume a and b are positive real numbers. Then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Proof: $\sqrt{\frac{a}{b}}$

$$= \left(\frac{a}{b}\right)^{1/2}$$ (convert radical to fractional exponent)

$$= \frac{a^{1/2}}{b^{1/2}}$$ (a law of exponents)

$$= \frac{\sqrt{a}}{\sqrt{b}}$$ (convert back to radical form)

Before we get to the heart of dividing radicals, let's recall the little formula $\left(\sqrt[n]{x}\right)^n = x$. Check out the following examples:

$$\left(\sqrt{u}\right)^2 = u \qquad \left(\sqrt[3]{w}\right)^3 = w \qquad \left(\sqrt[4]{xy}\right)^4 = xy$$

$$\sqrt{c}\sqrt{c} = \left(\sqrt{c}\right)^2 = c \qquad \sqrt{17} \times \sqrt{17} = \left(\sqrt{17}\right)^2 = 17$$

Now consider the number $\frac{1}{\sqrt{2}}$. It's just a fraction, and a calculator gives an approximate value of 0.707106781. But now consider the scenario where you have no calculator, but you need a decimal version of the fraction. Even if you could look up the positive square root of 2 somehow — and find that it's about 1.414213562 — you'd still be stuck with the following long division problem:

$$1.414213562\overline{\smash{)}1.000000000}$$

This is a killer long division problem — it would take a really long time to get a few digits of the answer.

But now watch this trick: I'm going to take the fraction $\frac{1}{\sqrt{2}}$ and multiply it by the fraction $\frac{\sqrt{2}}{\sqrt{2}}$ (which, of course, equals 1). Here we go:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \underbrace{\left[\frac{\sqrt{2}}{\sqrt{2}}\right]}_{=1} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We've shown that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. What's the purpose of all this? Consider the long division problem created by the second fraction:

$$2\,\overline{)\,1.414213562}$$

This is <u>much</u> simpler to carry out than the previous long division problem, and yet it yields the same answer. This is the historical reason that students, especially in the sciences, would convert $\frac{1}{\sqrt{2}}$ into $\frac{\sqrt{2}}{2}$ — it made calculations much simpler.

Are we going to perform this kind of maneuver — removing radicals from the denominator of a fraction — in our Algebra course? Yes, and for two reasons. One is historical; it's been done like this for centuries. As Tevya from *Fiddler on the Roof* might say, "We do it because it's our tradition!" A second reason: It's a lot easier when we look up answers in the "back of the book" if we can all agree on what form to leave our answers. We thus have a basic agreement in math: <u>Do not leave radicals in the denominator of a fraction</u>.

The process of converting a fraction <u>with</u> a radical in the denominator into an equivalent fraction <u>without</u> a radical in the denominator is called ***rationalizing the denominator***.

I'll let you in on a little secret: If you take Business Calculus, there will be problems where you will <u>want</u> to leave radicals in the denominator.

<u>EXAMPLE 2:</u>

A. Rationalize the denominator, which is a **monomial**:

$$\frac{3}{\sqrt{x}} = \frac{3}{\sqrt{x}}\underbrace{\left[\frac{\sqrt{x}}{\sqrt{x}}\right]}_{=1} = \frac{3\sqrt{x}}{\sqrt{x}\sqrt{x}} = \frac{3\sqrt{x}}{x}$$

B. Rationalize the denominator, which is a **binomial**:

For this example, the trick is in figuring out what, exactly, we should multiply the top and the bottom of the fraction by.

$$\frac{\sqrt{2}}{5-\sqrt{10}} = \frac{\sqrt{2}}{5-\sqrt{10}}\left[\frac{5+\sqrt{10}}{5+\sqrt{10}}\right] = \frac{5\sqrt{2}+\sqrt{20}}{25-10} = \frac{5\sqrt{2}+2\sqrt{5}}{15}$$

$$=1$$

Notes: Many students try multiplying top and bottom by $\sqrt{10}$, figuring that it would remove the $\sqrt{10}$. Try it and you'll see this procedure fail.

Other students think they're more clever and multiply top and bottom by $5-\sqrt{10}$. Go for it; you'll still have radicals in the bottom.

The quantity $5+\sqrt{10}$, which <u>does</u> eradicate the radical from the bottom, is called the ***conjugate*** of $5-\sqrt{10}$.

Homework

4. Simplify so that there are no radicals in the denominator:

a. $\frac{2}{\sqrt{8}}$ b. $\frac{x^2}{\sqrt{x}}$ c. $\frac{3}{5+\sqrt{2}}$ d. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

e. $\frac{1}{\sqrt{3}}$ f. $\frac{6}{\sqrt{9}}$ g. $\frac{8}{\sqrt{2}}$ h. $\frac{\sqrt{x}}{\sqrt{y}}$

i. $\frac{5}{1+\sqrt{5}}$ j. $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ k. $\frac{7}{3\sqrt{2}}$ l. $\frac{x}{n\sqrt{x}}$

m. $\frac{7}{1+\sqrt{7}}$ n. $\frac{a+\sqrt{b}}{a-\sqrt{b}}$ o. $\frac{a}{b\sqrt{c}}$ p. $\frac{7+\sqrt{103}}{7+\sqrt{103}}$

5. Two of the following are valid, while two of them are bogus. Which are which?

a. $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ b. $\sqrt{a-b} = \sqrt{a}-\sqrt{b}$

c. $\sqrt{ab} = \sqrt{a}\sqrt{b}$ d. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Practice Problems

6. a. Find the two fourth roots of 256.
   b. Find all the cube roots of -1000.
   c. Find all the square roots of -121.

7. Simplify each expression:

a. $\sqrt{7}\sqrt{5}$ b. $\sqrt{7}+\sqrt{5}$ c. $\frac{\sqrt{7}}{\sqrt{5}}$ d. $\sqrt{7}-\sqrt{5}$

e. $\sqrt{2}+\sqrt[3]{2}$ f. $\sqrt{28}+\sqrt{175}$ g. $\left(\sqrt{h}+7\right)^2$ h. $\left(3-\sqrt{t}\right)^2$

8. Simplify so that there are no radicals in the denominator:

a. $\frac{4}{\sqrt{35}-5}$ b. $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ c. $\frac{12}{\sqrt{9}+3}$

9. Simplify each expression:

a. $\sqrt{18}\sqrt{8}$ b. $\sqrt{7}-\sqrt{7}$ c. $\frac{1+\sqrt{7}}{\sqrt{7}}$ d. $\sqrt{50}-\sqrt{8}$

e. $\frac{3}{\sqrt{28}+5}$ f. $\frac{\sqrt{a}+b}{\sqrt{a}-b}$ g. $\frac{3-\sqrt{25}}{\sqrt{25}-3}$

10. Let $r_1 = \frac{3+\sqrt{11}}{4}$ and $r_2 = \frac{3-\sqrt{11}}{4}$.

Prove a. $r_1 + r_2 = \frac{3}{2}$ b. $r_1 r_2 = -\frac{1}{8}$

11. True/False:

a. 36 has two square roots.

b. $\sqrt{49} = \pm 7$

c. If $x \le 0$, then $\sqrt{x}$ is not a real number.

d. 64 has exactly one cube root.

e. $-\sqrt[4]{20}$ is a real number.

f. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

g. $\sqrt{ab} = \sqrt{a}\sqrt{b}$

h. $\sqrt[3]{18} = 3\sqrt{2}$

i. $\sqrt{20} + \sqrt{5} = 5$

j. $\sqrt{20} + \sqrt{5} = 3\sqrt{5}$

k. $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$

l. $\frac{7}{\sqrt{n}} = 7\sqrt{n}$

m. $\frac{1}{\sqrt{n}+1} = \frac{\sqrt{n}-1}{n-1}$

n. $\sqrt[3]{x^2} + \sqrt[3]{x} = x$

Solutions

1. a. $12\sqrt{x}$ b. As is c. $13\sqrt[3]{n}$ d. As is e. $68\sqrt{2}$
f. $5\sqrt[3]{3}$ g. $-3\sqrt{5}$ h. $32\sqrt[3]{7}$ i. $3\sqrt[3]{z^2}$ j. $12\sqrt{3}$
k. $10\sqrt{5}$ l. $\sqrt{11}$

2. a. $\sqrt{84} = 2\sqrt{21}$ b. $\sqrt[3]{27} = 3$ c. $\sqrt{14}$
d. 171 e. $\sqrt{28} = 2\sqrt{7}$ f. $\sqrt[3]{125} = 5$
g. $\sqrt[4]{16} = 2$ h. $\sqrt{xyz}$ i. $\sqrt[3]{8} = 2$

3. a. $\sqrt{n} + n$ b. $x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$
c. $4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$ d. $\left(\sqrt{p} - q\right)\left(\sqrt{p} - q\right) = p - 2q\sqrt{p} + q^2$
e. $7 - 4 = 3$ f. $x^2 - 2x\sqrt{n} + n$ g. $a - c$
h. $2\left(\sqrt{3} + 2\right) = 2\sqrt{3} + 4$ i. $\left(2\sqrt{x}\right)^2 = 4x$

4. a. $\dfrac{2}{\sqrt{8}}\left[\dfrac{\sqrt{8}}{\sqrt{8}}\right] = \dfrac{2\sqrt{8}}{8} = \dfrac{4\sqrt{2}}{8} = \dfrac{\sqrt{2}}{2}$ b. $x\sqrt{x}$

c. $\dfrac{3}{5+\sqrt{2}}\left[\dfrac{5-\sqrt{2}}{5-\sqrt{2}}\right] = \dfrac{15-3\sqrt{2}}{25-2} = \dfrac{15-3\sqrt{2}}{23}$ d. $\dfrac{x + 2\sqrt{xy} + y}{x - y}$

e. $\dfrac{\sqrt{3}}{3}$ f. 2 g. $4\sqrt{2}$ h. $\dfrac{\sqrt{xy}}{y}$

i. $\dfrac{5}{1+\sqrt{5}}\left[\dfrac{1-\sqrt{5}}{1-\sqrt{5}}\right] = \dfrac{5-5\sqrt{5}}{1-5} = \dfrac{5-5\sqrt{5}}{-4} = \dfrac{5-5\sqrt{5}}{-4}\left[\dfrac{-1}{-1}\right] = \dfrac{5\sqrt{5}-5}{4}$

The last step is optional. It was done just to remove the negative sign from the bottom of the fraction.

j. $-3 - 2\sqrt{2}$ k. $\dfrac{7}{3\sqrt{2}}\left[\dfrac{\sqrt{2}}{\sqrt{2}}\right] = \dfrac{7\sqrt{2}}{3 \cdot 2} = \dfrac{7\sqrt{2}}{6}$

l. $\dfrac{\sqrt{x}}{n}$ m. $\dfrac{7\sqrt{7}-7}{6}$ n. $\dfrac{a^2 + 2a\sqrt{b} + b}{a^2 - b}$ o. $\dfrac{a\sqrt{c}}{bc}$ p. 1

5. c. and d. are valid — a. and b. are bogus.

6. a. ± 4 b. -10 c. None

7. a. $\sqrt{35}$ b. As is c. $\frac{\sqrt{35}}{5}$ d. As is
e. As is f. $7\sqrt{7}$ g. $h+14\sqrt{h}+49$ h. $9-6\sqrt{t}+t$

8. a. $\frac{2\sqrt{35}+10}{5}$ b. $\frac{a-2\sqrt{ab}+b}{a-b}$ c. 2

9. a. 12 b. 0 c. $\frac{\sqrt{7}+7}{7}$ d. $3\sqrt{2}$
e. $2\sqrt{7}-5$ f. $\frac{a+2b\sqrt{a}+b^2}{a-b^2}$ g. -1

10. Do the arithmetic and it should work out.

11. a. T b. F c. F d. T e. T f. F g. T h. F
i. F j. T k. T l. F m. T n. F

"Every job is a self-portrait of the person who did it. Autograph your work with excellence."

– Unknown