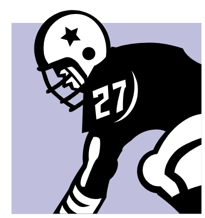
CH 35 – FUNCTIONS

❑ INTRODUCTION

The notion of a *function* is much more abstract than most of the algebra you've seen so far, so we'll start with three specific non-math examples.

❑ THE MEANING OF A FUNCTION



First Example: A football game is divided into four quarters. The following table shows each quarter together with the total number of points scored during that quarter.

Quarter	Number of Points Scored
1st	21
2nd	9
3rd	0
4th	25
Inputs	Outputs

This table of inputs and outputs is a ***function***, and here's why:

First, we have a set of **inputs**, the four quarters:

1st 2nd 3rd 4th

Second, we have a set of **outputs**, the number of points scored:

21 9 0 25

Third, there's a definite connection, or correspondence, between the inputs and the outputs. For example, the input "2nd" is associated with the output "9." What about the input "4th"? The output is 25.

Similarly, "21" is the output for the input "1st," while "3rd" produces the output "0."

Fourth, and most important, notice that each input has exactly one output. For instance, the input "2nd" has exactly one output, namely "9." There is no doubt that the output is 9 — just look at the table. This is the essence of a function, and so our table of quarters and points scored is a function.

Second Example: So, what isn't a function? Consider the following table:

City	Major League Baseball Teams
San Francisco	Giants
Oakland	Athletics
New York	Yankees and Mets

Inputs Outputs

The input San Francisco has exactly one output, Giants, because the city of San Francisco has exactly one major league baseball team. The input Oakland also has exactly one output associated with it. So far, so good, in terms of being a function. But check out New York. It has two baseball teams, the Yankees and the Mets. We thus have an input (New York) with more than one output (Yankees and Mets). This kills the function concept, and we conclude that our table of cities and baseball teams, although accurate and useful, is not a function.

To summarize this example, for something to be a function, there must be exactly one output for each input; this example has an input with two outputs. That's why the city/team pairings do NOT make a function.

Third Example:

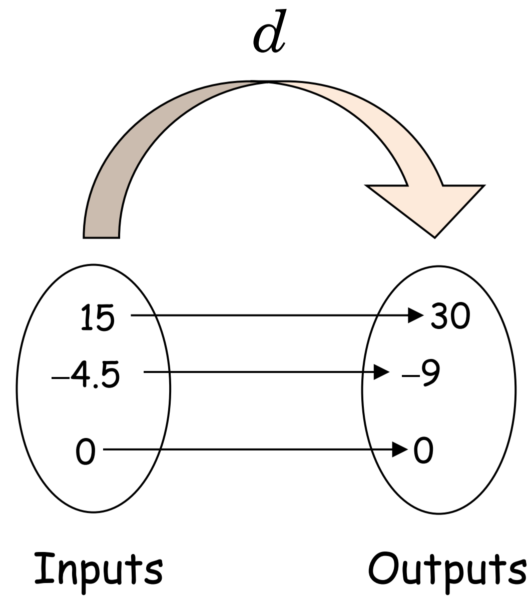
City	Population
Elmville	35,000
Gomerville	47,500
Moeville	35,000
Inputs	Outputs

Is this table a function? Does each input have exactly one output? YES, and I'll prove it to you. The input Elmville has one output (35,000), the input Gomerville has one output (47,500), and the input Moeville has one output (35,000). Does it matter that Elmville and Moeville have the same population? Not at all; it's just a coincidence. The fact is, every city has exactly one population associated with it; that is, each input has exactly one output. We must conclude that the population table is a function.

❑ *MATHEMATICAL EXAMPLES OF FUNCTIONS*

EXAMPLE 1: **Analyze the "doubling" function.**

Solution: Let d be the "doubling" function. This function takes an input, and produces an output that is double the input. For example, if the input is 15, then the output is double 15, or 30. That is, the function d takes an input of 15 and produces an output of 30. In the diagram below we see three inputs (15, –4.5, and 0) producing three outputs (30, –9, and 0).



We can write the fact that d takes 15 to 30 as follows:

$d(15) \; = \; 30$ read: "d of 15 = 30"

or, "d at 15 = 30"

Function Name Input Output

The name of a function can be anything we choose. It can consist of any letter (of any alphabet) or even a group of letters. For example, in computer programming, "SQRT" may be the name given to the square root function.

Here are some more examples of inputs and outputs for the doubling function, d:

$d(-3.25) \; = \; -6.5$ $d(\sqrt{2}) \; = \; 2\sqrt{2}$

$d(-10) = -20$ $d\left(\frac{\pi}{2}\right) \; = \; \pi$

To describe d as a formula (not all functions have formulas), we can write

$$d(x) = 2x \qquad \text{“}d \text{ of } x \text{ equals } 2x\text{”}$$

The output of *d* is twice the input.

Notice that any real number can be doubled, so any real number could be used as the input to this function.

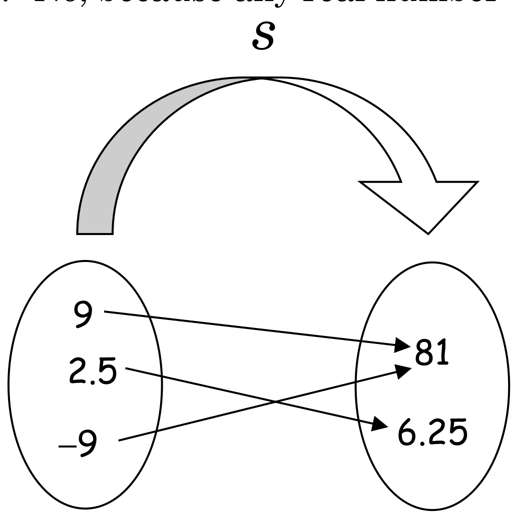
Final observation: Given an input to the function *d*, notice that there is **exactly one** output. For example, an input of 25 produces an output of 50, and only 50! No reasonable person could claim that the double of 25 is anything besides 50. This is the cornerstone of a function.

EXAMPLE 2:

A. Let s be the “squaring” function. For example $s(9) = 81$, $s(2.5) = 6.25$, and $s(-9) = 81$. As a formula, we write

$$s(x) = x^2 \qquad \text{“}s \text{ of } x\text{” is } x^2$$

Is there any number which would be illegal to use in this function? No, because any real number can be squared.



$s(9) = 81$

$s(2.5) = 6.25$

$s(-9) = 81$

Why is s a function? Because, given an input, there is exactly one output for it. Notice that even though both 9 and –9 produce the same output of 81, it nevertheless is still the case that 9 squared is 81, and only 81 — and that –9 squared is 81, and only 81. Each input has exactly one output: s is a function.

B. Let t be the "adding 10" function. Then $t(x) = x + 10$ is an appropriate formula; for example $t(23) = 33$. Any real number could be used for x, and t is a function because adding 10 to an input produces only one possible output.

C. Define r to be the "non-negative square root" function, $r(x) = \sqrt{x}$. For instance, $r(49) = 7$ and $r(2) = \sqrt{2}$. Recall that the formula represents the non-negative square root only.

What inputs are allowed in this function? Since negative numbers don't have real number square roots, all inputs must be at least zero. That is, x must be ≥ 0.

Also, r is a function because given any legal input ($x \geq 0$), we obtain exactly one output.

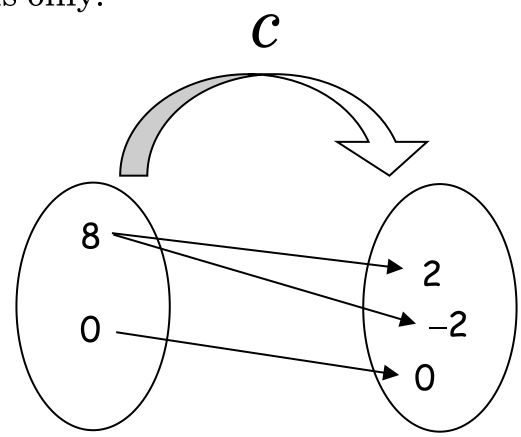
D. Let m be the "reciprocal" function; that is, $m(x) = \frac{1}{x}$. ("m of x is the reciprocal of x.) Thus,

$$m(7) = \frac{1}{7} \qquad m\left(-\frac{2}{3}\right) = -\frac{3}{2} \qquad m(0) = \text{undefined}$$

What can x be in this function? Well, the only number we can't divide by is 0, so x can be any real number <u>except</u> 0.

Starting with any legal input ($x \neq 0$), we obtain exactly one output. We conclude that m is a function.

E. Let c be defined by the formula $c(x) = \pm\sqrt[3]{x}$. Is c a function? Suppose we take the input $x = 8$. Ask a person what the output is, and he might say 2, since 2 is indeed the positive cube root of 8. But ask another person, and she might say –2, since –2 is certainly the negative cube root of 8. Who's right? They're both right! There's nothing illegal or immoral about our formula for c. The fact is, it's simply <u>not</u> a function — there's an input that produces more than one output. As we'll see a little later, the "$c(x)$" notation we used in the formula for this example is to be reserved for functions only.



Homework

1. Using the functions in Example 2, calculate each output:

a. The squaring function: $s(2/3)$ $s(-15)$ $s(\pi)$
Is there any number which cannot be used as an input to s?

b. The adding 10 function: $t(99)$ $t(-10)$ $t(1/2)$
Is there any number which cannot be used as an input to t?

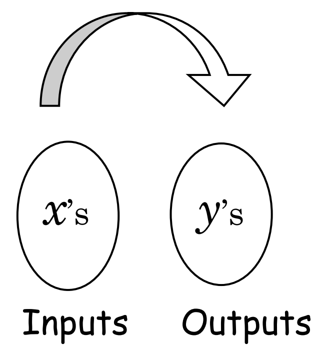
c. The square root function: $r(144)$ $r(1)$ $r(5)$
Is there any number which cannot be used as an input to r?

d. The reciprocal function: $m(1/4)$ $m(-3)$ $m(1)$
Is there any number which cannot be used as an input to m?

2. Let k represent the "cubing" function.

   a. An input of 10 produces what output?
   b. Calculate $k(4)$, $k(0)$, and $k(-5)$.
   c. Write a formula for k.
   d. Explain why k is a function.
   e. If the output was -216, what was the input?

❑ FORMULAS



When a function is written as a formula with x's and y's, we usually assume that the x's are the inputs and the y's are the outputs.

EXAMPLE 3:

A. $y = 2x + 7$ is a function.

Given an input x, there is exactly one output y. For example, if $x = 4$, then $y = 15$ and nothing else. [You may recall that this would be called a *linear* function, and its graph is a straight line.]

B. $y = x^4$ is a function.

Given an input x, there is exactly one output y. For instance, $x = 3$ produces $y = 81$. If $x = 0$, then $y = 0$. And if $x = -3$, then the unique output is 81. So, for any value of x, the formula produces exactly one value of y. This is why it's a function.

C. $y = -3x^2 + 7x - 19$ is a function. [In fact, it's a *quadratic* function.]

D. $y = |x|$ is a function.

If $x = 7$, then $y = 7$, exactly one output for the input of 7. If $x = -7$, then $y = 7$, exactly one output for the input of -7.

E. $x = |y|$ is not a function.

Why isn't it a function? We need to conjure up some input (an x) which has more than one output (y). Let's choose $x = 4$. This gives us the equation $4 = |y|$, and there are two solutions to this equation; namely, y could be 4, or y could be -4. We have more than one output for a single input. This formula is not a function.

F. $3x - 7y = 19$ is a function.

Here we could solve for y,

$$y = \frac{3}{7}x - \frac{19}{7},$$

and see that given an x, there's only one y for it, so this is a (linear) function.

G. $x^2 + y^2 = 4$ is not a function.

Let $x = 0$. Then $y^2 = 4 \Rightarrow y = \pm 2$. That is, an input of 0 results in two outputs, 2 and -2. Thus, this formula (which is a circle) is not a function.

H. $y = \pm\sqrt{x}$ is not a function.

If we choose 81 as the input, we get two outputs: ± 9, violating the fundamental notion of a function.

I. $y = 5$ is a function.

Given any input (the x), there's only one output, namely 5. Therefore, this horizontal line is a function.

J. $x = 2$ is <u>not</u> a function.

In fact, this example probably holds the world's record for being a non-function. After all, given an input (the only choice is 2), there are an <u>infinite</u> number of outputs (y can be anything). We thus see that a vertical line is not a function.

x	y
2	5
2	0
2	–3

Homework

3. Consider the formula $y = x^3$. Recall the x is an input and y is an output. When a value of x has been assigned, there's only one y value. What does this mean?

4. Consider $y^2 = x$. When $x = 25$, y has two values: 5 and –5. What does this mean?

5. Consider $x = |y+10|$.
   a. When $x = 20$, y has two values. Verify this fact.
   b. What does this mean?

6. Determine if the given formula is a function:

a. $y = -7x + 9$ b. $x = 2y + 5$ c. $y = x^5$

d. $y^2 = x$ e. $x^2 + y^2 = 14$ f. $y = |7x + 2|$

g. $x^2 - y^2 = 9$ h. $x = \pi$ i. $y = \sqrt{2}$

j. $y = \pm\sqrt[3]{x-1}$ k. $y = \dfrac{x+1}{x-3}$ l. $y = \sqrt{2x^2 + x + 1}$

m. $x = |y + 5|$ n. $x^2 + y^2 = 1$ o. $y = \sqrt[4]{1-x}$

p. $y = x^3 - x^2 + x$ q. $x - 5 = 0$ r. $y + 2\pi = 0$

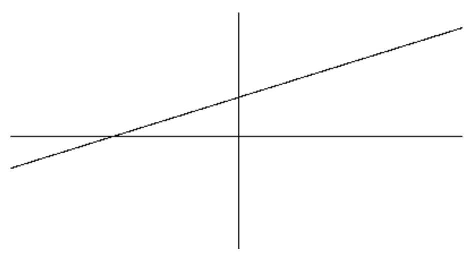
s. $3x - 7y = 8$ t. $y = |10 - x|$ u. $x = |1 - y|$

❑ *GRAPHS*

Now it's time to determine if a given graph is a function or not. We assume that the standard x- and y-axes are used, and as before, we agree that x is the input and y is the output. Recalling that a function must produce <u>exactly one</u> output for each legal input, we look at the following examples.

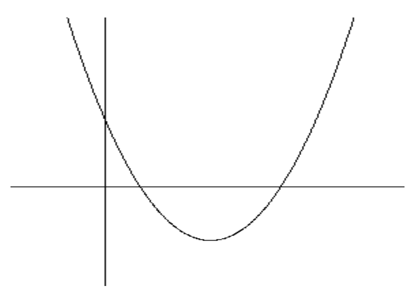
<u>EXAMPLE 4:</u>

A.



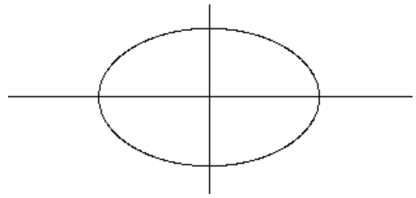
Choose any x-value on the x-axis — this is the input. Now go straight up or down till you get to the graph to find the y-value — this is the output. How many outputs did you get? You should have gotten exactly one output. Whatever legal x you use, there's exactly one y for it. The graph <u>is</u> a function.

B.



Pick any input on the x-axis. Go up or down until you hit the graph — the y-value is the output. Every legal value of x produces exactly one output. This graph <u>is</u> also a function.

c.



Choose a legal input (say, $x = 0$). Now go find the graph. This time we run into the graph twice, once going up and once going down. We have an x which has two different y's. That is, we have a legal input with more than one output. This graph is definitely not a function.

The following chart indicates other examples of graphs and if they are functions:

FUNCTIONS	NON-FUNCTIONS
single point	vertical line
horizontal line	circle
top-half of a circle	ellipse
non-vertical line	right-half of a circle
parabola opening up	parabola opening right

Homework

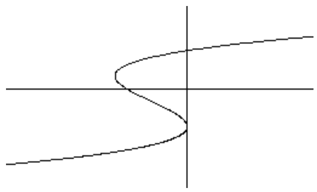
7. T/F: No line is a function.

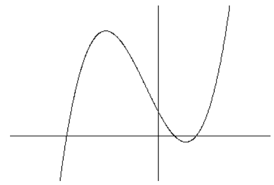
8. Which of the following are functions?

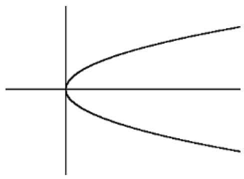
   a. horizontal line b. bottom half of a circle c. left half of a circle

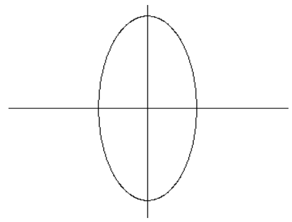
9. Explain why a graph consisting of a single point is a function.

10. Which of the following are functions?

a. 

b. 

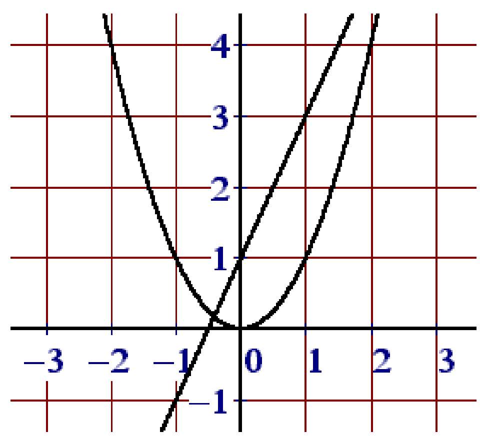
c. 

d. 

❑ *FUNCTION NOTATION*

You may be a little confused regarding notation. Earlier in the chapter we used letters like d, s, and k to represent the names of functions and wrote formulas like $d(x) = 2x$ to describe what a function does. But later we used the letter y, for example, $y = 2x + 3$. Both notations are needed in math and other disciplines, but the function notation with the parentheses is better. The next paragraph tries to explain why.

Suppose we are talking about two functions in the same problem, for example, the line $y = 2x + 1$ and the parabola $y = x^2$. I now ask you what the y-value is when $x = 1$. Is the answer 3 (from the line) or is the answer 1 (from the parabola)? It depends on which formula you chose. This is not an acceptable situation — we'd never be able to communicate this way. This is why we use the notation with the different letters and the parentheses.



If x = 1, what does y equal? It depends on which graph you're looking at.

Thus, if we give our line and parabola functions names like

$$f(x) = 2x+1 \quad \text{and} \quad g(x) = x^2$$

it's fair to ask you what $f(1)$ is, and you'll know for sure that it's 3, because I gave you both the input (the 1), and the function (f). On the other hand, if I asked what $g(1)$ is, you'd be confident that it's 1. Get it??

<u>EXAMPLE 5:</u> **Given the three functions**

$$\boldsymbol{f(x) = 3x-10},$$

$$\boldsymbol{g(x) = x^2 + 1},$$

$$\boldsymbol{h(x) = \frac{1}{x}},$$

calculate each of the functional values:

A. $f(5) = 3(5)-10 = 15-10 = \mathbf{5}$

B. $g(5) = 5^2 + 1 = 25 + 1 = \mathbf{26}$

C. $h(5) = \mathbf{\frac{1}{5}}$

D. $f(10) = 3(10)-10 = 30-10 = \mathbf{20}$

E. $g(\sqrt{7}) = (\sqrt{7})^2 + 1 = 7+1 = \mathbf{8}$

F. $h(0) = \frac{1}{0}$, which is **undefined**

Homework

11. Let $f(x) = x^2 + 3x$

$g(x) = 5 - x$

$h(x) = \sqrt[3]{x+1}$

Calculate:

a. $f(7)$	b. $g(7)$	c. $f(0)$	d. $g(0)$
e. $f(10)$	f. $g(10)$	g. $f(-5)$	h. $g(-5)$
i. $h(0)$	j. $h(-28)$	k. $h(-1)$	l. $f(-1)$

❑ *COMPOSITION OF FUNCTIONS*

Computer Language Functions

Let's look at how a computer language handles functions by looking at three functions: two that are built into the language, `abs` and `sqrt`, and one that we'll define ourselves.

You may have guessed that **`abs`** stands for absolute value. For example, the `abs` function produces the outputs for the given inputs:

`abs`(7) = 7 `abs`(–12) = 12 `abs`(0) = 0

As for the square root function, **`sqrt`**, a computer would calculate like this:

`sqrt`(81) = 9 `sqrt`(0) = 0 `sqrt`(–4) = *Error*

Our third example will be a *user-defined* function; this is a function that does not come pre-built into the language, but one that we create ourselves.

Notice that the square root of a negative number does not exist as a real number — hence the *Error* in the `sqrt` calculations above. So how can a programmer absolutely guarantee that her program will never accidentally try to take the square root of a negative number? Here's one way: Take the number's absolute value first, then feed that result (which can't be negative) into the square root function:

`sqrt`(`abs`(–9)) would produce an output of **3**.

Do you see why? Starting with the input, –9, <u>first</u> its absolute value is calculated, which results in the number 9. Second, the positive square root of the 9 is taken, with a final output of 3. Get it? What if we start with a positive number? No problem — for example,

`sqrt`(`abs`(100)) would produce an output of **100**.

The absolute value of 100 is still 100, whose square root is 10. Whether the input is positive, zero, or negative, our new function will always be able to take a square root, with no possible Error messages. And this new function will even work perfectly if the input is 0.

Just for the heck of it, let's reverse the order of the functions which comprise our new user-defined function, and see how it handles an input of –25:

`abs(sqrt(–25)) = abs(`***Error!***`)`

It appears that the order in which you carry out your functions might make everything fall apart, thus defeating the purpose of a user-defined function.

And we can even name our user-defined function (the one that works, not the one that starts with `abs`), and have it stored in the computer language, available anytime we need it. For instance, we might call the new function `RealSqrt`, meaning that our new function will always result in a real number and never produce an Error message — a good thing for a programmer to have. Depending on the computer language, it might be as simple to create as this:

```
Function RealSqrt(x)
    RealSqrt(x) = sqrt(abs(x))
End Function
```

Our new function, `RealSqrt`, is actually *composed* of two functions, `abs` and `sqrt`, performed one after the other. As such, we say that our new function is the ***composition*** of the two functions. So, if you wanted to use the function with an input of –100, you could write

`RealSqrt(–100)`, and the output would be `10`.

Mathematical Functions

Suppose $f(x) = x^2$ and $h(x) = x - 10$. Calculate the composition:

$$f\big(h(5)\big)$$

We calculate $h(5)$ first (since it's inside the parentheses), and then calculate f of whatever the $h(5)$ was:

First, $h(5) = 5 - 10 = -5$

Therefore, $f\big(h(5)\big) = f(-5) = (-5)^2 = \mathbf{25}$

We know that the commutative property of multiplication states that $ab = ba$ for any numbers a and b. Let's determine whether this same property holds for function composition. Let's keep the same functions, but calculate the composition in reverse order:

$$h\big(f(5)\big) = h(25) = \mathbf{15}$$

We got <u>different</u> answers (the 25 and the15) depending on the order in which applied the functions. We deduce that function composition is <u>not</u> a commutative operation. [After all, putting on your socks and then your shoes is obviously not the same as putting on your shoes and then your socks.]

Homework

12. If $f(x) = x^3$ and $g(x) = x + 7$,
calculate $f\big(g(3)\big)$ and $g\big(f(3)\big)$.

13. If $P(x) = \sqrt{x}$ and $Q(x) = \frac{1}{x}$,
calculate $P\left(Q\left(\frac{1}{9}\right)\right)$ and $Q\left(P\left(\frac{1}{9}\right)\right)$.

14. If $h(x) = 2x^2 + x$ and $k(x) = |-x - 10|$,
calculate $h\big(k(-4)\big)$ and $k\big(h(-4)\big)$.

Practice Problems

15. Let T be the "tripling" function.
    a. An input of 10 produces what output?
    b. Calculate $T(4)$, $T(0)$, and $T(-5)$.
    c. Write a formula for T.
    d. Explain why T is a function.
    e. If the output was -216, what was the input?

16. What is the only number that cannot be the input to the reciprocal function?

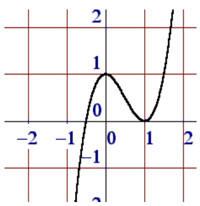
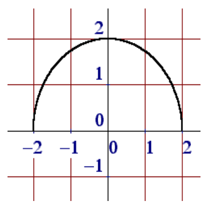
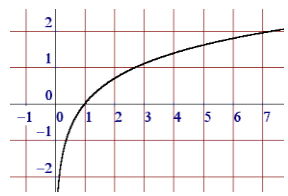
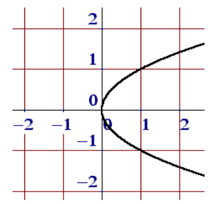
17. Consider the formula $x^2 + y^2 = 1$. Explain why it's not a function.

18. Consider the formula $y = \pm\sqrt{x}$. If $x = 9$, what is y? Is this a function?

19. Is it a function?

a. $x = \lvert y+1 \rvert$	b. $y = x^3$	c. $y = \pm\sqrt{x-3}$
d. $x = 5$	e. $y = -\pi$	f. $x^2 = y^2$
g. $7x - 9y = 10$	h. $y = \lvert x+5 \rvert$	i. $y = 7x^2 - \pi x^{10}$
j. $x^2 + y^2 = 49$	k. $x = 0$	l. $y = 0$

20. Which are functions?

21. True/False:

a. In the tripling function, if the input is 12, the output is 36.

b. In the cubing function, if the output is 125, the input was 5.

c. $x = y^4$ is a function.

d. $y = \lvert x-2 \rvert$ is a function.

e. $y = \pm\sqrt[3]{x-\pi}$ is a function.

f. $x = 9$ is a function.

g. $y = -\pi$ is a function.

h. $x^2 + y^2 = 2$ is a function.

i. Every line is a function.

j. Every semicircle is a function.

k. $x = \lvert y-1 \rvert$ is a function.

l. If $f(x) = \sqrt{x-10}$, then $f(35) = 5$.

Solutions

1.

a.	4/9	225	π^2	Every number can be an input.
b.	109	0	$10\frac{1}{2}$	Every number can have 10 added to it.
c.	12	1	$\sqrt{5}$	No negative number can be square rooted.
d.	4	–1/3	1	The only number without a reciprocal is 0.

2. a. $10^3 = 1000$ b. $k(4) = 4^3 = 64$; $k(0) = 0$; $k(-5) = -125$

c. $k(x) = x^3$

d. Given a number, cubing it produces only one number.

e. –6

3. It means that the formula represents a function, since each input has a unique output.

4. It means that the formula does <u>not</u> represent a function, since there's an input with more than one output.

5. a. If $x = 20$, then $20 = |y+10|$. This implies that

$y + 10 = 20$ or $y + 10 = -20$, giving two solutions: $y = 10, -30$.

b. One input (20) produced two outputs (10 and –30). Therefore, the formula does <u>not</u> represent a function.

6. a. Yes, given an input (x), there's only one output (y).

b. Yes, solve for y and it's just like part a.

c. Yes, one input produces exactly one output.

d. No, an x-value of 4 produces two y-values, namely 2 and –2.

e. No, an x-value of 0 produces two y-values.

f. Yes

g. No, if $x = 10$, then y has two values.

h. No, for $x = \pi$, there are an infinite number of y's.

i. Yes, given any input, the output must be $\sqrt{2}$, so there's a unique output for each input.

j. No, if $x = 9$, then y has two values, 2 and –2.

k. Yes, put in any legal value of x, and only one y-value will result.

l. Yes, it may be complicated but only one y-value will appear for a given x-value.
m. No, if $x = 2$, then y can be either -3 or -7.
n. No, let $x = 0$ and see what you get.
o. Yes, as long as an appropriate x is chosen, the 4th root produces exactly one answer.
p. Yes q. No r. Yes s. Yes t. Yes u. No

7. F (actually, most lines are functions)

8. a. and b.

9. When you choose the only legal x, you're already at the y-value, and there's only one.

10. b. only

11. a. $f(7) = 7^2 + 3(7) = 49 + 21 = 70$ b. $g(7) = 5 - 7 = -2$
c. 0 d. 5 e. 130 f. -5 g. 10 h. 10
i. 1 j. -3 k. 0 l. -2

12. 1000; 34 **13**. 3; 3 **14**. 70; 38

15. a. 30 b. 12; 0; -15 c. $T(x) = 3x$
d. For each input, there's exactly one output.
e. -72

16. 0

17. If $x = 0$, $y = \pm 1$; an input can produce two outputs, so it's not a function. Or, the graph is a circle, which is not a function.

18. $y = \pm 3$; it is not a function, since a single input of 9 produced two outputs.

19. a. No b. Yes c. No d. No e. Yes f. No
g. Yes h. Yes i. Yes j. No k. No l. Yes

20. All but the 4th

21. a. T b. T c. F d. T e. F f. F g. T h. F
i. F j. F k. F l. T

"I'm a great believer in luck,
and I find the harder I work,
the more I have of it."

– Thomas Jefferson