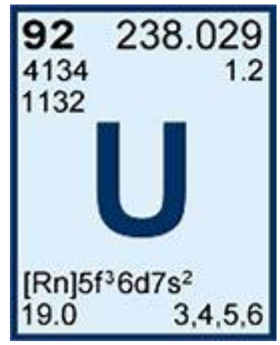
CH 38 – INTRO TO EXPONENTIAL EQUATIONS

❑ *INTRODUCTION*

Watching your investments grow, tracking populations, measuring the decay of a radioactive substance — these are the kinds of problems which lead to exponential equations.



Uranium, atomic number 92, decays exponentially – the less that remains, the less that decays.

❑ *EXAMPLES*

An ***exponential equation*** is an equation with the variable in the exponent, not something we're used to seeing. Let's start with two basic examples.

#1: Consider the exponential equation

$$5^x = 25$$

Five raised to what power equals 25? Well, 5 to the 2nd power is 25, so $\boldsymbol{x = 2}$. Not so tough.

#2: How about the equation

$$3^{2n-1} = 81?$$

You might go about solving this equation by asking yourself "3 to what power is 81?" Since 3 to the 4th power is 81, it follows that the exponent, $2n - 1$, must be equal to 4.

Writing this last phrase as an equation, we can find the value of n:

$$2n - 1 = 4 \Rightarrow 2n = 5 \Rightarrow \boldsymbol{n = \frac{5}{2}}.\ \text{Done!}$$

Check: Placing $n = \frac{5}{2}$ into the equation $3^{2n-1} = 81$ gives

$$3^{2\left(\frac{5}{2}\right)-1} \stackrel{?}{=} 81$$
$$3^{\cancel{2}\left(\frac{5}{\cancel{2}}\right)-1} \stackrel{?}{=} 81$$
$$3^{5-1} \stackrel{?}{=} 81$$
$$3^{4} \stackrel{?}{=} 81$$
$$81 = 81 \checkmark$$

EXAMPLE 1: **Solve for x: $3^{7x} = 3^{14}$**

Solution: Each side of the equation is an exponential expression. Notice that the bases are the same (the 3's), so the only way the two sides of the equation can be equal is if the exponents are equal. In other words, $7x$ must equal 14:

$$7x = 14,$$

from which we determine that

$$\boxed{x = 2}$$

EXAMPLE 2: **Solve for y: $5^{4y} = 25$**

Solution: We're not as lucky here as we were in Example 1 — the bases are not the same. But maybe we can make them the same. Suppose we think of 25 as 5^2. Then each side of the equation will have the same base, and we can set the exponents equal to each other to find the value of y. Let's try all of this:

$$5^{4y} = 25 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad 5^{4y} = 5^2 \qquad \text{(rewrite 25 with a base of 5)}$$

$$\Rightarrow \quad 4y = 2 \qquad \text{(set the exponents equal to each other, since the bases are the same)}$$

$$\Rightarrow \quad \boxed{y = \frac{1}{2}}$$

<u>EXAMPLE 3:</u> **Solve for z: $27^{-4z} = \frac{1}{9}$**

<u>Solution:</u> This one's terrible! The bases aren't the same, and it contains a fraction. But look at the 27 in the equation — it's equal to 3^3. And check out the 9 in the denominator — it can be written as 3^2. So maybe we can write everything in terms of the base **3**:

$$27^{-4z} = \frac{1}{9} \qquad \text{(the original equation)}$$

$$\Rightarrow \quad (3^3)^{-4z} = \frac{1}{3^2} \qquad \text{(express 27 and 9 as powers of 3)}$$

$$\Rightarrow \quad 3^{-12z} = 3^{-2} \qquad \text{(exponent rules)}$$

$$\Rightarrow \quad -12z = -2 \qquad \text{(bases are the same - set exponents equal)}$$

$$\Rightarrow \quad \boxed{z = \frac{1}{6}} \qquad \text{(solve for } z\text{)}$$

Homework

Solve each exponential equation:

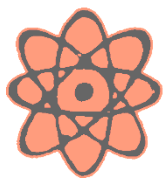
1. $16^{6p} = 8$
2. $4^{10q} = 16$
3. $125^{3t} = \frac{1}{5}$
4. $16^{6w} = 16$
5. $4^{-2h} = 64$
6. $25^{6b} = \frac{1}{25}$
7. $25^{8w} = 125$
8. $625^{-4x} = \frac{1}{625}$
9. $125^{-10a} = \frac{1}{25}$
10. $16^{3v} = 2$
11. $4^{9r} = \frac{1}{2}$
12. $9^{-4p} = 729$
13. $625^{-3y} = \frac{1}{25}$
14. $81^{9x} = \frac{1}{729}$
15. $25^{-2k} = \frac{1}{25}$
16. $16^{-5x} = 512$
17. $27^{8b} = \frac{1}{729}$
18. $81^{6a} = 3$
19. $81^{-5r} = 81$
20. $625^{6b} = \frac{1}{25}$
21. $27^{-5v} = 3$
22. $81^{-4k} = \frac{1}{9}$
23. $125^{5p} = \frac{1}{25}$
24. $125^{-10d} = \frac{1}{5}$

❑ *The Growth and Decay Formula*

Many things grow and shrink (decay) at a rate that's based on how much of the stuff there is at the moment.

For instance, the number of births in a population is based on the number of people living at the time (since the more people there are, the more <u>new</u> people there will be).

Another example is compound interest: As money accumulates (due to earned interest) in the account, the more interest the account will earn; that is, the interest will earn interest.



In science, we can observe the radioactive decay of an element like uranium. As the uranium disintegrates, less of it will decay, because as time goes on, there's less of it left to decay.

Here's a formula to help you predict how much of something there will be in the future. It works for population growth, continuous compounding of interest, and the decay of radioactive substances:

Let

A_0 = starting amount (read: "A sub zero" or "A naught")

A = ending amount

e = a constant whose value is **approximately** 2.718.

k = growth or decay rate (expressed as a decimal)

t = time

Then

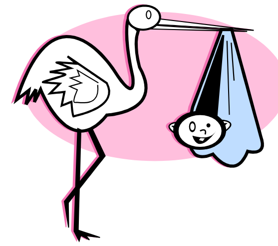
$$\boxed{A = A_0 e^{kt}}$$

Get your calculator out. Here's a quick review of the "exponent" button. To calculate $(3.2)^{4.15}$, try either

3.2 $\boxed{y^x}$ 4.15 = OR 3.2 $\boxed{\wedge}$ 4.15 =

The answer is about 124.845.

<u>EXAMPLE 4:</u> **Assuming an initial population of 1506, and a growth rate of 25% per year compounded continuously, predict the population in 10 years.**



<u>Solution:</u> Let's start by writing the formula that will solve this problem:

$$A = A_0 e^{kt}$$

The initial population is 1506; so $A_0 = 1506$.

The growth rate is 25%; thus $k = 0.25$.

We're talking about a period of 10 years; therefore $t = 10$.

Plug all these values into our formula (using 2.718 for e):

$$A = 1506(2.718)^{(0.25)(10)}$$

$$\Rightarrow \quad A = 1506(2.718)^{2.5}$$

$$\Rightarrow \quad A = 1506(12.18) \qquad \text{(we'll round this result to 2 digits)}$$

$$\Rightarrow \quad \boxed{A = 18{,}343}$$

<u>EXAMPLE 5:</u> **Assuming a starting investment of \$2401, and an annual interest rate of 13% compounded continuously, predict the value of the investment in 11 years.**

<u>Solution:</u> The formula we used for population growth works equally well for compound interest.

$$A = A_0 e^{kt}$$

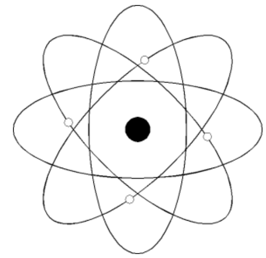
$$\Rightarrow \quad A = 2401(2.718)^{(0.13)(11)}$$

$$\Rightarrow \quad A = 2401(2.718)^{1.43}$$

$\Rightarrow \quad A = 2401(4.18)$ (we'll round this result to 2 digits)

$\Rightarrow \quad \boxed{A = \$10{,}036}$

<u>EXAMPLE 6:</u> **Starting with 624 grams of uranium, and assuming an annual <u>decay</u> rate of 5%, compute the number of grams of uranium remaining after 9 years.**



<u>Solution:</u> We use the same growth/decay formula except for one thing: Since the amount of uranium is shrinking (decaying) rather than growing, we will use a decay rate of **–5%** (that's <u>negative</u> 5 percent) in our formula:

$$A = A_0e^{kt}$$

$$\Rightarrow \quad A = 624(2.718)^{(-0.05)(9)}$$

$$\Rightarrow \quad A = 624(2.718)^{-0.45}$$

$\Rightarrow \quad A = 624(0.64)$ (we'll round this result to 2 digits)

$$\Rightarrow \quad \boxed{A = 399 \text{ grams}}$$

The number ***e*** used in the growth formula is one of the most important numbers in math, science, engineering, and business. Like the number π, this real number contains an infinite number of digits that never form a repeating pattern (called *irrational*).

Homework

<u>Note:</u> The result of raising 2.718 to the exponent is assumed to be rounded to 2 digits before the multiplication by A_0, as in the examples.

25. Starting with 178 grams of uranium, and assuming an annual decay rate of 17%, predict the number of grams remaining after 18 years.

26. Assuming an initial population of 3902, and a growth rate of 14% per year, determine the population in 10 years.

27. Assuming an initial investment of $1299, and an annual interest rate of 14% compounded continuously, predict the value of the investment in 4 years.

28. Assuming an initial investment of $9574, and an annual interest rate of 11% compounded continuously, compute the value of the investment in 2 years.

29. Starting with 416 grams of thorium, and assuming an annual decay rate of 3%, determine the number of grams remaining after 26 years.

30. Assuming an initial population of 2586, and a growth rate of 16% per year, predict the population in 4 years.

31. Assuming an initial population of 1422, and a growth rate of 12% per year, compute the population in 2 years.

32. Assuming an initial investment of $6897, and an annual interest rate of 14% compounded continuously, compute the value of the investment in 6 years.

33. Starting with 215 grams of plutonium, and assuming an annual decay rate of 15%, calculate the number of grams remaining after 26 years.

34. Assuming an initial population of 2159, and a growth rate of 6% per year, calculate the population in 11 years.

[Technically, the growth/decay formula we've been using applies only when the growth or decay is *continuous*, which means the growth or decay occurs at every single moment of time. This may not strictly be the case in all situations (for example, in the births of people), but let's not worry about it in this book. Let's just use the formula to solve the problems.]

❑ *THE BELL-SHAPED CURVE*

Some of you will take Statistics soon, and you will learn that the most important function in the course is a special bell-shaped curve called the ***standard normal*** curve, introduced in Chapter 1. It's an exponential function, and its formula is a bit complicated, but we have all the tools needed to find some of its points and then sketch it.

The formula for the standard normal curve is given by

$$\boxed{y = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}}$$

Note that this function is called an exponential function because the base is a constant (in this case, e) and the variable x occurs in the exponent. Let's decipher this formula before we try to plot any points. The coefficient of the exponential function is $\frac{1}{\sqrt{2\pi}}$, the base is e, and the exponent on the e is $-\frac{1}{2}x^2$.

Let's start calculating some (x, y) pairs for this function and see what kind of graph we get. Let's start by letting $x = 0$. This, of course, will yield the y-intercept.

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}}e^0 = \frac{1}{\sqrt{2\pi}}(1) \approx 0.3989$$

This gives us the approximate point **(0, 0.4)**. We're off to a good start.

Now we'll choose $x = 1$. Plugging this number into the function gives

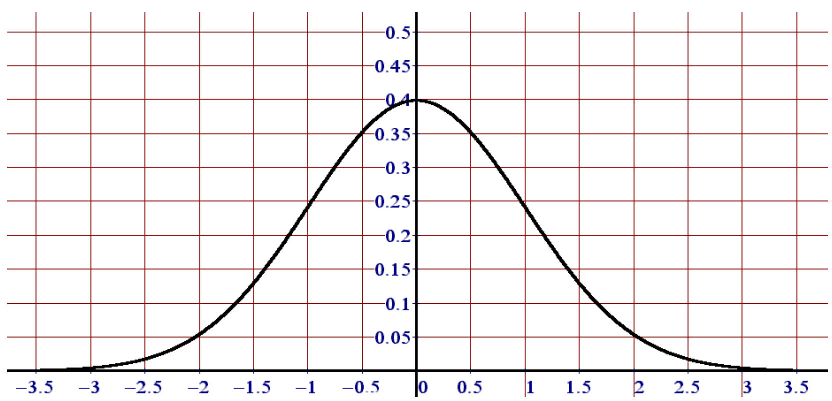
$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}(0.6065) \approx 0.2420$$

Our second point is therefore **(1, 0.24)**. Now for $x = -1$. You should be able to do this one in your head. First, the –1 is squared, giving 1 — but this is exactly what we had in the previous calculation, and so the y-value must also be 0.2420. Our third point is therefore **(–1, 0.24)**.

Choosing $x = 2$ (or –2) gives the following calculation for y:

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\pm 2)^2} = \frac{1}{\sqrt{2\pi}}e^{-2} = \frac{1}{\sqrt{2\pi}}(0.1353) \approx 0.0540$$

We now have two more points for our graph: **(2, 0.05)** and **(–2, 0.05)**. Continuing in this manner, we can find the points **(3, 0.0044)** and **(–3, 0.0044)**. It appears that as we get farther from the origin, the y-values are getting smaller and smaller, but always remaining positive. If we plot the seven points we've just computed and then connect them with a smooth curve, we get the following:



Practice Problems

35. Solve for x: $27^{x+1} = \frac{1}{9}$

36. Solve for x: $\frac{1}{125} = 25^{4-2x}$

37. Starting with 286 grams of uranium, and assuming an annual decay rate of 4%, predict the number of grams remaining after 26 years.

38. Assuming an initial population of 23,900 and a growth rate of 7% per year compounded continuously, predict the population in 12 years.

39. Assuming a starting investment of \$5575 and an annual interest rate of 12.5% compounded continuously, predict the value of the investment in 10 years.

Solutions

1. $p = \frac{1}{8}$	**2.** $q = \frac{1}{5}$	**3.** $t = -\frac{1}{9}$	**4.** $w = \frac{1}{6}$
5. $h = -\frac{3}{2}$	**6.** $b = -\frac{1}{6}$	**7.** $w = \frac{3}{16}$	**8.** $x = \frac{1}{4}$
9. $a = \frac{1}{15}$	**10.** $v = \frac{1}{12}$	**11.** $r = -\frac{1}{18}$	**12.** $p = -\frac{3}{4}$
13. $y = \frac{1}{6}$	**14.** $x = -\frac{1}{6}$	**15.** $k = \frac{1}{2}$	**16.** $x = -\frac{9}{20}$

17. $b = -\frac{1}{4}$	**18**. $a = \frac{1}{24}$	**19**. $r = -\frac{1}{5}$	**20**. $b = -\frac{1}{12}$
21. $v = -\frac{1}{15}$	**22**. $k = \frac{1}{8}$	**23**. $p = -\frac{2}{15}$	**24**. $d = \frac{1}{30}$
25. 9 g	**26**. 15,803	**27**. \$2273	**28**. \$11,968
29. 191 g	**30**. 4913	**31**. 1806	**32**. 16,001
33. 4 g	**34**. 4167	**35**. $x = -\frac{5}{3}$	**36**. $x = \frac{11}{4}$
37. 100 g	**38**. 55,448	**39**. \$19,457	

"*Education* is not merely a means for earning a living or an instrument for the acquisition of wealth. It is an initiation into life of spirit, a training of the human soul in the pursuit of truth and the practice of virtue."

– Vijaya Lakshmi Pandit