CH 40 – LOG AND EXPONENTIAL EQUATIONS

❑ INTRODUCTION

Do you recall the formula $A = A_0 e^{kt}$ for exponential growth and decay from Chapter 37? We used it to solve various problems where we wanted to determine either the final amount, A, or the initial amount, A_0, of whatever quantity is changing. But what if we wanted to find the time it takes for something to happen; that is, what if we're looking for t? Based on the algebra we've learned so far in this course, there's no way to isolate the t. We now remedy that issue.

First and foremost: Don't forget the definition of ***log***:

$$\log_b x = y \quad \text{means} \quad b^y = x$$

❑ SOLVING LOG EQUATIONS

EXAMPLE 1: **Solve for x: $\log_8 x - 2 = 0$**

Solution:

	$\log_8 x - 2 = 0$	(the original equation)
$\Rightarrow$	$\log_8 x = 2$	(first step to isolate the x)
$\Rightarrow$	$8^2 = x$	(change to exponential form)
$\Rightarrow$	$\boxed{x = 64}$	(calculate the value of x)

Quick Check: $\log_8 \mathbf{64}$ is 2, and 2 minus 2 equals 0. ✓

EXAMPLE 2: **Solve for y: $\log(3y + 5) = 1$**

Solution: Note: This is the common log, base 10.

	$\log(3y + 5) = 1$	(the original equation)
$\Rightarrow$	$10^1 = 3y + 5$	(change to exponential form)
$\Rightarrow$	$3y + 5 = 10$	(rearrange and simplify)
$\Rightarrow$	$3y = 5$	(subtract 5 from each side)
$\Rightarrow$	$\boxed{y = \frac{5}{3}}$	(divide each side by 3)

EXAMPLE 3: **Solve for x: $\ln(7 - 4x) = \frac{1}{2}$**

Solution: *ln* is just another log, and its base is understood to be e.

	$\ln(7 - 4x) = \frac{1}{2}$	(the original equation)
$\Rightarrow$	$e^{1/2} = 7 - 4x$	(convert to exponent form)
$\Rightarrow$	$4x = 7 - \sqrt{e}$	(move the $-4x$ and the $e^{1/2}$, and write the power of e in radical form)
$\Rightarrow$	$x = \frac{7 - \sqrt{e}}{4}$	(divide each side by 4)

So the exact solution is

$$\boxed{x = \frac{7 - \sqrt{e}}{4}}$$

We can use a calculator to approximate the solution as 1.338.

Homework

1. Solve each log equation:

a. $\log_6 x = 2$

b. $\log_5 x = 3$

c. $\log_2 x = \frac{1}{2}$

d. $\log_3 x = -4$

e. $\log_7(x+2) = 1$

f. $\log(2x-1) = 2$

g. $\ln x = 3$

h. $\ln(x-1) = 1$

i. $\ln(3x-5) = 0$

j. $\ln(2x+5) = \frac{1}{4}$

❑ *THE POWER PROPERTY OF LOGS*

To motivate this section, please consider trying to solve the exponential equation

$$3^x = 5$$

Could $x = 1$? No. $3^1 = 3$, too small.

Could $x = 2$? Also No. $3^2 = 9$, too big.

Since $3^1 = 3$ and $3^2 = 9$, it appears that x should be somewhere between 1 and 2; let's try $x = 1.5$. We'll calculate $3^{1.5}$ in two ways, just for practice:

$$3^{1.5} = 3^{1\frac{1}{2}} = 3^{3/2} = \sqrt{3^3} = \sqrt{27} = 3\sqrt{3}$$, which is clearly not 5.

And using a calculator, $3^{1.5} \approx 5.19615$, kind of close to 5, but certainly not close enough to claim we have a solution. We could keep guessing for a really long time — and we could even figure out a "solution" to any degree of accuracy we would want — but trust me: We would never get the exact answer by guessing. We need a new tool that will allow us to get a hold of that darned x up in the exponent. It's called the **Power Property of Logs**.

We know that $\log_{10} 1{,}000 = \mathbf{3}$ (since $10^3 = 1{,}000$). Now watch this:

	$\log_{10} 1{,}000$	
$=$	$\log_{10} 10^3$	(since $10^3 = 1{,}000$)
$=$	$3 \cdot \log_{10} 10$	**[just for giggles, bring the 3 down in front]**
$=$	$3 \cdot 1$	($\log_{10} 10 = 1$, since $10^1 = 10$)
$=$	$\mathbf{3}$	(do you really need a reason here?)

the same answer!

What did we do here? We took the exponent, the 3, and moved it down in front of the word *log*, making it a coefficient (3 times the log). Seems a bit tricky, but it worked. In fact, this always works:

$$\log_b x^n = n \log_b x$$

The Power Property of Logs

❑ *Solving Exponential Equations Using Logs*

EXAMPLE 4: **Solve for *x*: $3^x = 5$**

Solution: The variable is in the exponent. This is a dilemma. How do we get the unknown out of the exponent so that we can solve for it?

The Power Property of Logs comes to the rescue:

$$\log_b a^x = x \log_b a$$

It allows us to move the exponent to the front (making it a coefficient), but only if we're taking the log of an expression. So the procedure here will be to take a log (we'll choose *ln*, since it's on your calculator and *ln* is used in calculus), bring down the exponent, and then solve for it.

$$3^x = 5 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad \boldsymbol{ln}\ 3^x = \boldsymbol{ln}\ 5 \qquad \text{(take the } ln \text{ of both sides)}$$

$$\Rightarrow \quad x \ln 3 = \ln 5 \qquad \text{(the Power Property of Logs)}$$

$$\Rightarrow \quad x = \frac{\ln 5}{\ln 3} \qquad \text{(simple algebra – solve for } x\text{)}$$

$$\Rightarrow \quad x = \frac{1.609437912}{1.098612289} \qquad \text{(use your calculator)}$$

$$\Rightarrow \quad \boxed{x = 1.464973521}$$

Note that the exact answer, $x = \frac{\ln 5}{\ln 3}$, is an irrational number, while the answer in the box is a rational approximation of the exact answer, but is quite good enough for applications in business and science.

<u>EXAMPLE 5:</u> **Solve for *n*: $2^{3n+2} = 7$**

<u>Solution:</u>

$$2^{3n+2} = 7 \qquad \text{(the given equation)}$$

$$\Rightarrow \quad \boldsymbol{ln}\ 2^{3n+2} = \boldsymbol{ln} 7 \qquad \text{(take the } ln \text{ of each side)}$$

$$\Rightarrow \quad (3n+2)\ln 2 = \ln 7 \qquad \text{(the Power Property of Logs)}$$

[Notice the parentheses around the 3*n* + 2]

$$\Rightarrow \quad 3n(\ln 2) + 2\ln 2 = \ln 7 \qquad \text{(distribute)}$$

$$\Rightarrow \quad (3\ln 2)\,n = \ln 7 - 2\ln 2 \qquad \text{(subtract the constant } 2\ln 2\text{)}$$

$$\Rightarrow \quad n = \frac{\ln 7 - 2\ln 2}{3\ln 2}$$

(divide each side by $3\ln 2$ to get the <u>exact</u> answer)

$\Rightarrow$ $\boxed{n = 0.269118307}$ (use your calculator to get a rational approximation)

EXAMPLE 6: **Solve for *a*: $5^{2a-3} = 6^{a+1}$**

Solution:

$5^{2a-3} = 6^{a+1}$ (the original equation)

$\Rightarrow$ $\boldsymbol{ln}\ 5^{2a-3} = \boldsymbol{ln}\ 6^{a+1}$ (take the *ln* of each side)

$\Rightarrow$ $(2a-3)\ln 5 = (a+1)\ln 6$ (Power Property of Logs)

$\Rightarrow$ $2a\ln 5 - 3\ln 5 = a\ln 6 + \ln 6$ (distribute)

$\Rightarrow$ $2a\ln 5 - a\ln 6 = \ln 6 + 3\ln 5$ (variables to the left and constants to the right)

$\Rightarrow$ $a(2\ln 5 - \ln 6) = \ln 6 + 3\ln 5$ (factor out the variable)

$\Rightarrow$ $\boxed{a = \dfrac{\ln 6 + 3\ln 5}{2\ln 5 - \ln 6}}$ (divide to isolate the a)

This is the exact answer. A rational approximation would be $a = 4.638776$.

Homework

Solve each equation and round your answers to the nearest ten thousandths place:

2. $2^x = 72$	3. $5^{-y} = 3$	4. $3^{4n-1} = 5$
5. $3^{z+1} = 8^{3-z}$	6. $3^{5c} = 7^{10c}$	7. $e^{3x+4} = 25$
8. $3^{-n} = 43$	9. $7^{4-3x} = 2$	10. $e^x = 2^{x-6}$

❑ THE GROWTH AND DECAY FORMULA REVISITED

The growth and decay formula

$$A = A_0 e^{kt}$$

worked just fine back in Chapter 37 when we were searching for either the starting (initial) amount A_0 or the ending (final) amount A. But when the unknown was in the exponent, we were stuck. Now we're not stuck.

EXAMPLE 7: **Assuming an initial population of 7500 elephants, a final population of 12,000, and a time period of 7 years, find the annual growth rate.**

Solution: We will write the growth formula, substitute the given values, and then solve for the unknown k:

	$A = A_0 e^{kt}$	(the growth formula)
$\Rightarrow$	$12{,}000 = 7500 e^{k \cdot 7}$	(substitute the given values)
$\Rightarrow$	$e^{7k} = \frac{12{,}000}{7500}$	(isolate the e^{7k})
$\Rightarrow$	$e^{7k} = 1.6$	(calculator)
$\Rightarrow$	$\ln e^{7k} = \ln 1.6$	(take the *ln* of each side)
$\Rightarrow$	$7k \ln e = \ln 1.6$	(Power Property of Logs)
$\Rightarrow$	$7k = \ln 1.6$	($\ln e = 1$)
$\Rightarrow$	$k = \frac{\ln 1.6}{7}$	(solve for k)
$\Rightarrow$	$k = 0.067$	(calculator gives a decimal)

And therefore the annual growth rate is **6.7%**

<u>EXAMPLE 8:</u> **How long will it take for an investment of \$10,000 to reach a final amount of \$32,000 if the interest rate is 7.3% per year compounded continuously?**

<u>Solution:</u> The phrase "compounded continuously" justifies the use of our growth formula.

	$A = A_0 e^{kt}$	(the growth formula)
$\Rightarrow$	$32{,}000 = 10{,}000\,e^{0.073t}$	(remember: 7.3% = .073)
$\Rightarrow$	$e^{0.073t} = \dfrac{32{,}000}{10{,}000}$	(divide each side by 10,000)
$\Rightarrow$	$e^{0.073t} = 3.2$	(arithmetic)
$\Rightarrow$	$\ln e^{0.073t} = \ln 3.2$	(take the *ln* of each side)
$\Rightarrow$	$0.073t = \ln 3.2$	(Power Property of logs and ln e = 1)
$\Rightarrow$	$t = \dfrac{\ln 3.2}{0.073} = 15.933$	(solve for t)

Thus, the amount of time it will take to reach the goal is

15.933 years

EXAMPLE 9: **The decay rate of a radioactive substance is 12%/year, compounded continuously. How long will it take for 40 grams of the substance to decay, leaving 10 grams?**

Solution: $A = A_0e^{kt}$ (the growth formula)

In this problem, the starting amount is 40 g, the ending amount is 10 g, and the decay rate, k, is negative 0.12.

$\Rightarrow \quad 10 = 40e^{-0.12t}$ (note: $k < 0$)

$\Rightarrow \quad 0.25 = e^{-0.12t}$ (divide each side by 40)

$\Rightarrow \quad \ln 0.25 = \ln e^{-0.12t}$ (take the *ln* of each side)

$\Rightarrow \quad \ln 0.25 = -0.12t \ln e$ (Power Property of Logs)

$\Rightarrow \quad -1.386294 = -0.12t$ ($\ln e = 1$)

$\Rightarrow \quad t = \dfrac{-1.386294}{-0.12}$

$\Rightarrow \quad \boxed{11.6 \text{ years}}$

Homework

11. In the growth formula $A = A_0e^{kt}$, solve for
    a. A_0 b. k c. t

12. Find the interest rate if an investment of \$7200 reached a total of \$18,000 in 5 years.

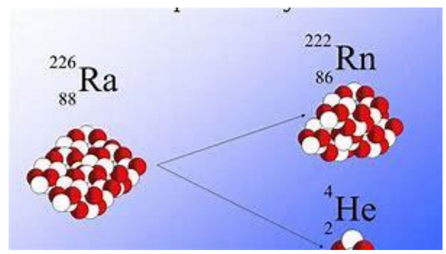
13. If the population is growing 9% per year, how long will it take for a population of 25,600 to reach a population of 100,000?

14. Find the annual growth rate if a population increased from 2000 to 7500 in a period of 9 years.

15. $25,000 is invested in a money market account paying 9.5% per year compounded continuously. How many years will it take for that investment to reach a total of $75,000?

16. The decay rate of a radioactive substance is 7%/year, compounded continuously. How long will it take for 88 kg of the substance to decay, leaving 41 kg?

17. Assuming an initial amount of 25 g of a radioactive substance, and assuming continuous radioactive decay, what is the decay rate if there are 7 g remaining after 12 years?



Solutions

1. a. 36 b. 125 c. $\sqrt{2}$ d. $\frac{1}{81}$ e. 5

f. $\frac{101}{2}$ g. e^3 h. $e + 1$ i. 2 j. $\frac{\sqrt[4]{e} - 5}{2}$

2. 6.1699 **3.** –0.6826

4. $3^{4n-1} = 5 \Rightarrow \ln(3^{4n-1}) = \ln 5 \Rightarrow (4n-1)\ln 3 = \ln 5$

$$\Rightarrow n = \frac{\frac{\ln 5}{\ln 3}+1}{4} \approx .6162$$

5. 1.6173

6. $3^{5c} = 7^{10c} \Rightarrow \ln(3^{5c}) = \ln(7^{10c}) \Rightarrow 5c\ln 3 = 10c\ln 7$

$\Rightarrow 5c\ln 3 - 10c\ln 7 = 0 \Rightarrow c(5\ln 3 - 10\ln 7) = 0$

$$\Rightarrow c = \frac{0}{5\ln 3 - 10\ln 7} \Rightarrow c = 0$$

7. –0.2604 **8.** –3.4236 **9.** 1.2146 **10.** –13.5533

11. a. $A_0 = \dfrac{A}{e^{kt}}$

b. $A = A_0e^{kt} \Rightarrow \dfrac{A}{A_0} = e^{kt} \Rightarrow \ln\dfrac{A}{A_0} = \ln e^{kt}$

$\Rightarrow \ln A - \ln A_0 = kt \Rightarrow k = \dfrac{\ln A - \ln A_0}{t}$

c. $t = \dfrac{\ln A - \ln A_0}{k}$

12. 18% **13.** 15 yrs **14.** 14.7% **15.** 11.56 yrs

16. 10.91 yrs **17.** 10.6%

"Strive for progress, not perfection."

– Anonymous