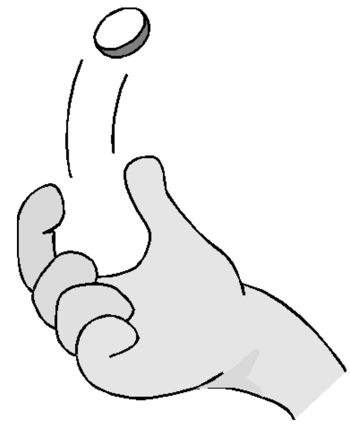
CH 44 – THE BINOMIAL THEOREM

❑ INTRODUCTION



Imagine expanding $(a + b)^{100}$ using regular algebra. This chapter will show us a neat shortcut for solving this kind of problem. The concepts involved in raising a binomial to a high power have applications in probability, from the flipping of coins to the issue of quality control in the manufacture of light bulbs.

The goal of the Binomial Theorem is to expand powers of binomials, things like $(a + b)^{100}$, without actually multiplying anything out. Let's start with some expansions we already know how to do using brute-force algebra.

$(a + b)^0 = 1$ (anything [but 0] to the zero power is 1)

$(a + b)^1 = a + b$ (anything to the 1st power is itself)

$(a + b)^2 = a^2 + 2ab + b^2$ (just multiply $a + b$ by $a + b$)

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (multiply $a^2 + 2ab + b^2$ by $a + b$)

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ (etc.)

$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ (etc.)

Homework

1. Verify the calculations for each expansion:

   $(a+b)^2$ $(a+b)^3$ $(a+b)^4$ $(a+b)^5$

2. How many terms are there in each expansion?

   a. $(a+b)^0$ b. $(a+b)^1$ c. $(a+b)^2$

   d. $(a+b)^3$ e. $(a+b)^4$ f. $(a+b)^5$

3. a. How many terms are there in the expansion of $(a+b)^{100}$?

   b. How many terms are there in the expansion of $(a+b)^n$?

4. a. Notice the term $10a^3b^2$ in the expansion of $(a+b)^5$. What is the *sum* of the exponents on the a and the b?

   b. In the same expansion, the 5th term is $5ab^4$. What is the *sum* of the exponents on the a and the b?

5. a. See the first term in the expansion of $(a+b)^4$? It's a^4. What is the *sum* of the exponents on the a and the b? [Hint: Maybe there's no explicit b in the term a^4, but what power of b could be placed next to the a^4 so that it's still a^4? That is, $a^4b^? = a^4$.]

   b. What is the *sum* of the exponents for <u>any</u> term in the expansion of $(a+b)^4$?

6. What is the *sum* of the exponents for <u>any</u> term in the expansion of $(a+b)^n$?

7. If the expansion of $(a+b)^k$ has a term of the form $ca^{44}b^{33}$, where c is a constant, what is the value of k?

❑ OBSERVATIONS ON THE EXPANSIONS

1. The number of terms in each expansion is one more than the exponent on the binomial. For example, the expansion of $(a + b)^5$ has 6 terms, and the expansion of $(a + b)^n$ has $n + 1$ terms.

2. In any given term, the sum of the exponents always equals the exponent on the binomial. For instance, looking at the expansion of $(a + b)^5$ from page 1:

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Let's "patch up" the expansion of $(a + b)^5$ to include both the a and the b in every term, we get

$$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

We notice that in any term, the sum of the exponents is always 5. This always works: In every term in the expansion of $(a + b)^n$, the sum of the exponents is always n.

3. The exponents on the a go down while the exponents on the b go up. Again, look at the form in the previous observation:

$$(a + b)^5 = a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + a^0b^5$$

Notice that the exponents on the a go down: 5, 4, 3, 2, 1, 0, while the exponents on the b go up: 0, 1, 2, 3, 4, 5.

EXAMPLE 1: **Apply the three observations above to the expansion of $(a + b)^{100}$.**

Solution: First, there will be 101 terms in the expansion. Second, in any term the sum of the exponents will be 100. For instance, the term with a^{40} in it will also have b^{60} in it. Third, the exponents on the a will go 100, 99, 98, . . ., 2, 1, 0 while the exponents on the b will go 0, 1, 2, . . . , 98, 99, 100. In short, using boxes for the unknown coefficients (the front numbers), the expansion of $(a + b)^{100}$ looks like this:

$$\Box a^{100} + \Box a^{99}b + \Box a^{98}b^2 + \cdots + \Box a^2b^{98} + \Box ab^{99} + \Box b^{100}$$

PASCAL'S TRIANGLE

Now let's focus on the ***coefficients*** in the expansions (remember that the term a^4, for instance, is really $1a^4$):

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1

Our goal is to extend this triangle of coefficients further, merely by looking at the rows of the triangle we currently have. Then we'll know what the coefficients are for even higher powers of $a + b$. Check out the 6 in the triangle; it's the sum of the 3 and the 3 above it. Look at the first 10; it's the sum of the 4 and the 6 above it. This pattern generates ***Pascal's Triangle***. Let's extend the triangle even further:

1	← Row 0
1 1	← Row 1
1 2 1	← Row 2
1 3 3 1	← Row 3
1 4 6 4 1	
1 5 10 10 5 1	
1 6 15 20 15 6 1	
1 7 21 35 35 21 7 1	
1 8 28 56 70 56 28 8 1	
1 9 36 84 126 126 84 36 9 1	

Each entry of PASCAL'S TRIANGLE (starting in Row 2)
is the <u>sum</u> of the two entries above it.

Homework

8. Consider the expansion of $(a + b)^{23}$. How many terms are there in the expansion? In the term containing a^{15}, what is the exponent on the b?

9. There are 102 terms in the expansion of $(x + y)^n$. What is n?

10. Let's call the bottom row in the portion of Pascal's Triangle listed above the "9th row." Find the "10th row" of Pascal's Triangle.

❑ *An Example of the Binomial Theorem*

Let's use the three observations mentioned earlier along with Pascal's Triangle to deduce the expansion of

$$(a + b)^7$$

1. There will be **8** terms in the expansion.

2. The sum of the exponents in each term will be **7**.

3. The exponents on the a will go from 7 down to 0, while the exponents on the b will go from 0 up to 7.

   Also, the relevant row of Pascal's Triangle is

   $$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

   There are two ways we can know this. First, it's the only row with 8 entries in it, and we need 8 coefficients for the 8 terms of the expansion. Moreover, the second number in the row, the **7**, matches the exponent in the expression $(a + b)^{\mathbf{7}}$.

Let's try it: $(a + b)^7$

$$= \underline{1}a^7b^0 + \underline{7}a^6b^1 + \underline{21}a^5b^2 + \underline{35}a^4b^3 + \underline{35}a^3b^4 + \underline{21}a^2b^5 + \underline{7}a^1b^6 + \underline{1}a^0b^7$$

The underlined coefficients come from Row 7 of Pascal's Triangle.

$$= \boxed{\boldsymbol{a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7}}$$

Is the Binomial Theorem worth knowing? Try expanding $(a + b)^7$ without it.

Homework

11. Find the first two terms and the last two terms of $(a + b)^{100}$.

12. Even though we know how to expand $(a + b)^2$ from basic algebra, show that the Binomial Theorem produces the same result.

13. Expand $(m + z)^8$.

14. Expand $(x + y)^{10}$.

Practice Problems

15. How many terms are there in the expansion of $(a + b)^{200}$?

16. What is the <u>sum</u> of the exponents for <u>any</u> term in the expansion of $(a + b)^{123}$?

17. Consider the expansion of $(a + b)^{250}$.
    a. How many terms are there in the expansion?
    b. In the term containing a^{100}, what is the exponent on the b?
    c. What is the first term of the expansion?
    d. What is the second term of the expansion?
    e. What is the second-to-last term of the expansion?
    f. What is the last term of the expansion?

18. Expand $(u + w)^9$.

Solutions

1. They're boring, but do them!

2. a. 1 b. 2 c. 3 d. 4 e. 5 f. 6

3. a. 101 b. $n + 1$

4. a. $3 + 2 = 5$ b. 5

5. a. 4 b. 4

6. n

7. $k = 77$

8. 24; 8

9. $n = 101$

10. 1 10 45 120 210 252 210 120 45 10 1

11. $a^{100} + 100a^{99}b + \cdots + 100ab^{99} + b^{100}$

12. To expand $(a + b)^2$ by the Binomial Theorem, we know that the relevant row of Pascal's Triangle is "1 2 1." We also know that the exponents on the a go from 2 down to 0, while the exponents on the b go from 0 up to 2. Putting it all together:

$$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2 = a^2 + 2ab + b^2,$$ as it should be.

13. $m^8 + 8m^7z + 28m^6z^2 + 56m^5z^3 + 70m^4z^4 + 56m^3z^5 + 28m^2z^6 + 8mz^7 + z^8$

14. $x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + 252x^5y^5 + 210x^4y^6$
$+ 120x^3y^7 + 45x^2y^8 + 10xy^9 + y^{10}$

15. 201 **16.** 123

17. a. 251 b. 150 c. a^{250} d. $250a^{249}b$
e. $250ab^{249}$ f. b^{250}

18. $u^9 + 9u^8w + 36u^7w^2 + 84u^6w^3 + 126u^5w^4 + 126u^4w^5 + 84u^3w^6$
$+ 36u^2w^7 + 9uw^8 + w^9$

"Cherish your visions and your dreams, as they are the children of your soul, the blueprints of your ultimate achievements."

– Napoleon Hill