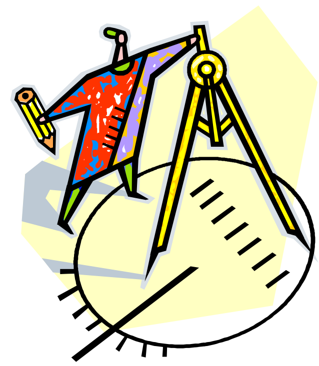
Ch 33 – The Circle

❑ Introduction



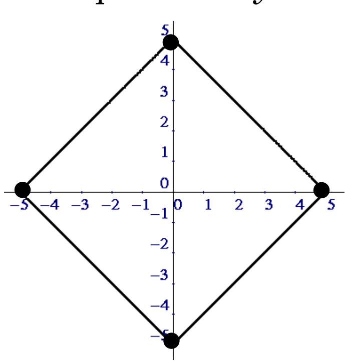
We've seen four basic graphs in this class so far: square roots and absolute values from Chapter 2, and in-depth studies of lines and parabolas. In this chapter we analyze "nature's perfect shape," the circle. Whereas the equation of a line has no variable squared, and a parabola has one variable squared, we will soon see that the equation of a circle has both variables squared.

❑ Graphing a Circle

<u>EXAMPLE 1:</u> **Graph:** $x^2 + y^2 = 25$

<u>Solution:</u> The graph is certainly not a line, since the variables are squared. Nor is it a parabola, because the equation of a parabola must have exactly one variable squared. Let's plot points and see what we get (as if we didn't already know).

Let's first check out the **intercepts**. If we set $x = 0$, the resulting equation is $y^2 = 25$, whose two solutions are $y = \pm 5$. Thus, there are two y-intercepts, **(0, 5)** and **(0, –5)**. You can calculate the x-intercepts to be **(5, 0)** and **(–5, 0)**. We now have four points on our graph, but it's not clear how to connect them — maybe the graph looks like a diamond? We'll find some other points; for example, if we let $x = 3$, then



Are four points enough to make an accurate graph?

$$3^2 + y^2 = 25 \Rightarrow 9 + y^2 = 25$$

$\Rightarrow \quad y^2 = 16 \;\Rightarrow\; y = \pm 4$

$\Rightarrow$ **(3, 4)** and **(3, –4)** are on the graph.

If x is chosen to be –3, then

$$(-3)^2 + y^2 = 25 \;\Rightarrow\; 9 + y^2 = 25 \;\Rightarrow\; y^2 = 16 \;\Rightarrow\; y = \pm 4$$

$\Rightarrow$ **(–3, 4)** and **(–3, –4)** are also on the graph.

Now let $x = 4$:

$$4^2 + y^2 = 25 \;\Rightarrow\; 16 + y^2 = 25 \;\Rightarrow\; y^2 = 9 \;\Rightarrow\; y = \pm 3$$

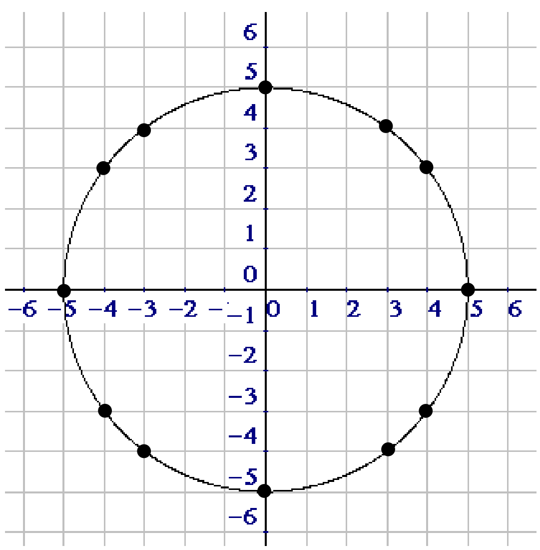
$\Rightarrow$ **(4, 3)** and **(4, –3)** are on the graph.

Our last choice for x will be –4:

$$(-4)^2 + y^2 = 25 \;\Rightarrow\; 16 + y^2 = 25 \;\Rightarrow\; y^2 = 9 \;\Rightarrow\; y = \pm 3$$

$\Rightarrow$ **(–4, 3)** and **(–4, –3)** are also on the graph.

This set of 12 points should be enough data to get a decent picture, which, sure enough, looks like a circle:



Summary: The graph of

$$x^2 + y^2 = 25$$

is a circle with its center at the origin and with a radius of 5.

[Notice that the radius, 5, is the positive square root of the 25 on the right side of the equation.]

Homework

1. For each circle, determine the four **intercepts**:

   a. $x^2 + y^2 = 1$ b. $x^2 + y^2 = 2$ c. $x^2 + y^2 = 4$

   d. $x^2 + y^2 = 7$ e. $x^2 + y^2 = 49$ f. $x^2 + y^2 = 60$

2. Given the circle and the *x*-value (i.e., an input), find the ***y*-values** (i.e., the outputs):

   a. $x^2 + y^2 = 100;\ x = 6$ b. $x^2 + y^2 = 100;\ x = -8$

   c. $x^2 + y^2 = 1;\ x = 1$ d. $x^2 + y^2 = 169;\ x = 5$

   e. $x^2 + y^2 = 169;\ x = -12$ f. $x^2 + y^2 = 169;\ x = -5$

   g. $x^2 + y^2 = 10;\ x = 2$ h. $x^2 + y^2 = 12;\ x = -2$

   i. $x^2 + y^2 = 21;\ x = 4$ j. $x^2 + y^2 = 15;\ x = -4$

EXAMPLE 2:

A. Consider the circle $x^2 + y^2 = 81$. Using Example 1 as a guide, we infer that its center is the origin and its radius is 9 (the positive square root of 81).

B. Now look at the circle $x^2 + y^2 = 13$. The center is (0, 0), and the radius is $\sqrt{13}$.

C. If the center of a circle is the origin, and if its radius is 15, what is the equation of the circle? $x^2 + y^2 = 15^2$, which comes out to $\boldsymbol{x^2 + y^2 = 225}$.

D. What is the equation of the circle with center (0, 0) and radius $3\sqrt{7}$?

$x^2 + y^2 = \left(3\sqrt{7}\right)^2$, which is $\boldsymbol{x^2 + y^2 = 63}$.

Note: $\left(3\sqrt{7}\right)^2 = 3^2 \cdot \sqrt{7}^2 = 9 \cdot 7 = 63$

Homework

3. Find the **center** and **radius** of each circle:

a. $x^2 + y^2 = 25$ b. $x^2 + y^2 = 144$

c. $x^2 + y^2 = 1$ d. $x^2 + y^2 = 17$

e. $x^2 + y^2 = 27$ f. $x^2 + y^2 = 200$

g. $x^2 + y^2 = 0$ h. $x^2 + y^2 = -9$

i. $x + y = 10$ j. $x^2 + y = 49$

Note: Working with the radius can be confusing. If given the circle equation in standard form, the radius of the circle is found by taking the positive square root of the number to the right of the equals sign. On the other hand, if you know the radius, you square it when you put it into the formula.

4. Find the **equation** of the circle with center at the origin and the given radius:

a. $r = 10$ b. $r = 25$ c. $r = 1$

d. $r = 16$ e. $r = \sqrt{11}$ f. $r = \sqrt{18}$

g. $r = 4\sqrt{5}$ h. $r = 3\sqrt{7}$ i. $r = A$

5. The ***unit circle*** is the circle whose center is at the origin and whose radius is 1.

a. Find the equation of the unit circle.
b. What is the area of the unit circle? $[A = \pi r^2]$
c. What is the circumference of the unit circle? $[C = 2\pi r]$

6. Describe the graph of each equation:

a. $x^2 + y^2 = 29$ b. $x^2 + y^2 = 0$ c. $x^2 + y^2 = -81$

❑ *Circles With Center Off the Origin*

EXAMPLE 3: **Graph:** $(x - 2)^2 + (y + 1)^2 = 25$

Solution: First, it might be clear that the center of the circle is not the origin. Second, our best guess right now is that the radius is 5 (being the positive square root of 25). Let's check out these theories.

It's hard to know what values of x we should choose to find our points to plot, but here's a neat trick to find four useful points.

We'll start with $\underline{x = 2}$ (this rids us of the first term) and then solve for y in the circle equation $(x - 2)^2 + (y + 1)^2 = 25$:

$$\begin{aligned}
x = 2 \quad &\Rightarrow \quad (\mathbf{2} - 2)^2 + (y + 1)^2 = 25 \\
&\Rightarrow \quad 0^2 + (y + 1)^2 = 25 \\
&\Rightarrow \quad (y + 1)^2 = 25 \\
&\Rightarrow \quad y + 1 = \pm 5 \\
&\Rightarrow \quad y = -1 \pm 5 \\
&\Rightarrow \quad y = 4 \text{ or } -6
\end{aligned}$$

Since letting $x = 2$ produced two y-values, we have the two points **(2, 4)** and **(2, –6)** on our circle.

Next we'll let $\underline{y = -1}$ (this annihilates the second term) and then solve for x in the circle equation $(x - 2)^2 + (y + 1)^2 = 25$:

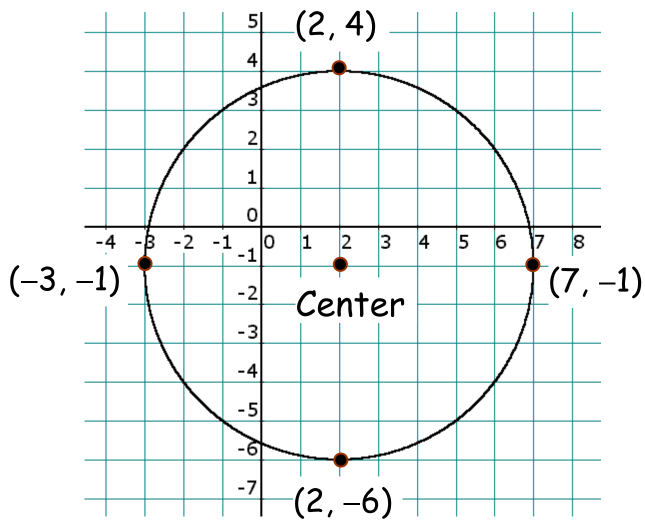
$$\begin{aligned}
y = -1 \quad &\Rightarrow \quad (x - 2)^2 + (\mathbf{-1} + 1)^2 = 25 \\
&\Rightarrow \quad (x - 2)^2 + 0^2 = 25 \\
&\Rightarrow \quad (x - 2)^2 = 25 \\
&\Rightarrow \quad x - 2 = \pm 5 \\
&\Rightarrow \quad x = 2 \pm 5 \\
&\Rightarrow \quad x = 7 \text{ or } -3
\end{aligned}$$

By letting $y = -1$, we determined that two more points on the circle are **(7, –1)** and **(–3, –1)**.

What have we done here? By choosing clever values of x and y, we have discovered the following four points on our circle:

$(2, 4) \quad (2, -6) \quad (7, -1) \quad (-3, -1)$

Let's plot these four points, connect them to make a circle, and then figure out the center and the radius of the circle.



Can you see that the center of the circle is the point (2, −1), and that the radius is 5? Let's summarize:

> The circle $(x - 2)^2 + (y + 1)^2 = 25$ has its center at (2, −1) and has a radius of 5.

Do you see any connections here? The x-coordinate of the center (the 2) is the *opposite* of the number following the x in the circle equation. Also, the y-coordinate of the center (the −1) is the *opposite* of the number following the y in the circle equation. And last, the radius of the circle, 5, is the positive square root of the 25 on the right side of the circle equation, just as we expected. The following table shows the relationships between the equation of a circle and the circle's center and radius. Study it carefully.

Equation of Circle	Center	Radius
$(x-7)^2+(y-4)^2 = 121$	(7, 4)	11
$(x+3)^2+(y+5)^2 = 49$	(–3, –5)	7
$(x-1)^2+(y+12)^2 = 21$	(1, –12)	$\sqrt{21}$
$(x+11)^2+(y-9)^2 = 93$	(–11, 9)	$\sqrt{93}$
$x^2+(y-8)^2 = 1$	(0, 8)	1
$(x+13)^2+y^2 = 50$	(–13, 0)	$5\sqrt{2}$
$x^2+y^2 = 108$	(0, 0)	$6\sqrt{3}$

Homework

7. Find the **center** and **radius** of each circle:

a. $x^2+y^2 = 64$

b. $x^2+y^2 = 24$

c. $x^2+(y-7)^2 = 4$

d. $(x-3)^2+y^2 = 5$

e. $(x+1)^2+(y+8)^2 = 60$

f. $(x-2)^2+(y-3)^2 = 99$

g. $(x+5)^2+(y-3)^2 = 144$

h. $(x-1)^2+(y+11)^2 = 48$

i. $(x+3)+(y-1) = 9$

j. $(x-3)^2+(y+2) = 7$

8. Find the **equation** of the circle with the given center and radius:

a. C(0, 0) $r = 7$

b. C(0, 0) $r = \sqrt{10}$

c. C(3, 0) $r = 3$

d. C(0, 4) $r = 1$

e. C(0, –2) $r = \sqrt{3}$

f. C(–12, 0) $r = 12$

g. C(2, 7) $r = 10$

h. C(–1, –3) $r = 2\sqrt{3}$

i. C(3, –4) $r = 3\sqrt{5}$

j. C(–2, 9) $r = 5\sqrt{7}$

k. C(1, 2) $r = 0$

l. C(–3, 5) $r = -9$

❑ *A NEW TWIST FOR THE CIRCLE*

Now for a tricky circle question: Find the center and radius of the circle

$$x^2 + y^2 + 8x - 6y + 9 = 0$$

This circle is not in the standard form we've been using to extract the center and radius. So, how in the heck do we convert this circle equation containing no parentheses into the proper form with parentheses? Any ideas before you read on?

EXAMPLE 4: **Graph the circle $x^2 + y^2 + 8x - 6y + 9 = 0$.**

Solution: The graph will depend on finding the center and radius of the circle. To find these, we need to convert the given equation of the circle into standard form.

Remember the "magic number" we used when we solved quadratic equations by completing the square? We use the same trick here, except that we will complete the square in both variables. Cool! We get to calculate two magic numbers.

Start with the given equation of the circle:

$$x^2 + y^2 + 8x - 6y + 9 = 0$$

Rearrange the terms, putting the x-terms next to each other and the y-terms next to each other:

$$x^2 + 8x + y^2 - 6y + 9 = 0$$

Take the constant 9 to the other side of the equation:

$$x^2 + 8x + y^2 - 6y = -9$$

Now calculate the two "magic numbers":

Half of 8 is 4, and $4^2 = \mathbf{16}$. This is the magic number for x.

Half of –6 is –3, and $(-3)^2 = \mathbf{9}$. This is the magic number for y.

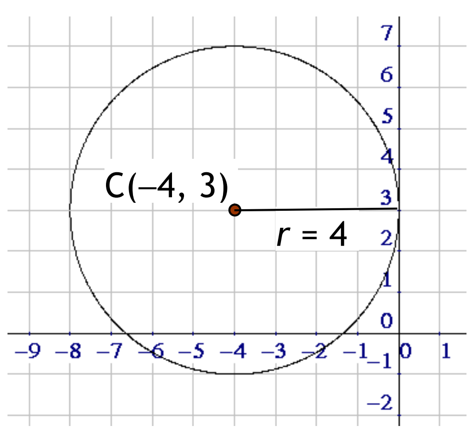
Now we add the magic numbers to both sides of the equation so that perfect square trinomials will be formed on the left side:

$$\underbrace{x^2 + 8x + \boxed{16}}_{\text{factorable}} + \underbrace{y^2 - 6y + \boxed{9}}_{\text{factorable}} = -9 + \boxed{16} + \boxed{9}$$

Now factor the first three terms, then factor the next three terms, and then do the arithmetic on the right side of the equation:

$$(x + 4)^2 + (y - 3)^2 = 16$$

We made it! Now that the circle equation is in standard form, we can read the center and radius directly from the equation. The center is (–4, 3) and the radius is 4 (the positive square root of 16). This info is just what we need to graph our circle:



Homework

9. Find the **center** and **radius** of each circle:

a. $x^2 + y^2 + 8x - 10y - 40 = 0$

b. $x^2 + y^2 - 16x + 8y + 31 = 0$

c. $x^2 + y^2 + 12x + 14y + 49 = 0$

d. $x^2 + y^2 + 14x + 24 = 0$

e. $x^2 + y^2 - 6y - 216 = 0$

f. $x^2 + y^2 + 8x + 6y + 10 = 0$

g. $x^2 + y^2 - 2x - 4y - 3 = 0$

h. $x^2 + y^2 + 4x - 10y + 12 = 0$

i. $x^2 + y^2 - 20x + 6y + 101 = 0$

j. $x^2 + y^2 + 2x - 4y + 25 = 0$

Practice Problems

10. Describe the graph of each equation:

a. $x^2 + y^2 = 1{,}000{,}000$ b. $x^2 + y^2 = 1$

c. $x^2 + y^2 = 0$ d. $x^2 + y^2 = -9$

11. Consider the equation $x^2 + y^2 = k$. Describe the graph of this equation

a. if $k > 0$ b. if $k = 0$ c. if $k < 0$

12. Find the center and radius of the circle $x^2 + y^2 = 20$.

13. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.

14. Matching:

____	$y = 3$	A. parabola
____	$2x^2 - 3y = 10$	B. circle
____	$x^2 + (y - 1)^2 = 2$	C. horizontal line
____	$x - y + 10 = 0$	D. non-horizontal line

15. True/False:

a. Every circle has at least one intercept.

b. The radius of the circle $x^2 + y^2 = 1$ is 1.

c. The graph of $y = x^2 - 9x + 17$ is a parabola.

d. $x^2 + y^2 + 4 = 3$ is a circle.

e. $x^2 + y^2 = 1$ is called the unit circle.

f. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is $(-2, -5)$.

g. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.

h. The graph of $10x^2 + 10y^2 = 37$ is a circle.

i. The graph of $10x^2 - 10y^2 = 37$ is a circle.

j. The graph of $10x^2 + 9y^2 = 37$ is a circle.

k. The area of the circle $x^2 + y^2 = 25$ is 25π.

Solutions

1. a. $(\pm1, 0)$ $(0, \pm1)$ b. $(\pm\sqrt{2}, 0)$ $(0, \pm\sqrt{2})$

c. $(\pm2, 0)$ $(0, \pm2)$ d. $(\pm\sqrt{7}, 0)$ $(0, \pm\sqrt{7})$

e. $(\pm 7, 0)$ $(0, \pm 7)$ f. $(\pm 2\sqrt{15}, 0)$ $(0, \pm 2\sqrt{15})$

2. a. ± 8 b. ± 6 c. 0 d. ± 12 e. ± 5
f. ± 12 g. $\pm\sqrt{6}$ h. $\pm 2\sqrt{2}$ i. $\pm\sqrt{5}$ j. Does not exist

3. a. C(0, 0) $r = 5$ b. C(0, 0) $r = 12$ c. C(0, 0) $r = 1$
d. C(0, 0) $r = \sqrt{17}$ e. C(0, 0) $r = 3\sqrt{3}$ f. C(0, 0) $r = 10\sqrt{2}$
g. It's not a circle; the graph is just the origin.
h. It's not a circle; there are two reasons. First, the radius would be $\sqrt{-9}$, which is not a real number. Second, if two numbers are squared and then added together, there's no way that sum can be negative.
i. It's a line, not a circle.
j. It's not a circle. What is it?

4. a. $x^2 + y^2 = 100$ b. $x^2 + y^2 = 625$ c. $x^2 + y^2 = 1$
d. $x^2 + y^2 = 256$ e. $x^2 + y^2 = 11$ f. $x^2 + y^2 = 18$
g. $x^2 + y^2 = 80$ h. $x^2 + y^2 = 63$ i. $x^2 + y^2 = A^2$

5. a. $x^2 + y^2 = 1$ b. $A = \pi r^2 = \pi(1^2) = \pi(1) = \pi$
c. $C = 2\pi r = 2\pi(1) = 2\pi$

6. a. A circle with center (0, 0) and radius $\sqrt{29}$
b. The graph is a single point, the origin.
c. There is no graph.

7. a. C(0, 0) $r = 8$ b. C(0, 0) $r = 2\sqrt{6}$
c. C(0, 7) $r = 2$ d. C(3, 0) $r = \sqrt{5}$
e. C(–1, –8) $r = 2\sqrt{15}$ f. C(2, 3) $r = 3\sqrt{11}$
g. C(–5, 3) $r = 12$ h. C(1, –11) $r = 4\sqrt{3}$
i. Trick: It's a line j. Trick: It's a parabola

8. a. $x^2 + y^2 = 49$ b. $x^2 + y^2 = 10$
c. $(x - 3)^2 + y^2 = 9$ d. $x^2 + (y - 4)^2 = 1$

e. $x^2 + (y + 2)^2 = 3$

f. $(x + 12)^2 + y^2 = 144$

g. $(x - 2)^2 + (y - 7)^2 = 100$

h. $(x + 1)^2 + (y + 3)^2 = 12$

i. $(x - 3)^2 + (y + 4)^2 = 45$

j. $(x + 2)^2 + (y - 9)^2 = 175$

k. Just the point (1, 2)

l. Not a circle; no graph at all

9. a. C(–4, 5) $r = 9$

b. C(8, –4) $r = 7$

c. C(–6, –7) $r = 6$

d. C(–7, 0) $r = 5$

e. C(0, 3) $r = 15$

f. C(–4, –3) $r = \sqrt{15}$

g. C(1, 2) $r = 2\sqrt{2}$

h. C(–2, 5) $r = \sqrt{17}$

i. C(10, –3) $r = 2\sqrt{2}$

j. Not a circle

10. a. A circle with center at the origin and a radius of 1,000.

b. A circle with center at the origin and a radius of 1.

c. The point (0, 0), and that's it.

d. Since the sum of two squares is <u>never</u> negative, there is NO graph.

11. a. If $k > 0$, the graph is a circle with center at the origin and radius $\sqrt{k}$.

b. If $k = 0$, the graph is just the single point (0, 0); i.e., the origin.

c. If $k < 0$, the graph is empty (there's no graph at all).

12. C(0, 0); $r = 2\sqrt{5}$

13. C(5, –1); $r = \sqrt{23}$

14. C, A, B, D

15. a. F b. T c. T d. F e. T f. F

g. T h. T i. F j. F k. T

“Only those who dare
to fail greatly
can ever achieve greatly.”

– Robert F. Kennedy