CH 36 – DOMAIN

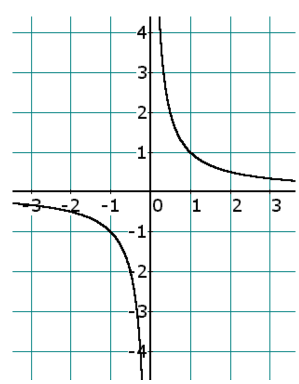
❑ INTRODUCTION

Have you ever gotten an ERROR message on your calculator? You may have tried to divide by zero — or perhaps you attempted to compute the square root of a negative number. In either case, you tried to use a number outside the ***domain*** of the function you were trying to calculate.

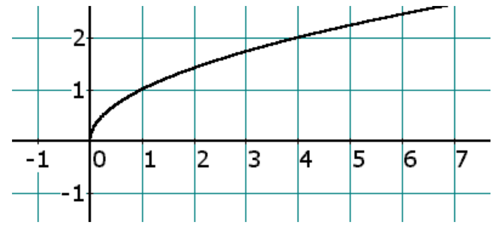
❑ PRELIMINARY EXAMPLES

We have a pretty good notion of what a function is. There's a set of *inputs*, which are in some way associated with a set of *outputs*. Although this association is usually given by a formula (e.g., $y = x^2$), don't forget about the four quarters of the football game and the graphs in the previous chapter. Most importantly, remember that this association between the inputs and the outputs is such that each element in the set of inputs is associated with exactly one output. In this chapter, we focus on what inputs are allowed in a function.

First Example: Consider the function $y = \frac{1}{x}$. What inputs are allowed in this function? That is, what can x legally be? Well, x had better not be 0, since division by 0 is undefined. But if x is any number other than 0, no problems will occur. So x can be **any real number except 0**.



The set of "all real numbers except 0" can be written $\mathbb{R} - \{0\}$.



Second Example: Now consider the function $y = \sqrt{x}$. We ask the same question: What x's are allowed in this formula? In other words, what kinds of real numbers are we allowed to take the square root of? The answer is: We can take the square root of any number that is zero or greater — which is equivalent to saying that we cannot take the square root of a negative number. Thus, x can be any number that is **greater than or equal to 0,** which we also write as $\boldsymbol{x \geq 0}$.

Homework

1. Let $y = 2x + 10$. Which of the following are legal inputs for x?

   a. 30 b. $-\pi$ c. $\sqrt{7}$ d. 0

2. Let $y = \dfrac{3}{x-2}$. Which of the following are legal inputs for x?

   a. 3 b. 2 c. π d. 0

3. Let $y = \sqrt{x}$. Which of the following are legal inputs for x?

   a. 7 b. 0 c. -9 d. 99

4. Let $y = \dfrac{10}{x^2 - 144}$. Which of the following are legal inputs for x?

   a. 12 b. 20 c. 0 d. -12

5. Let $y = \dfrac{x}{x^2 + 9}$. Which of the following are legal inputs for x?

   a. 3 b. 0 c. -3 d. 2

❑ *THE DOMAIN OF A FUNCTION*

> The **domain** of a function is the set of all legal **inputs** .

Now we need a simple name for the set of legal inputs to a function. We'll call this set of inputs the ***domain*** of the function. Why is the domain an important idea to consider? When a number outside the domain is introduced into the function (like in a spreadsheet or a programming language), total havoc can result.

For example, if our computer application tries to let $x = 0$ in the function $f(x) = \frac{1}{x}$, we're sunk; the computer will stop execution of the program and give an error message (or possibly even freeze up!). And if we allow $x = -4$ in the function $g(x) = \sqrt{x}$, then we're really up a creek, since $\sqrt{-4}$ is not a real number, and real numbers are all we have at our disposal in this course.

To review, the **domain** of a typical function found in an algebra course is the set of <u>all real numbers that are legal for ***x*** to be in the formula</u>. For example, the function $g(x) = x^2$ would have a domain of $\mathbb{R}$, all the real numbers (since any real number can be squared no problem). But in the formula $y = \frac{3}{x-7}$, x can be any real number <u>except</u> 7 (why?), and therefore the domain is $\mathbb{R} - \{7\}$, all the real numbers with 7 removed.

Homework

6. Consider the function $f(x) = \sqrt{2x-8}$. Calculate each functional value:

   a. $f(12)$ b. $f(36)$ c. $f(4.5)$ d. $f(4)$ e. $f(3)$

   From these results we see that the values 12, 36, 4.5, and 4 are in the **domain** of the function f, while 3 is not.

7. Let h be the function defined by $h(x) = \dfrac{1}{x^2 - 9}$. Calculate each functional value:

x	-4	-3	-2	0	1	3	4	5
$h(x)$								

From this table of inputs and outputs, take a guess what the domain of the function h is.

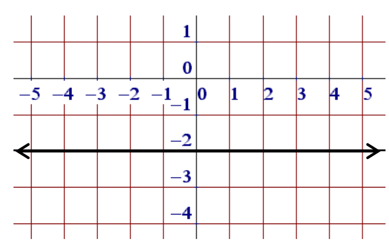
8. Consider the function given by the formula $g(x) = x^3 - x^2$. What is the domain of g?

9. What is the domain of the function $y = \dfrac{2}{x+3}$?

❑ CALCULATING DOMAINS OF POLYNOMIALS AND FRACTIONS

I. $y = -2$

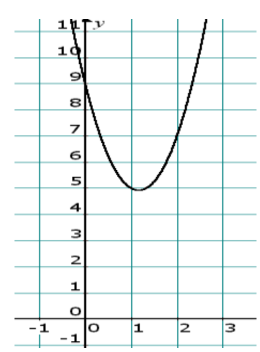
In this equation, the x isn't even mentioned. There can, therefore, be no restrictions on x. That is, x can be any real number. Thus, the domain is



$$\boxed{\mathbb{R}}$$

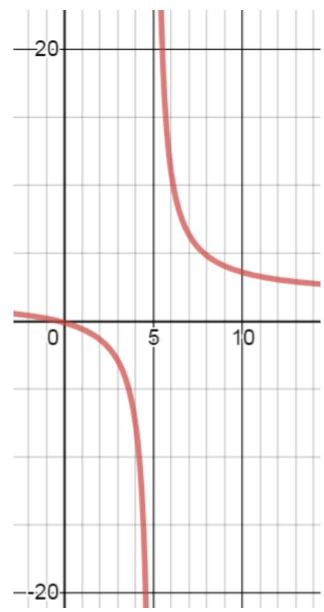
II. $y = 3x^2 - 7x + 9$

We ask ourselves: What are the legal x's? Well, x can be anything, since the operations in the formula could not possibly be a cause for concern. The domain is therefore



$$\boxed{\mathbb{R}}$$

III. $y = \dfrac{7x+2}{4x-20}$



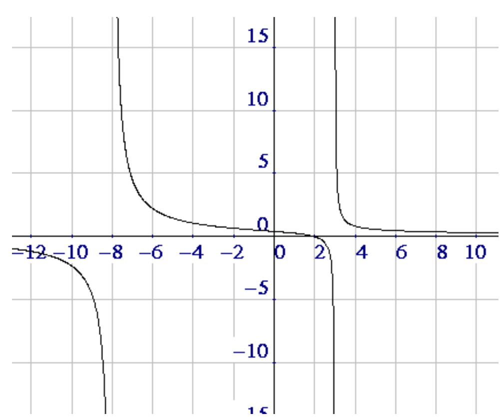
Now we've got something interesting to look at. Question: What can go wrong in a division problem? Answer: The possibility of dividing by zero. We must make sure that x is <u>never</u> allowed to be a number which would make the denominator zero. So we find out what value(s) of x <u>would</u> make the bottom zero, and then <u>don't allow those x's to be in the domain</u>. Setting the bottom to zero gives

$$4x - 20 = 0$$
$$\Rightarrow \quad 4x = 20$$
$$\Rightarrow \quad x = 5$$

Thus, if $x = 5$, the denominator is zero, which is absolutely forbidden! So, the domain of this function is the set of <u>all</u> real numbers <u>except</u> 5:

$$\boxed{\mathbb{R} - \{5\}}$$

IV. $g(x) = \dfrac{5x-10}{x^2+5x-24}$



As in the previous example, we must make certain that the bottom of the fraction is <u>never</u> zero; so we'll see what values of x make it zero, and then <u>exclude</u> such values from our domain:

$$x^2 + 5x - 24 = 0$$
$$\Rightarrow \quad (x+8)(x-3) = 0$$

$$\Rightarrow \quad x + 8 = 0 \text{ or } x - 3 = 0$$
$$\Rightarrow \quad x = -8 \text{ or } x = 3$$

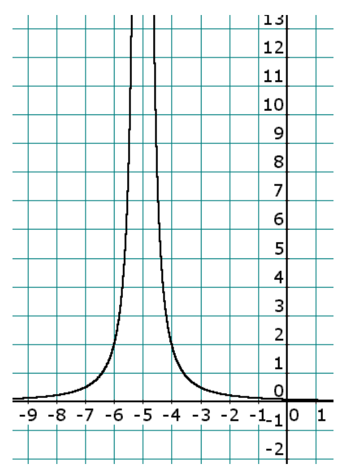
We conclude that the domain is all real numbers except –8 and 3:

$$\boxed{\mathbb{R} - \{-8, 3\}}$$

V. $f(x) = \dfrac{2}{x^2 + 10x + 25}$

Again we determine what values of x would make the bottom zero, and then exclude these values from the domain:

$$x^2 + 10x + 25 = 0$$
$$\Rightarrow \quad (x + 5)(x + 5) = 0$$
$$\Rightarrow \quad x + 5 = 0 \text{ or } x + 5 = 0$$
$$\Rightarrow \quad x = -5 \text{ or } x = -5$$



We see that the only value of x that is not allowed in the domain is $x = -5$. Hence, the domain of the function f is all real numbers except –5:

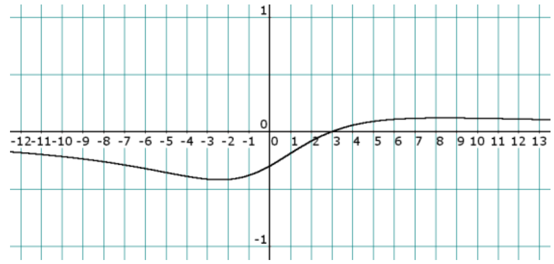
$$\boxed{\mathbb{R} - \{-5\}}$$

VI. $h(x) = \dfrac{2x - 6}{x^2 + 20}$

Let's see what makes the denominator zero (so we can exclude it from the domain.)

$$x^2 + 20 = 0$$
$$\Rightarrow \quad x^2 = -20$$
$$\Rightarrow \quad x = \pm\sqrt{-20}\text{, which are not real numbers.}$$

What do we conclude here? We tried to figure out what values of x would make the denominator zero, and thus be excluded from the domain. But there aren't any values of x that make the denominator zero, so there is nothing to exclude from the domain. Therefore, every real number is allowed in the function. The domain of h is



$$\boxed{\mathbb{R}}$$

Homework

Find the domain of each function:

10. $f(x) = \pi$

11. $y = \dfrac{x^2 - 9}{9x - 7}$

12. $g(x) = \dfrac{2x + 1}{x^2 - 100}$

13. $y = \dfrac{x^2 - 25}{x^2 + 49}$

14. $y = \sqrt{2} + \sqrt{3}$

15. $y = \sqrt{3}$

16. $f(x) = \dfrac{2x - 3}{2x^2 + 3x}$

17. $g(x) = \dfrac{3}{2x - x^2}$

18. $y = x^3 - 8$

19. $y = \dfrac{9x - 7}{2x + 9}$

20. $f(x) = \dfrac{5x + 25}{x^2 - 144}$

21. $y = \dfrac{x}{x^2 + 3}$

22. $f(x) = \sqrt{2}\,x + 9$

23. $g(x) = \dfrac{x - 2}{x^2 - 81}$

24. $y = \dfrac{3}{2x^2 - 5x - 3}$

25. $h(x) = 9$

26. $y = \dfrac{1}{x^2 + 10x + 25}$

27. $y = \dfrac{2x - 7}{8x^2 - 10x - 3}$

28. $f(x) = \dfrac{x + 10}{x^2 + 100}$

29. $R(x) = \dfrac{x^2 + 5x + 1}{4x^2 - x - 3}$

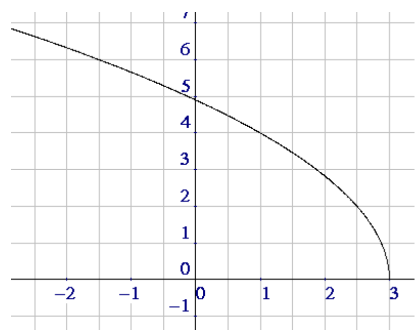
30. $y = \dfrac{3x + 7}{10x^2 - 20x}$

31. $g(x) = \dfrac{x^2 - 14}{14x^2 - 42x}$

❑ *CALCULATING DOMAINS OF LINEAR SQUARE ROOTS*

Find the domain of $f(x) = \sqrt{24 - 8x}$.

<u>Solution:</u> In this formula, we ask ourselves if anything could possibly go wrong using certain values of x. Well, the square root of a negative number doesn't exist in the real numbers, so certainly something could go wrong. For instance, if $x = 4$, then we would obtain $\sqrt{-8}$, not a real number. So $x = 4$ is <u>not</u> allowed to be in the domain.



Now we need a statement which describes the legal x's. How about this:

The square root of a quantity is defined (as a real number) only when that quantity is greater than or equal to zero.

So, in this example, the radicand, $24 - 8x$, must be ≥ 0.

$24 - 8x \ge 0$ (the radicand must be 0 or positive)

$\Rightarrow -8x \ge -24$ (subtract 24 from each side)

$\Rightarrow x \le 3$ (divide by −8, and <u>reverse</u> the inequality)

Therefore, the domain is the set of all real numbers which are less than or equal to 3:

$$\boxed{x \le 3}$$

Homework

Find the domain of each function:

32. $f(x) = \sqrt{3x+12}$

33. $g(x) = \sqrt{8-2x}$

34. $y = \sqrt{7x-15}$

35. $y = \sqrt{-4x+1}$

36. $h(x) = \sqrt{17-17x}$

37. $y = \sqrt{4x+12}$

38. $y = \sqrt{4x+1}$

39. $f(x) = \sqrt{10-10x}$

40. $g(x) = \sqrt{20x}$

41. $k(x) = \sqrt{20-x}$

42. $y = \sqrt{-3x-15}$

43. $y = \sqrt{-30-7x}$

Practice Problems

Find the domain of each function:

44. $y = \sqrt{2}$

45. $y = \dfrac{2x-6}{x^2+4x+3}$

46. $f(x) = \pi$

47. $y = \pi x^3 - 17x + 1$

48. $y = \dfrac{x+2}{x-3}$

49. $f(x) = \dfrac{3}{x^2-x-20}$

50. $y = \dfrac{20}{x^2-x}$

51. $h(x) = \dfrac{\pi}{13x^2+26x}$

52. $g(x) = \sqrt{18-9x}$

53. $h(x) = \dfrac{4-x}{x^2-400}$

54. $f(x) = \dfrac{13x-13}{9x^2-42x+49}$

55. $y = \dfrac{17x-17}{25x^2+20x+4}$

56. $y = \dfrac{x^2-5x+6}{50}$

57. $y = \dfrac{\sqrt{2}}{3}$

58. $f(x) = \sqrt{12x-16}$

59. $g(x) = \sqrt{10-5x}$

60. $y = \dfrac{x^2-49}{2x+17}$

61. $y = \dfrac{\sqrt{3}}{9x^2-49}$

62. $h(x) = \sqrt{3-4x}$

63. $f(x) = 13x+26$

64. $y = \dfrac{x-3}{3x^2-12x}$

65. $g(x) = \dfrac{12x-24}{x^2+1}$

66. $y = \dfrac{\sqrt{2}}{x^2-1}$

67. $f(x) = \dfrac{x^2-x-1}{14x^2-23x-10}$

68. True/False:

a. The domain of $f(x) = 3x + 4$ is $\mathbb{R}$.

b. The domain of $y = (x - 7)^2$ is $\mathbb{R} - \{7\}$.

c. The domain of $y = 9$ is $\mathbb{R}$.

d. The domain of $h(x) = \dfrac{1}{x^2}$ is $\mathbb{R}$.

e. The domain of $y = \dfrac{3x-9}{5x-20}$ is $\mathbb{R} - \{4\}$.

f. The domain of $f(x) = \sqrt{7x-63}$ is $x \ge 9$.

g. The domain of $f(x) = \dfrac{x^2+17x-\pi}{1000}$ is $\mathbb{R}$.

h. The domain of $g(x) = \dfrac{2x+1}{x^2-5x+6}$ is $\mathbb{R} - \{3\}$.

i. The domain of $y = \sqrt{17-34x}$ is $x \le \dfrac{1}{2}$.

j. The domain of $f(x) = \dfrac{x}{x^2+4}$ is $\mathbb{R} - \{\pm 2\}$.

k. The domain of $h(x) = \dfrac{x^2-5}{x^2-10x}$ is $\mathbb{R} - \{0, 10\}$

Solutions

1. All four numbers are legal inputs, since all four numbers could be multiplied by 2 and then have 10 added on with no problem.

2. 3 is a legal input, since $\frac{3}{3-2} = \frac{3}{1} = 3$; no muss, no fuss.

2 is <u>not</u> legal, since $\frac{3}{2-2} = \frac{3}{0} = \textit{Undefined}$.

π is legal because nothing can go wrong.

0 is legal because $\frac{3}{-2}$ is a perfectly fine number.

3. $\sqrt{7}$ is a perfectly fine real number, so 7 is a legal input.
$\sqrt{0}$ is also a real number (= 0), so 0 is legal.
$\sqrt{-9}$ is <u>not</u> a real number, so this number doesn't exist in our current world; hence, –9 is <u>not</u> a legal input.
$\sqrt{99}$ is some real number (9 point something), so 99 is legal.

4. When $x = 12$, we get $12^2 - 144 = 144 - 144 = 0$ in the denominator. Zero in the denominator? I don't think so! Thus, 12 is <u>not</u> a legal input. 20 is O.K. to use for x, just as 0 is fine for x. So 20 and 0 are legal inputs. But when $x = -12$, we're in the same boat we were when x was 12. This is because when $x = -12$ we get $144 - 144 = 0$ in the denominator, which is certainly not allowed. Therefore, –12 is <u>not</u> legal.

5. All four values of x are legal, since none of them makes the denominator zero.

6. a. 4 b. 8 c. 1 d. 0 e. Undefined

7. 1/7; Undefined; –1/5; –1/9; –1/8; Undefined; 1/7; 1/16
When $x = 3$ or -3, the function is undefined due to dividing by zero. It seems that no other inputs would produce a zero in the denominator. So

our guess would be that the domain of h is all real numbers <u>except</u> 3 and −3. This domain can be written $\mathbb{R} - \{3, -3\}$, or $\mathbb{R} - \{\pm 3\}$

8. $\mathbb{R}$ **9**. $\mathbb{R} - \{-3\}$ **10**. $\mathbb{R}$ **11**. $\mathbb{R} - \left\{\frac{7}{9}\right\}$ **12**. $\mathbb{R} - \{\pm 10\}$

13. $\mathbb{R}$, and here's why: No matter what x is, x^2 is at least 0 (since x^2 is never negative). Now add 49 to something that is at least 0, and you have a number which is at least 49. So the denominator $x^2 + 49$ can <u>never</u> be 0. Since dividing by 0 is the only critical issue in this function, the domain is all real numbers.

14. $\mathbb{R}$ **15**. $\mathbb{R}$ **16**. $\mathbb{R} - \left\{0, -\frac{3}{2}\right\}$ **17**. $\mathbb{R} - \{0, 2\}$

18. $\mathbb{R}$ **19**. $\mathbb{R} - \left\{-\frac{9}{2}\right\}$ **20**. $\mathbb{R} - \{\pm 12\}$ **21**. $\mathbb{R}$

22. $\mathbb{R}$ **23**. $\mathbb{R} - \{\pm 9\}$ **24**. $\mathbb{R} - \left\{3, -\frac{1}{2}\right\}$ **25**. $\mathbb{R}$

26. $\mathbb{R} - \{-5\}$ **27**. $\mathbb{R} - \left\{-\frac{1}{4}, \frac{3}{2}\right\}$ **28**. $\mathbb{R}$ **29**. $\mathbb{R} - \left\{1, -\frac{3}{4}\right\}$

30. $\mathbb{R} - \{0, 2\}$ **31**. $\mathbb{R} - \{0, 3\}$ **32**. $x \ge -4$ **33**. $x \le 4$

34. $x \ge \frac{15}{7}$ **35**. $x \le \frac{1}{4}$ **36**. $x \le 1$ **37**. $x \ge -3$

38. $x \ge -\frac{1}{4}$ **39**. $x \le 1$ **40**. $x \ge 0$ **41**. $x \le 20$

42. $x \le -5$ **43**. $x \le -\frac{30}{7}$ **44**. $\mathbb{R}$ **45**. $\mathbb{R} - \{-1, -3\}$

46. $\mathbb{R}$ **47**. $\mathbb{R}$ **48**. $\mathbb{R} - \{3\}$ **49**. $\mathbb{R} - \{5, -4\}$

50. $\mathbb{R} - \{0, 1\}$ **51**. $\mathbb{R} - \{0, -2\}$ **52**. $x \le 2$ **53**. $\mathbb{R} - \{\pm 20\}$

54. $\mathbb{R} - \left\{\frac{7}{3}\right\}$ **55**. $\mathbb{R} - \left\{-\frac{2}{5}\right\}$ **56**. $\mathbb{R}$ **57**. $\mathbb{R}$

58. $x \ge \frac{4}{3}$ **59**. $x \le 2$ **60**. $\mathbb{R} - \left\{-\frac{17}{2}\right\}$ **61**. $\mathbb{R} - \left\{\pm\frac{7}{3}\right\}$

62. $x \le \frac{3}{4}$ **63**. $\mathbb{R}$ **64**. $\mathbb{R} - \{0, 4\}$ **65**. $\mathbb{R}$

66. $\mathbb{R} - \{\pm 1\}$ **67**. $\mathbb{R} - \left\{2, -\frac{5}{14}\right\}$

68. a. T b. F c. T d. F e. T f. T g. T h. F
i. T j. F k. T

"Education is not the filling of a pail, but the lighting of a fire."