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# CH 37 – RATIONAL FUNCTIONS

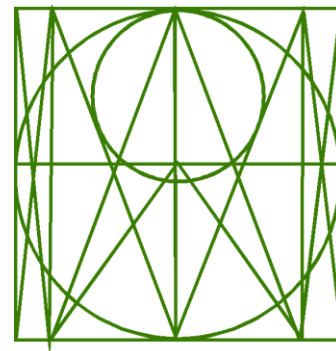
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## □ INTRODUCTION

Remember what we call a number like  $\frac{2}{7}$ ? This is called a *rational number* because it is the *ratio* of two integers. In a like manner, a rational function is the ratio of two special functions called ***polynomials***. Since a rational function is essentially a fraction, we will have to avoid dividing by zero, which means the domain of a such a function may not be all real numbers.



## □ POLYNOMIAL FUNCTIONS

Each of the following is a polynomial function:

$$y = 7 \quad (\text{a } \textit{linear} \text{ function – it's a horizontal line})$$

$$y = -3x + \sqrt{2} \quad (\text{a } \textit{linear} \text{ function – it's a line with slope } -3)$$

$$y = 2x^2 - x + 9 \quad (\text{a } \textit{quadratic} \text{ function – it's a parabola})$$

$$f(x) = \sqrt[4]{2}x^3 - x^2 \quad (\text{a } \textit{cubic} \text{ function})$$

$$P(x) = -\pi x^4 + 5x^2 + 8 \quad (\text{a } \textit{quartic} \text{ function})$$

$$Q(x) = \frac{2}{3}x^5 + 1 \quad (\text{a } \textit{quintic} \text{ function})$$

The key to any ***polynomial function*** is that all the exponents on the  $x$  come from the set of whole numbers:  $\{0, 1, 2, 3, \dots\}$ . The coefficients (the numbers in front of the variables), on the other hand, can come from anywhere in  $\mathbb{R}$ , the set of real numbers.

Consider the quartic (4<sup>th</sup> degree) polynomial

$$y = -2\pi x^4 + \frac{9}{10}x^3 - 17x^2 + \sqrt{2}.$$

First look at the exponents; they are all whole numbers. Even the last term of the polynomial,  $\sqrt{2}$ , can be written as  $\sqrt{2}x^0$ , and so even the exponent on this last term is a whole number. Thus, all the exponents (4, 3, 2, and 0) come from the whole numbers, while all the coefficients ( $-2\pi$ ,  $\frac{9}{10}$ ,  $-17$ ,  $\sqrt{2}$ ) come from  $\mathbb{R}$ . Considering the definition of polynomial, the given function is indeed a polynomial.

Each of the following is not a polynomial function:

$$y = \frac{1}{x} \quad \left(\frac{1}{x} = x^{-1} \text{ and } -1 \text{ is not a whole number}\right)$$

$$y = \sqrt{x} \quad \left(\sqrt{x} = x^{1/2} \text{ and } \frac{1}{2} \text{ is not a whole number}\right)$$

$$f(x) = \frac{1}{\sqrt[3]{x}} \quad \left(\frac{1}{\sqrt[3]{x}} = x^{-1/3} \text{ and } -\frac{1}{3} \text{ is not a whole number}\right)$$

$$g(x) = |x - 1| \quad \left(\text{no absolute values allowed around the } x\right)$$

$$E(x) = 2^x \quad \left(\text{since } x \text{ is in the exponent, it can be any number}\right)$$

$$T(x) = \sin x \quad \left(\text{it's on your calculator, but it's not a polynomial}\right)$$

$$y = \log x \quad \left(\text{a log function can never be a polynomial}\right)$$

$$x^2 + y^2 = 25 \quad \left(\text{it's a circle – it's not a function of any kind}\right)$$

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## Homework

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1. Explain why  $y = \pi x^5 - \sqrt{2}x^3 + \frac{1}{4}x - 17.5$  is a polynomial function.
2. Explain why  $h(x) = \sqrt[3]{5}x^4 - \sqrt{2}x + \frac{1}{2}$  is not a polynomial function.

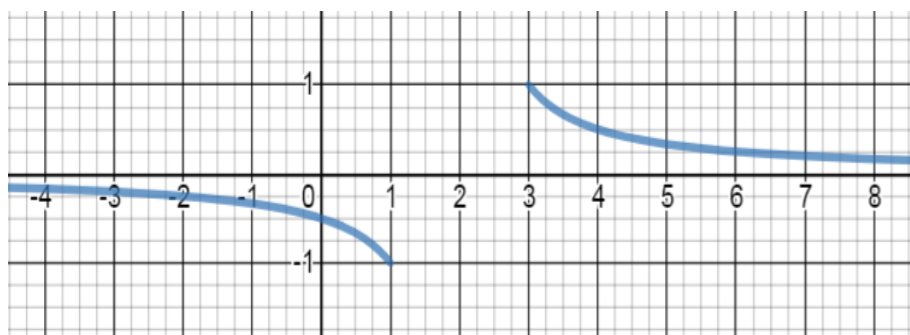


Intercepts come next. If  $x = 0$ , then  $y = \frac{1}{0-2} = -\frac{1}{2}$ . Thus,  $(0, -\frac{1}{2})$  is the  $y$ -intercept. To find an  $x$  intercept, set  $y = 0$ . This gives  $0 = \frac{1}{x-2} \Rightarrow 0(x-2) = \frac{1}{x-2}(x-2) \Rightarrow 0 = 1$ , which has no solution. Thus, there are **no**  $x$ -intercepts.

Now for some ordered pairs that satisfy the formula  $y = \frac{1}{x-2}$ :

If we plot these points and connect them with a smooth curve, we would get the following graph:

$x$	$y$
-3	$-\frac{1}{5}$
-2	$-\frac{1}{4}$
-1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	-1
2	Und.
3	1
4	$\frac{1}{2}$
5	$\frac{1}{3}$
6	$\frac{1}{4}$
7	$\frac{1}{5}$

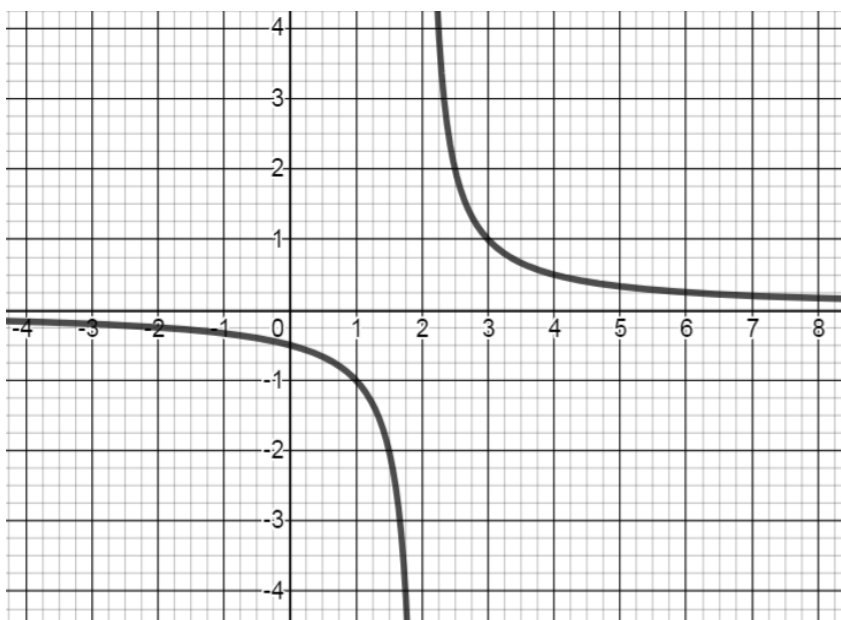


What some students do at this point is to lazily connect the points  $(3, 1)$  and  $(1, -1)$  with a straight line. Talk about jumping to conclusions! Our domain of  $\mathbb{R} - \{2\}$  implies that  $x$  cannot be 2 in this function; the straight-line trick won't work. So we agree that a major chunk of the graph is missing. How do we get a better picture of the graph? We try some  $x$ -values that are near 2:

$$\left(1\frac{1}{2}, -2\right) \quad \left(1\frac{3}{4}, -4\right) \quad \left(1\frac{7}{8}, -8\right)$$

$$\left(2\frac{1}{2}, 2\right) \quad \left(2\frac{1}{4}, 4\right) \quad \left(2\frac{1}{8}, 8\right)$$

Adding these points to our previous attempt at a graph gives us a much better picture:



This graph has some real cool **limits**. Suppose we let  $x$  approach  $\infty$ . The  $y$ -values are positive (the curve is above the  $x$ -axis), but are getting smaller and smaller, approaching zero. Thus, as  $x \rightarrow \infty$ ,  $y \rightarrow 0$ .

Now let  $x$  approach  $-\infty$ . The  $y$ -values are negative but are rising toward zero. Therefore, as  $x \rightarrow -\infty$ ,  $y \rightarrow 0$ .

The number 2 seems to be an interesting  $x$ -value. Although  $x$  can never be 2 in this function, it looks like the curve is getting closer and closer to the vertical line  $x = 2$ . In fact, if we let  $x$  approach 2 from the right, the curve is growing taller and taller, and so we have the limit: As  $x \rightarrow 2$  (from the right),  $y \rightarrow \infty$ .

Now let  $x$  approach 2 from the left. This time the curve is dropping rapidly, toward negative infinity. This observation yields the limit: As  $x \rightarrow 2$  (from the left),  $y \rightarrow -\infty$ .

Let's summarize the four limits we've deduced:

$$\text{As } x \rightarrow \infty, y \rightarrow 0$$

$$\text{As } x \rightarrow -\infty, y \rightarrow 0$$

$$\text{As } x \rightarrow 2 \text{ (from the right), } y \rightarrow \infty$$

$$\text{As } x \rightarrow 2 \text{ (from the left), } y \rightarrow -\infty$$

Do you see that as you move far to the right or far to the left, the curve gets closer and closer to the  $x$ -axis? We say that the line  $y = 0$  (which is the  $x$ -axis) is a **horizontal asymptote**.

Now look at the region of the graph near  $x = 2$ . The curve gets closer and closer to the vertical line  $x = 2$  (in fact, on both sides of the vertical line). We call the line  $x = 2$  a **vertical asymptote**.

**EXAMPLE 2:**      **Graph:**  $y = \frac{2x-1}{x+2}$

**Solution:** First we find the **domain**. Recall that this function will be undefined when the denominator is zero, which occurs when  $x = -2$ . Thus, the domain is  $\mathbb{R} - \{-2\}$ .

Now let's explore the **intercepts**:

If  $x = 0$ , then  $y = \frac{2(0)-1}{0+2} = -\frac{1}{2}$ . There's a  $y$ -intercept at  $(0, -\frac{1}{2})$ .

If  $y = 0$ , then  $0 = \frac{2x-1}{x+2} \Rightarrow 2x-1 = 0 \Rightarrow x = \frac{1}{2}$ . So  $(\frac{1}{2}, 0)$  is an  $x$ -intercept.

It's time for some more ordered pairs for this function. Use your calculator to verify each of the following:

$(-1, -3)$     $(1, 0.33)$     $(3, 1)$     $(5, 1.29)$     $(10, 1.58)$

$(15, 1.71)$     $(20, 1.77)$     $(100, 1.95)$     $(1000, 1.995)$

What's happening as  $x$  grows very large? It appears that  $y$  is approaching 2. That is, as  $x \rightarrow \infty$ ,  $y \rightarrow 2$ .

Now we'll let  $x$  go the other direction:

$(-3, 7)$     $(-5, 3.67)$     $(-10, 2.63)$     $(-20, 2.28)$

$(-100, 2.05)$     $(-1000, 2.01)$

These points show that as  $x \rightarrow -\infty$ ,  $y \rightarrow 2$ .

Finally, here are some ordered pairs for  $x$ 's near  $-2$  (the only real number not in the domain):

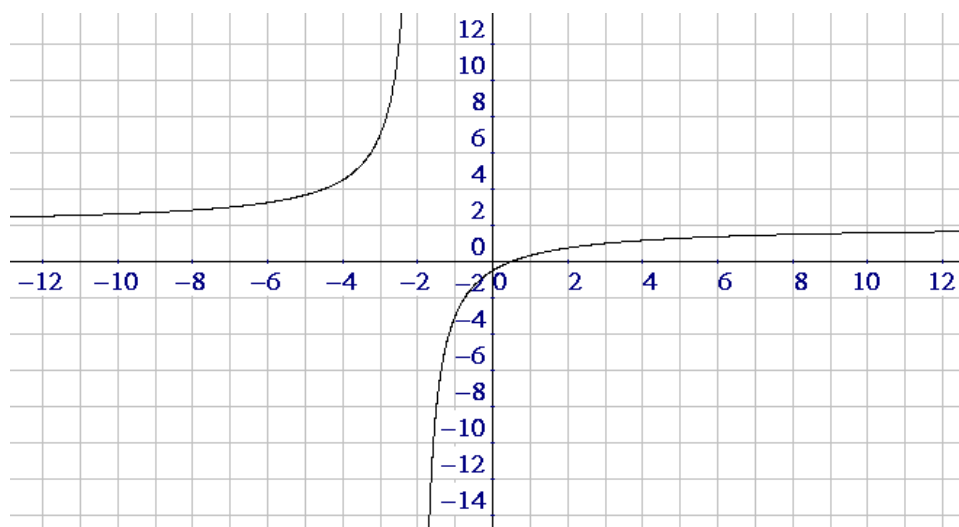
$$(-1.5, -8) \quad (-1.9, -48) \quad (-1.99, -498)$$

Thus, as  $x \rightarrow -2$  from the right,  $y \rightarrow -\infty$ .

$$(-2.5, 12) \quad (-2.1, 52) \quad (-2.01, 502)$$

Therefore, as  $x \rightarrow -2$  from the left,  $y \rightarrow \infty$ .

Plotting as many of the calculated points as possible, and taking into account the four limits we've found, the following graph emerges:



We can now be reasonably sure of the **asymptotes**. Either by recalling the limits described above or by looking at the graph, we conclude that there's a vertical asymptote at  $x = -2$  and a horizontal asymptote at  $y = 2$ .

**EXAMPLE 3:**      **Graph:**  $y = \frac{4}{1+x^2}$

**Solution:** Why is this function rational? Because it's the ratio  $P/Q$  of two polynomial functions: the constant polynomial  $P(x) = 4$  and the quadratic polynomial  $Q(x) = 1 + x^2$ .

To find the **domain**, set the denominator to zero to see what's not in the domain:  $1 + x^2 = 0$ . This equation has no solution in  $\mathbb{R}$ , since solving it leads to  $x = \pm\sqrt{-1}$ , which are not real numbers. In fact, for any value of  $x$ , the quantity  $1 + x^2$  is at least 1 (why?), so it certainly can't be zero. Since the denominator can never be zero, there's nothing to be excluded from the domain, and therefore the domain is  $\mathbb{R}$ . We can also figure that the graph will not have a **vertical asymptote**, since the denominator can never be 0.

Notice that if we put in some positive value of  $x$ , we'll get a certain  $y$ -value. Now look at what will happen if we put  $-x$  (the negative version of  $x$ ) into the formula. Since the  $x$  is being squared in the formula, we will get the same  $y$ -value. This implies that the left side of the graph will be mirror image of the right side. We say that the graph possesses ***y-axis symmetry***.

Now we seek the **intercepts**. Set  $x = 0$  to get  $y = 4$ , and so the  $y$ -intercept is  $(0, 4)$ . Now set  $y = 0$ , giving

$$0 = \frac{4}{1+x^2} \Rightarrow 0(1+x^2) = \frac{4}{1+x^2}(1+x^2) \Rightarrow 0 = 4.$$

This absurd result indicates that the equation has no solution; hence, there are no  $x$ -intercepts.

It's time for some ordered pairs for this function:

$$(1, 2) \quad (2, 0.8) \quad (3, 0.4) \quad (4, 0.24) \quad (10, 0.04) \quad (200, 0.0001)$$



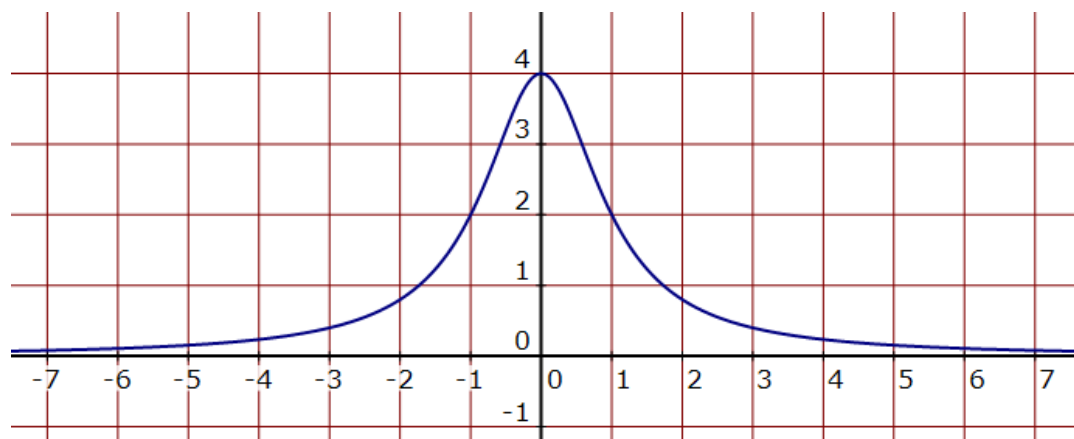
These points suggest the limit: As  $x \rightarrow \infty, y \rightarrow 0$ . This implies that  $y = 0$  is a horizontal **asymptote**.

Here are some more ordered pairs, designed to see what happens as we approach the  $y$ -axis from the right:

(0.75, 2.56) (0.5, 3.2) (0.25, 3.76) (0.1, 3.96) (0.02, 3.998)

These points give us the limit: As  $x \rightarrow 0$  (from the right),  $y \rightarrow 4$ .

If we plot all the points calculated so far, and if we recall the  $y$ -axis symmetry, we get the following graph:



We determined at the outset that the domain of this rational function is  $\mathbb{R}$ . Is it clear from the graph that this is indeed the case?

## Homework

5. Consider the rational function in Example 2. Without referring to the graph, prove that  $y$  can have the value 2.01, but  $y$  can never have the value 2.

6. Find the domain:

a.  $y = \frac{2x+7}{9}$

b.  $f(x) = \frac{2x-3}{4+x}$

c.  $g(x) = \frac{3x}{2x-10}$

d.  $R(x) = \frac{x-1}{-7x+4}$

e.  $f(x) = \frac{x^2-9}{x^2-100}$

f.  $g(x) = \frac{8x-16}{x^2+25}$

7. Find the intercepts:

a.  $f(x) = \frac{x-4}{x-2}$

b.  $y = \frac{3}{5x-15}$

c.  $y = \frac{2x+1}{x-3}$

d.  $g(x) = \frac{5-x}{6x+1}$

8. Find the asymptotes:

a.  $R(x) = \frac{8x+1}{4x-4}$

b.  $y = \frac{2x-3}{2x+1}$

c.  $y = \frac{3x-7}{x+2}$

d.  $h(x) = \frac{2x+7}{4x-4}$

9. Find the domain, the intercepts, and the asymptotes of

$$y = \frac{1}{4+x^2}.$$

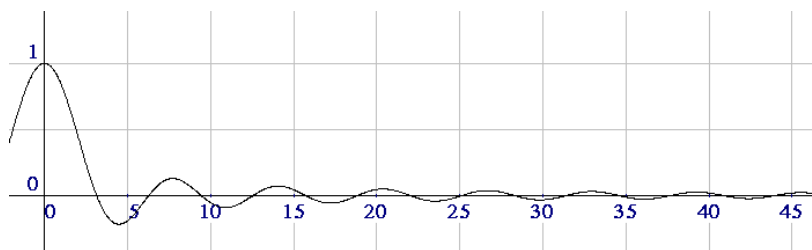
10. Perform a complete analysis of the function  $y = \frac{2}{x-3}$ .

11. Perform a complete analysis of the function  $y = \frac{3x-5}{x-2}$ .

12. Perform a complete analysis of the function  $y = \frac{2}{2+x^2}$ .

13. Perform a complete analysis of the function  $y = \frac{-1}{x+1}$ .

14. Consider the graph



Explain why the horizontal line  $y = 0$  (that is, the  $x$ -axis) is a horizontal asymptote for the curve.

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## Practice Problems

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15. Explain why  $f(x) = \sqrt{7}x^{10} + \pi x^7 - 6x - 1$  is a polynomial. What is its degree?
16. Explain why  $y = 3x^5 - \sqrt{x} + \pi$  is not a polynomial.
17. a. A horizontal line (is, is not) a polynomial.  
 b. The function  $y = \frac{1}{x}$  (is, is not) a polynomial.  
 c. What is the degree of the polynomial  $y = 7x - \pi$ ?  
 d. Is a circle a polynomial?
18. Consider the rational function  $y = \frac{7}{2x - 8}$ .
- Find the domain.
  - Find all the intercepts.
  - Find all the asymptotes.
  - Calculate  $y$  if  $x = 4.1$ .

19. Find all the intercepts and asymptotes of  $r(x) = \frac{8x+6}{2x-3}$ , and graph.
20. Graph  $y = \frac{-2x-2}{x-1}$ .  
 As  $x \rightarrow 1$  (from the right),  $y \rightarrow \underline{\hspace{1cm}}$ . As  $x \rightarrow 1$  (from the left),  $y \rightarrow \underline{\hspace{1cm}}$ .  
 As  $x \rightarrow \infty$ ,  $y \rightarrow \underline{\hspace{1cm}}$ . As  $x \rightarrow -\infty$ ,  $y \rightarrow \underline{\hspace{1cm}}$ .
21. Graph  $y = \frac{5}{2+x^2}$ . Discuss domain, range, symmetry, and asymptotes. As  $x \rightarrow \infty$ ,  $y \rightarrow \underline{\hspace{1cm}}$ . As  $x \rightarrow -\infty$ ,  $y \rightarrow \underline{\hspace{1cm}}$ .  
 As  $x \rightarrow 0$  (from the right),  $y \rightarrow \underline{\hspace{1cm}}$ . As  $x \rightarrow 0$  (from the left),  $y \rightarrow \underline{\hspace{1cm}}$ .
22. True/False:
- $y = \sqrt[3]{7}x^{10} - \pi x^3 + \sqrt{2}$  is a polynomial.
  - $y = \frac{1}{x^5}$  is a polynomial.
  - The graph of  $f(x) = \frac{1}{2x+10}$  has a vertical asymptote at  $x = -5$ .
  - The graph of  $g(x) = \frac{10x+9}{5x-11}$  has a horizontal asymptote at  $y = 10$ .
  - The domain of the function  $y = \frac{6}{1+x^2}$  is  $\mathbb{R} - \{\pm 1\}$ .
  - For the graph of  $y = \frac{3x+1}{x-\pi}$ , as  $x \rightarrow \infty$ ,  $y \rightarrow 3$ .

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## Solutions

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- All coefficients are from  $\mathbb{R}$ , and all exponents are from  $\mathbb{W}$ .
- The middle term is  $\sqrt{2}x^{1/2}$ , and  $\frac{1}{2} \notin \mathbb{W}$ .

3. a. 0      b. 3      c. 10      d. 1
4. a.  $\mathbb{R}$       b. False      c. True      d. False
5.  $y = \frac{2x-1}{x+2} \Rightarrow 2.01 = \frac{2x-1}{x+2} \Rightarrow 2.01x + 4.02 = 2x - 1 \Rightarrow x = -502$ .  
So,  $(-502, 2.01)$  is on the graph, and indeed  $y$  can be 2.01.

Now let's pretend that  $y$  could be 2; then

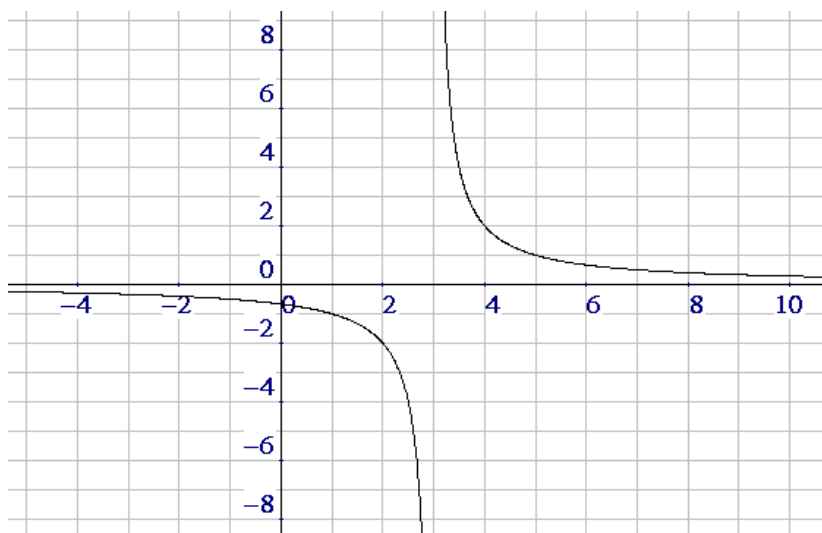
$2 = \frac{2x-1}{x+2} \Rightarrow 2x + 4 = 2x - 1 \Rightarrow 4 = -1 \Rightarrow$  no solution. Thus, there is no  $x$  which will make  $y = 2$ .

6. a.  $\mathbb{R}$     b.  $\mathbb{R} - \{-4\}$     c.  $\mathbb{R} - \{5\}$     d.  $\mathbb{R} - \left\{\frac{4}{7}\right\}$     e.  $\mathbb{R} - \{\pm 10\}$     f.  $\mathbb{R}$
7. a.  $(4, 0)$   $(0, 2)$     b.  $(0, -\frac{1}{5})$     c.  $(-\frac{1}{2}, 0)$   $(0, -\frac{1}{3})$     d.  $(5, 0)$   $(0, 5)$
8. a.  $x = 1$   $y = 2$     b.  $x = -\frac{1}{2}$   $y = 1$     c.  $x = -2$   $y = 3$     d.  $x = 1$   $y = \frac{1}{2}$
9. Since the only way the formula can be messed up is by dividing by 0, and since the denominator can never be zero (verify this yourself), the domain is  $\mathbb{R}$ .

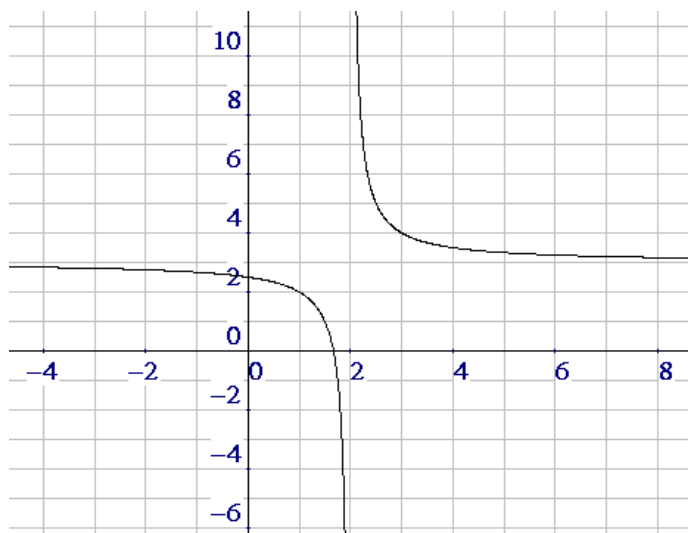
Setting  $x = 0$  gives a  $y$ -value of  $1/4$ , so the  $y$ -intercept is  $(0, \frac{1}{4})$ . If you set  $y = 0$ , you'll get no solution for  $y$ . Thus, there is no  $x$ -intercept.

There are no vertical asymptotes, since the denominator is never zero. Letting  $x$  approach either  $\infty$  or  $-\infty$ ,  $y$  approaches 0. Thus, a horizontal asymptote is  $y = 0$  (the  $x$ -axis).

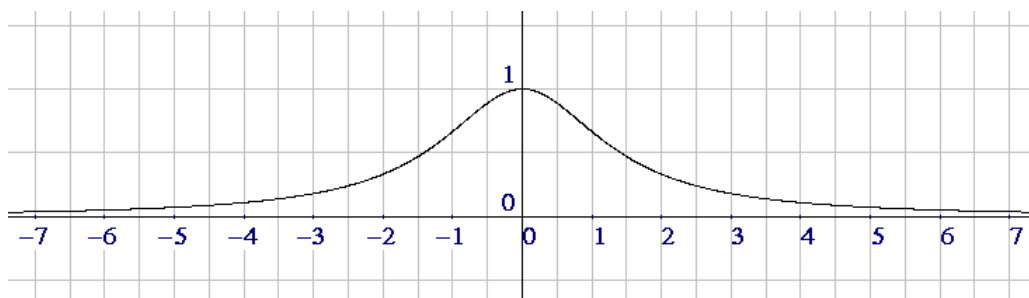
10.

Domain =  $\mathbb{R} - \{3\}$  $x$ -int: none $y$ -int:  $(0, -\frac{2}{3})$ As  $x \rightarrow 3$  (from the right),  $y \rightarrow \infty$ As  $x \rightarrow 3$  (from the left),  $y \rightarrow -\infty$ As  $x \rightarrow \infty$ ,  $y \rightarrow 0$ As  $x \rightarrow -\infty$ ,  $y \rightarrow 0$ vert asy:  $x = 3$ horiz asy:  $y = 0$ 

11.

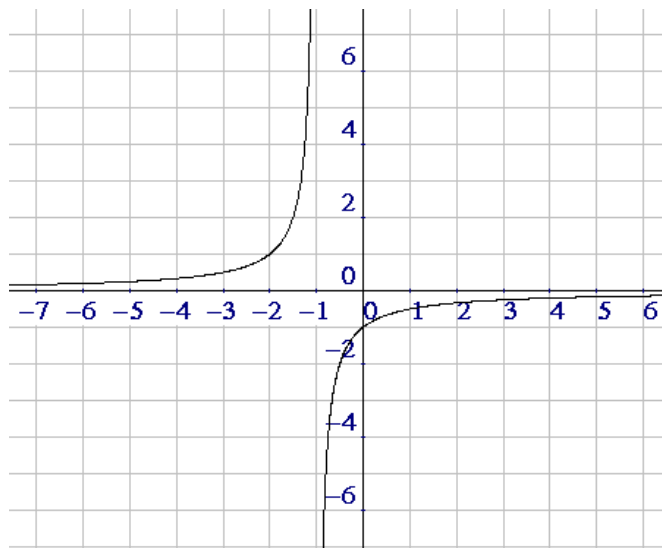
Domain =  $\mathbb{R} - \{2\}$  $x$ -int:  $(\frac{5}{3}, 0)$  $y$ -int:  $(0, \frac{5}{2})$ As  $x \rightarrow 2$  (from the right),  $y \rightarrow \infty$ As  $x \rightarrow 2$  (from the left),  $y \rightarrow -\infty$ As  $x \rightarrow \infty$ ,  $y \rightarrow 3$ As  $x \rightarrow -\infty$ ,  $y \rightarrow 3$ vert asy:  $x = 2$ horiz asy:  $y = 3$ 

12.



Domain =  $\mathbb{R}$      $x$ -int: none     $y$ -int: (0, 1)  
 As  $x \rightarrow \infty, y \rightarrow 0$     As  $x \rightarrow -\infty, y \rightarrow 0$   
 vert asy: none    horiz asy:  $y = 0$   
 maximum point at (0, 1)

13.

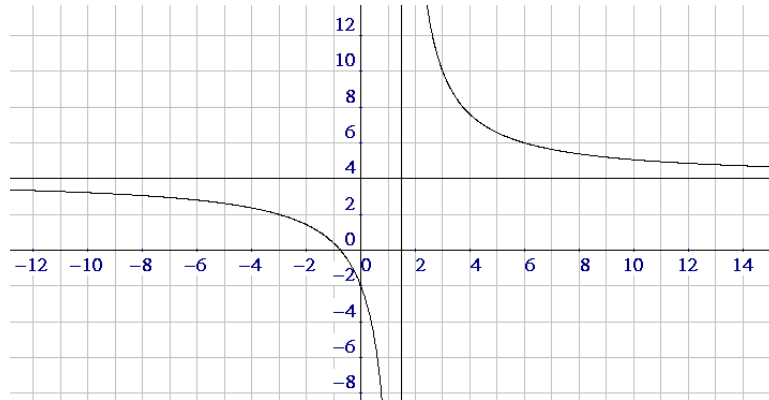


Domain =  $\mathbb{R} - \{-1\}$   
 $x$ -int: none  
 $y$ -int: (0, -1)  
 As  $x \rightarrow -1$  (from the right),  
 $y \rightarrow -\infty$   
 As  $x \rightarrow -1$  (from the left),  
 $y \rightarrow \infty$   
 As  $x \rightarrow \infty, y \rightarrow 0$   
 As  $x \rightarrow -\infty, y \rightarrow 0$   
 vert asy:  $x = -1$   
 horiz asy:  $y = 0$

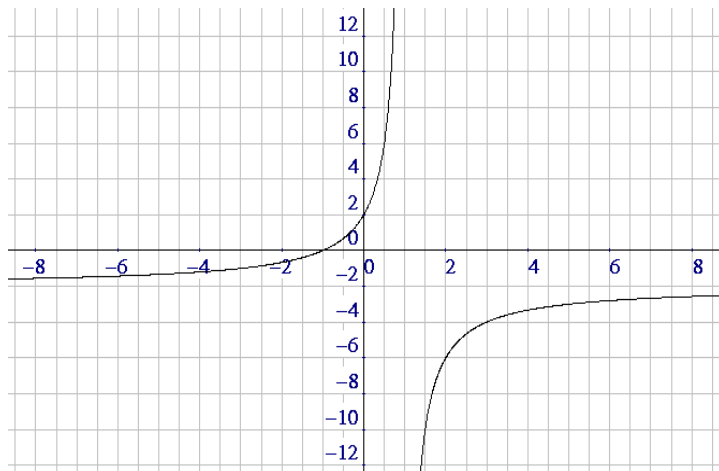
14. Because of the limit: As  $x \rightarrow \infty, y \rightarrow 0$ . Even though the graph intersects its own horizontal asymptote infinitely often, the curve nevertheless continues to get closer and closer to the  $x$ -axis (the line  $y = 0$ ), and this is ultimately what is meant by a horizontal asymptote.
15.  $f$  is a polynomial because the coefficients are real numbers and the exponents (the 10, 7 and 1) are whole numbers. Its degree is 10.
16. Look at the middle term; it can be written as  $x^{1/2}$ , a term whose exponent is not from the whole numbers.
17. a. is      b. is not ( $1/x = x^{-1}$ )      c. 1      d. It's not even a function, let alone the special function called a polynomial.

18. a.  $\mathbb{R} - \{4\}$       b.  $(0, -7/8)$       c.  $x = 4$  and  $y = 0$  d. 35

19. Intercepts:  $(0, -2)$  and  $(-\frac{3}{4}, 0)$ ; vert asy:  $x = \frac{3}{2}$ ; horiz asy:  $y = 4$

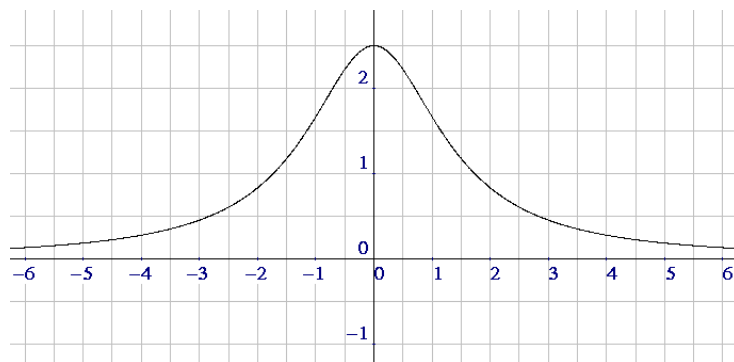


20.



Limits:  $-\infty$ ;  $\infty$ ;  $-2$ ;  $-2$

21.



Domain =  $\mathbb{R}$

Range =  $(0, \frac{5}{2}]$

y-axis symmetry

No vert asy

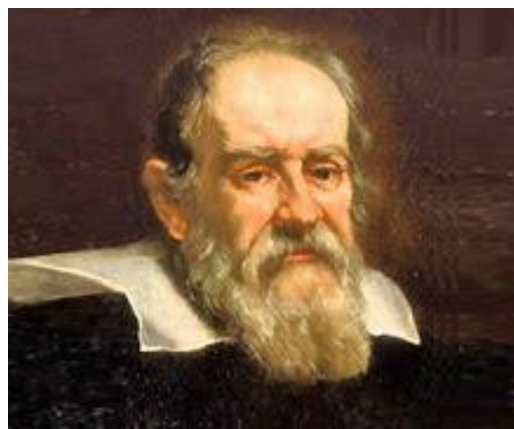
Horiz asy:  $y = 0$

Limits:  $0$ ;  $0$ ;  $\frac{5}{2}$ ;  $\frac{5}{2}$



22. a. T   b. F   c. T   d. F   e. F   f. T

*“ The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language.”*



*Galileo Galiliei*