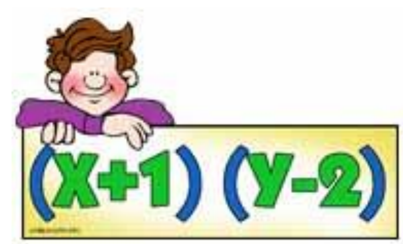
FACTORING QUADRATICS – THE REAL DEAL

❑ INTRODUCTION

Hopefully you have grasped the concept of factoring (see *Factoring – Introduction to Factoring Quadratics*). We now present a series of examples which try to turn the random method into something a little more systematic; but no matter what method is used, it's essentially a matter of *trial and error*. (Some teachers call it *guess and check*.)



❑ EXAMPLES

EXAMPLE 1: **Factor: $6x^2 - 7x - 5$**

Solution: We first think about the ways that $6x^2$ can be broken up. We see $6x \cdot x$ and $3x \cdot 2x$.

Using $6x$ and x, we start with

$$(6x \_\_\_\_)(x \_\_\_\_)$$

Now put numbers in the slots whose product is –5:

$(6x + \mathbf{5})(x - \mathbf{1}) = 6x^2 - 6x + 5x - 5 = 6x^2 - x - 5$ ☹

$(6x - \mathbf{5})(x + \mathbf{1}) = 6x^2 + 6x - 5x - 5 = 6x^2 + x - 5$ ☹

$(6x + \mathbf{1})(x - \mathbf{5}) = 6x^2 - 30x + x - 5 = 6x^2 - 29x - 5$ ☹

$(6x - \mathbf{1})(x + \mathbf{5}) = 6x^2 + 30x - x - 5 = 6x^2 + 29x - 5$ ☹

That's it for the –5; we've tried everything.

Now we'll start again, using $3x$ and $2x$ in the front slots:

$$(3x \;\_\_\_\_\;)(2x \;\_\_\_\_\;)$$

Now try the same combinations as above for the –5:

$$(3x + \mathbf{5})(2x - \mathbf{1}) \;=\; 6x^2 - 3x + 10x - 5 \;=\; 6x^2 + 7x - 5 \quad ☹$$

$$(3x - \mathbf{5})(2x + \mathbf{1}) \;=\; 6x^2 + 3x - 10x - 5 \;=\; \underline{6x^2 - 7x - 5} \quad ☺$$

Although there are other ways to arrange the 5 and the 1, there's no need to go any further; we've found what we were looking for.

Therefore, the final factorization of $6x^2 - 7x - 5$ is

$$\boxed{(3x - 5)(2x + 1)}$$

EXAMPLE 2: **Factor: $n^2 - 5n + 6$**

Solution: As before, we focus on the first and last terms. There's only one way to break up the n^2, namely $n \cdot n$. So we start with

$$(n \;\_\_\_\_\;)(n \;\_\_\_\_\;)$$

But there are a few ways to break up the 6. Noting that 6 is positive — and therefore the signs of its factors must be the same — let's begin experimenting:

$$(n + 6)(n + 1) \;=\; n^2 + n + 6n + 6 \;=\; n^2 + 7n + 6 \quad ☹$$

$$(n + 1)(n + 6) \;=\; n^2 + 6n + n + 6 \;=\; n^2 + 7n + 6 \quad ☹$$

$$(n - 6)(n - 1) \;=\; n^2 - n - 6n + 6 \;=\; n^2 - 7n + 6 \quad ☹$$

$$(n - 1)(n - 6) \;=\; n^2 - 6n - n + 6 \;=\; n^2 - 7n + 6 \quad ☹$$

$$(n + 3)(n + 2) \;=\; n^2 + 2n + 3n + 6 \;=\; n^2 + 5n + 6 \quad ☹$$

$$(n + 2)(n + 3) \;=\; n^2 + 3n + 2n + 6 \;=\; n^2 + 5n + 6 \quad ☹$$

$$(n - 3)(n - 2) \;=\; n^2 - 2n - 3n + 6 \;=\; \underline{n^2 - 5n + 6} \quad ☺$$

and thus our final factorization is

$$(n - 3)(n - 2)$$

Did we really need all those attempts to find the answer? NO.

Isn't it possible to see that the factors of 6 must be negative, considering the $-5n$ term? OF COURSE.

You need to use only as many options as needed until you reach the required goal. Some students can just "see" what combination works, while others need a lot of experimenting. The more you practice, the quicker you'll find the right factors.

EXAMPLE 3: **Factor: $y^2 + 7y + 14$**

Solution: There's really only one way to begin; we split the y^2 term into y and y:

$$(y \_\_\_\_)(y \_\_\_\_)$$

Now let's work on the 14:

$(y + 14)(y + 1) = y^2 + y + 14y + 14 = y^2 + 15y + 14$ ☹

$(y + 1)(y + 14) = y^2 + 14y + y + 14 = y^2 + 15y + 14$ ☹

$(y - 14)(y - 1) = y^2 - y - 14y + 14 = y^2 - 15y + 14$ ☹

$(y - 1)(y - 14) = y^2 - 14y - y + 14 = y^2 - 15y + 14$ ☹

$(y + 7)(y + 2) = y^2 + 2y + 7y + 14 = y^2 + 9y + 14$ ☹

$(y + 2)(y + 7) = y^2 + 7y + 2y + 14 = y^2 + 9y + 14$ ☹

$(y - 7)(y - 2) = y^2 - 2y - 7y + 14 = y^2 - 9y + 14$ ☹

$(y - 2)(y - 7) = y^2 - 7y - 2y + 14 = y^2 - 9y + 14$ ☹

Nothing but sad faces, and we've tried every possible arrangement. This can mean only one thing: There's no way to factor the expression $y^2 + 7y + 14$. The expression is

Not factorable

$y^2 + 7y + 14$ is called **prime** because, like the prime number 13, it can't be factored.

EXAMPLE 4: **Factor:** $w^2 - 25$

Solution: This quadratic expression has only two terms, but which one is missing? When a quadratic starts with w^2, we expect the next term to contain a w. So it's the middle term that is missing. But the middle term is not the one we focus on anyway, so let's begin the usual process:

The binomial $w^2 - 25$ is called a **difference of squares**, because it's w-squared minus 5-squared.

$(w + 25)(w - 1) = w^2 - w + 25w - 25 = w^2 + 24w - 25$ ☹

$(w - 25)(w + 1) = w^2 + w - 25w - 25 = w^2 - 24w - 25$ ☹

$(w + 1)(w - 25) = w^2 - 25w + w - 25 = w^2 - 24w - 25$ ☹

$(w - 1)(w + 25) = w^2 + 25w - w - 25 = w^2 + 24w - 25$ ☹

$(w + 5)(w - 5) = w^2 - 5w + 5w - 25 = \underline{w^2 - 25}$ ☺

Therefore, the expression $w^2 - 25$ factors into

$(w + 5)(w - 5)$

EXAMPLE 5: **Factor: $9u^2 + 12u + 4$**

Solution: The quadratic term, $9u^2$, can be broken down two ways, $9u \times u$ and $3u \times 3u$.

Let's start the multiplications using $9u$ and u:

$(9u + 4)(u + 1) = 9u^2 + 9u + 4u + 4 = 9u^2 + 13u + 4$ ☹

$(9u + 1)(u + 4) = 9u^2 + 36u + u + 4 = 9u^2 + 37u + 4$ ☹

$(9u + 2)(u + 2) = 9u^2 + 18u + 2u + 4 = 9u^2 + 20u + 4$ ☹

That's about it for the $9u$ and u combination. Now for $3u$ and $3u$:

$(3u + 4)(3u + 1) = 9u^2 + 3u + 12u + 4 = 9u^2 + 15u + 4$ ☹

$(3u + 1)(3u + 4) = 9u^2 + 12u + 3u + 4 = 9u^2 + 15u + 4$ ☹

$(3u + 2)(3u + 2) = 9u^2 + 6u + 6u + 4 = \underline{9u^2 + 12u + 4}$ ☺

Eureka! The factorization of $9u^2 + 12u + 4$ is $(3u + 2)(3u + 2)$, which we can write more succinctly as

$$\boxed{(3u+2)^2}$$

EXAMPLE 6: **Factor: $a^2 + 49$**

Solution: Let's start experimenting right away:

$(a + 7)(a + 7) = a^2 + 7a + 7a + 49 = a^2 + 14a + 49$ ☹

$(a + 49)(a + 1) = a^2 + a + 49a + 49 = a^2 + 50a + 49$ ☹

$(a + 1)(a + 49) = a^2 + 49a + a + 49 = a^2 + 50a + 49$ ☹

$(a - 7)(a - 7) = a^2 - 7a - 7a + 49 = a^2 - 14a + 49$ ☹

$(a - 49)(a - 1) = a^2 - a - 49a + 49 = a^2 - 50a + 49$ ☹

$(a - 1)(a - 49) = a^2 - 49a - a + 49 = a^2 - 50a + 49$ ☹

Have we tried everything? It appears we have, and no smiley face! (As we saw before, not every expression can be factored.) So we say that $a^2 + 49$ is

Not factorable

First Notice: Even though $a^2 + 49$ is not factorable, $a^2 - 49$ is factorable, since $a^2 - 49 = (a + 7)(a - 7)$. The difference between a plus sign and a minus sign makes all the difference in the world — so be careful!

Second Notice: Some students jump to the conclusion that when two terms are connected by a plus sign, the expression is not factorable. But consider $4x^2 + 16$. It may not factor with two sets of parentheses like we're learning in this chapter, but it does have a greatest common factor (GCF) of 4, which can be factored out to produce $4(x^2 + 4)$. Thus $4x^2 + 16$ is factorable.

Homework

1. Factor each expression:

a. $2x^2 + 3x + 1$ b. $3n^2 - 7n + 2$ c. $5a^2 + 3a - 2$

d. $3m^2 - 11m - 20$ e. $4x^2 - 3x - 1$ f. $6u^2 + 7u - 10$

g. $4z^2 - 4z - 3$ h. $6y^2 - 5y - 6$ i. $7n^2 - 45n + 18$

j. $a^2 + 16a + 63$ k. $2x^2 + x - 3$ l. $7z^2 - 12z + 5$

2. Factor each expression:

a. $x^2 + 5x + 6$ b. $x^2 - 5x + 6$ c. $x^2 - 5x - 6$

d. $x^2 + 5x - 6$ e. $n^2 + 10n + 9$ f. $z^2 - 4z - 5$

g. $t^2 - 20t + 96$ h. $u^2 - 6u - 16$ i. $Q^2 + 34Q - 72$

j. $x^2 + 25x + 156$ k. $a^2 - 17a + 70$ l. $t^2 + t - 110$

3. Factor each expression:

a. $x^2 + 8x + 16$
b. $y^2 - 10y + 25$
c. $a^2 + 18a + 81$
d. $b^2 - 20b + 100$
e. $4z^2 + 4z + 1$
f. $9n^2 - 24n + 16$
g. $25x^2 - 30x + 9$
h. $x^2 + 6x + 36$
i. $2t^2 + 33t + 100$
j. $16a^2 + 8a + 1$
k. $9u^2 - 12u + 4$
l. $w^2 + 50w + 625$

4. Factor each expression:

a. $p^2 - 1$
b. $c^2 - 4$
c. $R^2 - 16$
d. $z^2 - 36$
e. $x^2 - 25$
f. $y^2 - 81$
g. $n^2 - 10$
h. $w^2 + 16$
i. $a^2 - 144$
j. $e^2 - 72$
k. $m^2 + 100$
l. $W^2 - 1$
m. $x^2 - 9$
n. $a^2 - 49$
o. $g^2 - 64$
p. $y^2 + 4$

5. Factor each expression:

a. $4x^2 - 9$
b. $9y^2 - 49$
c. $u^2 - 2$
d. $v^2 + 1$
e. $16z^2 - 49$
f. $49w^2 - 16$
g. $49a^2 - 144$
h. $121b^2 - 64$
i. $9x^2 + 2$
j. $1 - x^2$
k. $16 - n^2$
l. $25 - 4g^2$
m. $9 + t^2$
n. $144N^2 - 169$
o. $225a^2 - 1$

6. Factor each expression:

a. $3x^2 + 10x - 8$
b. $t^2 - 121$
c. $y^2 + 10y + 25$
d. $16a^2 - 121$
e. $b^2 - 20$
f. $n^2 + 121$
g. $x^2 + 3x + 1$
h. $12q^2 - 23q + 5$
i. $6a^2 - 13a + 6$
j. $x^2 + 14x + 13$
k. $4y^2 - 49$
l. $9Q^2 + 12Q + 4$
m. $25z^2 - 10z + 1$
n. $16x^2 + 34x - 15$
o. $16x^2 + 118x - 15$
p. $16x^2 - 77x - 15$
q. $16x^2 - 72x + 45$
r. $16a^2 - 8a + 1$
s. $x^2 + 7x + 5$
t. $8c^2 + 2c - 21$
u. $8c^2 - 13c - 21$
v. $3a^2 - 5a - 12$
w. $6x^2 + 17x - 14$
x. $2y^2 + 7y - 9$

Review Problems

7. Factor each expression:

a. $x^2 + 17x + 72$ b. $y^2 - 9y + 8$ c. $N^2 + 100$

d. $N^2 - 100$ e. $x^2 - 18x + 81$ f. $a^2 + 10a + 25$

g. $t^2 + 4t - 45$ h. $a^2 - 21a - 22$ i. $a^2 - 9a - 22$

j. $2x^2 - 9x - 5$ k. $6x^2 + x - 40$ l. $6x^2 - 13x - 15$

m. $6x^2 + 11x - 17$ n. $R^2 - 144$ o. $25n^2 - 30n + 9$

p. $T^2 + 144$ q. $49w^2 + 70w + 25$ r. $9a^2 - 9a + 2$

s. $9a^2 - 19a + 2$ t. $x^2 + 16x - 36$ u. $x^2 + 37x + 36$

v. $30c^2 - 11c + 1$ w. $16a^2 - 6a - 1$ x. $36q^2 + 60q + 25$

y. $x^2 + 8x + 18$ z. $32n^2 - 66n + 27$

Solutions

1. a. $(2x + 1)(x + 1)$ b. $(3n - 1)(n - 2)$ c. $(5a - 2)(a + 1)$

d. $(3m + 4)(m - 5)$ e. $(4x + 1)(x - 1)$ f. $(6u - 5)(u + 2)$

g. $(2z + 1)(2z - 3)$ h. $(2y - 3)(3y + 2)$ i. $(7n - 3)(n - 6)$

j. $(a + 9)(a + 7)$ k. $(2x + 3)(x - 1)$ l. $(7z - 5)(z - 1)$

2.
a. $(x + 3)(x + 2)$ b. $(x - 2)(x - 3)$ c. $(x - 6)(x + 1)$
d. $(x + 6)(x - 1)$ e. $(n + 9)(n + 1)$ f. $(z - 5)(z + 1)$
g. $(t - 12)(t - 8)$ h. $(u - 8)(u + 2)$ i. $(Q + 36)(Q - 2)$
j. $(x + 13)(x + 12)$ k. $(a - 10)(a - 7)$ l. $(t + 11)(t - 10)$

3.
a. $(x + 4)^2$ b. $(y - 5)^2$ c. $(a + 9)^2$
d. $(b - 10)^2$ e. $(2z + 1)^2$ f. $(3n - 4)^2$
g. $(5x - 3)^2$ h. Not factorable i. $(2t + 25)(t + 4)$
j. $(4a + 1)^2$ k. $(3u - 2)^2$ l. $(w + 25)^2$

4.
a. $(p + 1)(p - 1)$ b. $(c + 2)(c - 2)$ c. $(R + 4)(R - 4)$
d. $(z + 6)(z - 6)$ e. $(x + 5)(x - 5)$ f. $(y + 9)(y - 9)$
g. Not factorable h. Not factorable i. $(a + 12)(a - 12)$
j. Not factorable k. Not factorable l. $(W + 1)(W - 1)$
m. $(x + 3)(x - 3)$ n. $(a + 7)(a - 7)$ o. $(g + 8)(g - 8)$
p. Not factorable

5.
a. $(2x + 3)(2x - 3)$ b. $(3y + 7)(3y - 7)$ c. Not factorable
d. Not factorable e. $(4z + 7)(4z - 7)$ f. $(7w + 4)(7w - 4)$
g. $(7a + 12)(7a - 12)$ h. $(11b + 8)(11b - 8)$ i. Not factorable
j. $(1 + x)(1 - x)$ k. $(4 + n)(4 - n)$ l. $(5 + 2g)(5 - 2g)$
m. Not factorable n. $(12N + 13)(12N - 13)$ o. $(15a + 1)(15a - 1)$

6.
a. $(3x - 2)(x + 4)$ b. $(t + 11)(t - 11)$ c. $(y + 5)^2$
d. $(4a + 11)(4a - 11)$ e. Not factorable f. Not factorable
g. Not factorable h. $(3q - 5)(4q - 1)$ i. $(2a - 3)(3a - 2)$
j. $(x + 1)(x + 13)$ k. $(2y + 7)(2y - 7)$ l. $(3Q + 2)^2$
m. $(5z - 1)^2$ n. $(8x - 3)(2x + 5)$ o. $(8x - 1)(2x + 15)$
p. $(16x + 3)(x - 5)$ q. $(4x - 3)(4x - 15)$ r. $(4a - 1)^2$

s. Not factorable
t. $(4c + 7)(2c - 3)$
u. $(8c - 21)(c + 1)$
v. $(3a + 4)(a - 3)$
w. $(2x + 7)(3x - 2)$
x. $(2y + 9)(y - 1)$

7.
a. $(x + 9)(x + 8)$
b. $(y - 1)(y - 8)$
c. Not factorable
d. $(N + 10)(N - 10)$
e. $(x - 9)^2$
f. $(a + 5)^2$
g. $(t + 9)(t - 5)$
h. $(a - 22)(a + 1)$
i. $(a - 11)(a + 2)$
j. $(2x + 1)(x - 5)$
k. $(3x + 8)(2x - 5)$
l. $(6x + 5)(x - 3)$
m. $(6x + 17)(x - 1)$
n. $(R + 12)(R - 12)$
o. $(5n - 3)^2$
p. Not factorable
q. $(7w + 5)^2$
r. $(3a - 1)(3a - 2)$
s. $(9a - 1)(a - 2)$
t. $(x + 18)(x - 2)$
u. $(x + 36)(x + 1)$
v. $(5c - 1)(6c - 1)$
w. $(8a + 1)(2a - 1)$
x. $(6q + 5)^2$
y. Not factorable
z. $(16n - 9)(2n - 3)$

"The school is the last expenditure on which America should be willing to economize."

Franklin D. Roosevelt