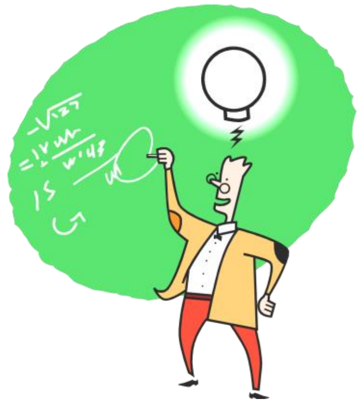
PREPARING FOR THE QUADRATIC FORMULA

❑ INTRODUCTION



We're pretty good by now at solving equations like $2(3x - 4) + 8 = -10(x + 1)$, and you've probably had your fill of word problems which can be solved by such equations. But some of the most important applications in algebra, science, and business involve equations where the variable is *squared*. For instance, the following is called a ***quadratic equation***, due simply to the fact that the variable is being squared:

$$x^2 - 10x + 16 = 0$$

Is it clear that x^2 and $-10x$ are UNLIKE terms?

Now that's an equation!

❑ CHECKING THE SOLUTIONS OF A QUADRATIC EQUATION

First, let's verify that $\underline{x = 2}$ is a solution of the above equation (don't worry about where the 2 came from):

$$x^2 - 10x + 16 = 0$$

$$\mathbf{2}^2 - 10(\mathbf{2}) + 16 \stackrel{?}{=} 0$$

$$4 - 20 + 16 \stackrel{?}{=} 0$$

$$-16 + 16 \stackrel{?}{=} 0$$

$$0 = 0 \quad ✓$$

Fine -- we have a solution. Here comes the (possibly) surprising fact: This equation has another solution, namely $\underline{x = 8}$. Watch this:

$$x^2 - 10x + 16 = 0$$
$$\mathbf{8}^2 - 10(\mathbf{8}) + 16 \stackrel{?}{=} 0$$
$$64 - 80 + 16 \stackrel{?}{=} 0$$
$$-16 + 16 \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$

One equation with <u>two</u> solutions? Yep, that's what we have. This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: We call it a ***quadratic equation***.

Homework

1. For each quadratic equation, verify that the two given solutions are really solutions:

    a. $x^2 + 3x - 10 = 0$ $\quad x = -5;\ x = 2$

    b. $n^2 - 25 = 0$ $\quad n = 5;\ n = -5$

    c. $a^2 + 7a = -12$ $\quad a = -3;\ a = -4$

    d. $w^2 = 7w + 18$ $\quad w = 9;\ w = -2$

    e. $2y^2 + 8y = 0$ $\quad y = -4;\ y = 0$

❑ *QUADRATIC EQUATIONS*

Recall the equation from part d. of the homework:

$$w^2 = 7w + 18$$

Because the variable w is squared, this is an example of a ***quadratic equation*** (from "quadrus," Latin for a 4-sided *square*). We can write this quadratic equation in **standard form** by subtracting $7w$ from each side of the equation and then subtracting 18 from each side of the equation:

$$w^2 - 7w - 18 = 0 \quad \longleftarrow \text{ A quadratic equation in } \textbf{\textit{standard form}}$$

Notice that the quadratic (squared) term is written first, the linear term (the one with the variable to the 1st power) comes next, and the constant term comes last; the right side of the equation is 0. We also note that the coefficient of the squared term is 1, the coefficient of the linear term is –7, and the constant term is –18.

> The ***coefficient*** tells us how many of a particular thing we have. If we have $9n$, then we have 9 n's, and so the coefficient of $9n$ is 9. The coefficient of the term $-7xy$ is –7, and the coefficient of $13w^2$ is 13.
>
> Sometimes the coefficient is "implied." For example, the coefficient of u^2 is 1 (since $u^2 = 1u^2$), and the coefficient of $-m$ is –1 (since $-m = -1m$).

We can solve some quadratic equations in our head. For instance, consider the quadratic equation

$$x^2 = 25$$

What number, when squared, equals 25? Recalling the homework above, we're not surprised to learn that there are two solutions to this equation: 5 and –5. After all, $5^2 = 25$, and $(-5)^2 = 25$. We conclude that the solutions of the quadratic equation $x^2 = 25$ are $x = 5$ and $x = -5$. So, as we study this and future chapters, keep in mind that a quadratic equation may (or may not) have two different solutions.

❑ *THE GENERAL QUADRATIC EQUATION*

Consider a general quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

where x represents the variable (the unknown) and a, b, and c are numbers. Notice that the squared term is first, the linear term is second, the constant term is third, and there's a 0 on the right side of the equation. The examples below show some quadratic equations along with their corresponding values of a, b, and c.

$3x^2 - 7x + 10 = 0$	$\Rightarrow$	$a = 3$	$b = -7$	$c = 10$
$y^2 - 9 = 0$	$\Rightarrow$	$a = 1$	$b = 0$	$c = -9$
$-4z^2 + 19z = 0$	$\Rightarrow$	$a = -4$	$b = 19$	$c = 0$
$2w^2 = 8 - 5w$	$\Rightarrow$	$a = 2$	$b = 5$	$c = -8$

To find the values of a, b, and c for this last equation, we need to rewrite the quadratic equation in standard form: $2w^2 + 5w - 8 = 0$, from which the values of a, b, and c can be determined.

Homework

2. For each quadratic equation, first make sure that it's in standard form ($ax^2 + bx + c = 0$), and then determine the values of a, b, and c:

a. $3x^2 + 9x + 17 = 0$

b. $2n^2 - 8n + 14 = 0$

c. $6y^2 + 2y - 2 = 0$

d. $5t^2 - 13t - 1 = 0$

e. $12a^2 + 13 = 0$	f. $x^2 - 13x = 0$
g. $u^2 - u + 1 = 0$	h. $-2w^2 - 19 = 0$
i. $-w^2 + 14w = 0$	j. $-z^2 - 99 = 0$
k. $2x^2 = 4x + 3$	l. $-3y^2 - 2y = -5$
m. $c^2 + 4 = 7c$	n. $6m^2 = 1 + m$
o. $18k^2 = 0$	p. $-3x^2 = -3x$

❑ *ORDER OF OPERATIONS*

Remember that exponents are done before multiplication, which itself comes before any addition or subtraction. For example,

	$12^2 - 4(2)(3)$	
=	$144 - 4(2)(3)$	(exponent first)
=	$144 - 24$	(multiplication second)
=	**120**	(subtraction last)

For a second example,

	$(-9)^2 - 4(3)(-5)$	
=	$81 - 4(3)(-5)$	(exponent first)
=	$81 - (-60)$	(multiplication second)
=	$81 + 60$	(subtracting a negative is adding)
=	**141**	(addition last)

❑ *SQUARE ROOTS*

$\sqrt{36} = 6$ $\qquad$ $-\sqrt{144} = -12$

$\sqrt{1} = 1$

$\sqrt{225} = 15$

$\sqrt{-9}$ does not exist (in Algebra 1)

$\sqrt{0} = 0$

$\sqrt{50} \approx 7.0711$

is approximately equal to

❑ *OPPOSITES*

Recall that the *opposite* of *N* is –*N*. So we note the following:

1) The opposite of a positive number is a negative number; for example, $-(+7) = -7$. Thus, if $b = 7$, then $-b = -7$.

2) The opposite of a negative number is a positive number; for example, $-(-12) = 12$. Thus, if $b = -12$, then $-b = 12$.

3) The opposite of a squared quantity (that isn't 0) is negative; for example, $-9^2 = -81$. On the other hand, don't forget that $(-9)^2 = 81$.

4) The opposite of 0 is 0.

❑ *PLUS/MINUS*

The symbol "$\pm$" is read "***plus or minus***" and is just a short way of indicating two numbers at once. For example, if you want to indicate the two numbers 7 and –7, you may write just ±7. Another example might be $\pm\sqrt{81}$, which stands for the two numbers 9 and –9. But $\pm\sqrt{0}$ would only be 0, since +0 and –0 are really just two ways to express 0.

The equation $x^2 = 25$ from a few pages back yielded the fact that x could be 5 or –5, so we could write the solution as $x = \pm 5$.

Let's work out a specific problem containing the "plus or minus" sign.

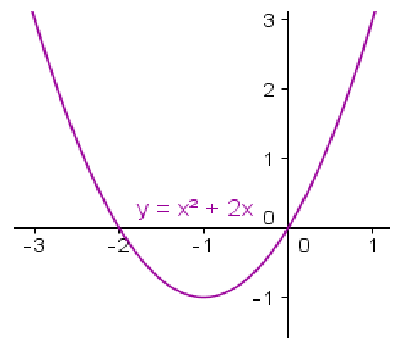
Simplify: $\dfrac{10 \pm \sqrt{25}}{5}$

Taking the square root yields $\dfrac{10 \pm 5}{5}$.

Using the plus sign, we get $\dfrac{10+5}{5} = \dfrac{15}{5} = \mathbf{3}$,

while using the minus set yields $\dfrac{10-5}{5} = \dfrac{5}{5} = \mathbf{1}$.

Thus, the expression $\dfrac{10 \pm \sqrt{25}}{5}$ is merely a strange way to represent the numbers 3 and 1.



Homework

3. Evaluate each expression:

a. $13^2 - 4(3)(2)$

b. $0^2 - 4(2)(-1)$

c. $(-3)^2 - 4(-1)(-2)$

d. $(-5)^2 - 4(1)(0)$

e. $0^2 - 4(17)(0)$

f. $(-4)^2 - (-4)(-5)$

4. Evaluate each expression:

a. $\sqrt{49}$

b. $-\sqrt{6+3}$

c. $\sqrt{16} + \sqrt{9}$

d. $\sqrt{-25}$

e. $\sqrt{(-32)(-2)}$

f. $\sqrt{(-5)^2 - 4(1)(6)}$

g. $\sqrt{6^2 - 4(1)(9)}$

h. $\sqrt{(-10)^2 - 4(8)(2)}$

i. $\sqrt{1^2 - 4(1)(1)}$

j. $\sqrt{(-7)^2 - 4(4)(-2)}$

5. Evaluate each expression (the leading minus sign represents the *opposite* of the quantity which follows it):

a. $-(13)$ b. $-(-5)$ c. $-(-(-7))$

d. $-(+7)$ e. $-(-1)$ f. $-(0)$

6. What number or numbers are represented by each of the following?

a. ± 121 b. $\pm\sqrt{121}$ c. $\pm\sqrt{0}$ d. $\pm\sqrt{-1}$

7. Simplify each expression:

a. $100+\sqrt{49}$ b. $-5-\sqrt{121}$ c. $\dfrac{3+\sqrt{16}}{2}$ d. $\dfrac{-8-\sqrt{4}}{10}$

e. $7-\sqrt{49}$ f. $-10+\sqrt{100}$ g. $\dfrac{4\pm\sqrt{16}}{6}$ h. $\dfrac{-1\pm\sqrt{81}}{9}$

8. Evaluate the expression $b^2 - 4ac$ for the given values of a, b, and c:

a. $a = -10,\ b = -5,\ c = 3$ b. $a = 4,\ b = 6,\ c = -1$

c. $a = 5,\ b = 0,\ c = 7$ d. $a = -2,\ b = -3,\ c = 0$

9. Evaluate the expression $\pm\sqrt{b^2 - 4ac}$ for the given values of a, b, and c:

a. $a = 2,\ b = -7,\ c = -15$ b. $a = 9,\ b = 30,\ c = 25$

c. $a = 1,\ b = 0,\ c = -81$ d. $a = 2,\ b = -3,\ c = 0$

❑ THE ULTIMATE EXAMPLE

For our final example of this chapter, let's evaluate the expression

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{for the values } a = 1,\ \ b = -5,\ \ c = 6$$

First change every variable to a pair of parentheses -- this is not required, but it's a handy little trick to help us avoid errors:

$$= \frac{-(\ \ ) \pm \sqrt{(\ \ )^2 - 4(\ \ )(\ \ )}}{2(\ \ )}$$

$$= \frac{-(\mathbf{-5}) \pm \sqrt{(\mathbf{-5})^2 - 4(\mathbf{1})(\mathbf{6})}}{2(\mathbf{1})} \qquad \text{(substitute the given values)}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{2} \qquad \text{(do exponents and multiplying)}$$

$$= \frac{5 \pm \sqrt{1}}{2} \qquad \text{(finish the inside of the radical)}$$

$$= \frac{5 \pm 1}{2} \qquad \text{(the positive square root of 1 is 1)}$$

$$= \frac{5+1}{2} \ \ \textit{or} \ \ \frac{5-1}{2} \qquad \text{(split the plus/minus sign)}$$

$$= \frac{6}{2} \ \ \textit{or} \ \ \frac{4}{2} \qquad \text{(simplify the numerators)}$$

$$= \boxed{3 \ \ \textit{or} \ \ 2} \qquad \text{(and finish up)}$$

There's no homework for this section, but don't be ☹ — there will be plenty of these problems when you study the Quadratic Formula.

Solutions

1. For each quadratic equation, substitute each "solution" (separately) into the original equation. Then work the arithmetic on each side of the equation separately. [Do NOT swap things back and forth across the equals sign.] In all cases, the two sides should balance at the end of the calculations.

2.

a. 3, 9, 17	b. 2, −8, 14	c. 6, 2, −2	d. 5, −13, −1
e. 12, 0, 13	f. 1, −13, 0	g. 1, −1, 1	h. −2, 0, −19
i. −1, 14, 0	j. −1, 0, −99	k. 2, −4, −3	l. −3, −2, 5
m. 1, −7, 4	n. 6, −1, −1	o. 18, 0, 0	p. −3, 3, 0

3. a. 145 b. 8 c. 1 d. 25 e. 0 f. −4

4. a. 7 b. −3 c. 7 d. Does not exist e. 8
f. 1 g. 0 h. 6 i. Does not exist j. 9

5. a. −13 b. 5 c. −7 d. −7 e. 1 f. 0

6. a. 121, −121 b. 11, −11 c. 0 d. Does not exist

7. a. 107 b. −16 c. $\frac{7}{2}$ d. −1 e. 0

f. 0 g. $\frac{4}{3}, 0$ h. $\frac{8}{9}, -\frac{10}{9}$

8. a. $b^2 - 4ac$

$= \quad (\ \ )^2 - 4(\ \ )(\ \ )$

Converting each variable to a set of parentheses is a handy way to make sure everything is written properly.

$= \quad (-5)^2 - 4(-10)(3)$

Note: The parentheses around the –5 are required, because as we've learned, $(-5)^2 = 25$, while the expression -5^2 is equal to –25.

$= \quad 25 - (-120) \ = \ 25 + 120 \ = \ 145$

b. 52 c. –140 d. 9

9. a. $\pm\sqrt{b^2 - 4ac}$

$= \pm\sqrt{(\ \ )^2 - 4(\ \ )(\ \ )}$

$= \pm\sqrt{(-7)^2 - 4(2)(-15)}$

$= \pm\sqrt{49 - (-120)}$

$= \pm\sqrt{49 + 120}$

$= \pm\sqrt{169}$

$= \pm 13$

b. 0 c. ± 18 d. ± 3

> "*Education* is what you get when you read the fine print. *Experience* is what you get if you don't."
>
> – ***Pete Seeger***