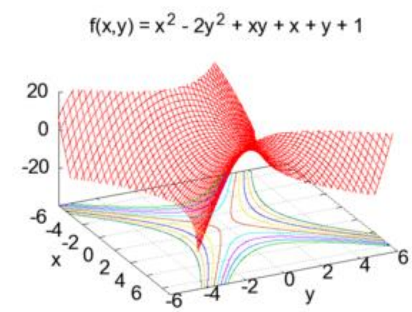
THE QUADRATIC FORMULA: RATIONAL SOLUTIONS

❑ INTRODUCTION



In the Introduction to the previous chapter we considered the quadratic equation $x^2 - 10x + 16 = 0$. We verified in detail that this equation had two solutions: $x = 8$ and $x = 2$. But no mention was made as to how these two solutions were obtained. The purpose of this chapter is to solve these kinds of equations. There are a few ways to solve this equation in Algebra 1. this is one of them.

❑ THE QUADRATIC FORMULA

Here is the formula that solves any quadratic equation:

> The quadratic equation $ax^2 + bx + c = 0$
>
> has the solutions: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

You may well be asking, "*Where the heck did that horrible-looking formula come from*?" Later in the book, after we've covered more algebra topics, we'll be able to **Derive the Quadratic Formula** (that is, create it from scratch). For now, just memorize it and enjoy it!

Homework

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Which symbol in the Quadratic Formula is responsible for possibly giving us two solutions to a quadratic equation?

2. How many solutions of a quadratic equation will there be if the quantity inside the square root sign (called the ***radicand***) is 100?

3. How many solutions of a quadratic equation will there be if the radicand is 0?

4. How many solutions of a quadratic equation will there be if the radicand is –9?

5. Consider the possibility that the value of a is 0 in the quadratic equation $ax^2 + bx + c = 0$. Does the Quadratic Formula apply? First, consider what happens to the quadratic equation if $a = 0$. Second, analyze what happens if we actually allow a to be 0 in the Quadratic Formula.

❑ *EXAMPLES OF USING THE QUADRATIC FORMULA*

EXAMPLE 1: **Solve for x: $x^2 - 9x - 10 = 0$**

Solution: We see that this is a quadratic equation in standard form, and in fact $a = 1$, $b = -9$, and $c = -10$. By the Quadratic Formula, x has (possibly) two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-10)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{81 + 40}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{121}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm 11}{2}$$

Using the plus sign: $x = \frac{9+11}{2} = \frac{20}{2} = 10$

Using the minus sign: $x = \frac{9-11}{2} = \frac{-2}{2} = -1$

Final answer: $\boxed{x = 10, -1}$

Let's check these solutions:

Letting $x = 10$ in the original equation,

$$x^2 - 9x - 10 = \mathbf{10}^2 - 9(\mathbf{10}) - 10 = 100 - 90 - 10 = 0 \ \checkmark$$

Letting $x = -1$,

$$x^2 - 9x - 10 = (\mathbf{-1})^2 - 9(\mathbf{-1}) - 10 = 1 + 9 - 10 = 0 \ \checkmark$$

EXAMPLE 2: **Solve for n: $6n^2 + 40 = -31n$**

Solution: The first step is to transform this quadratic equation into standard form. Adding $31n$ to each side (and putting the $31n$ between the $6n^2$ and the 40) produces

$$6n^2 + 31n + 40 = 0$$ [It's now in standard form.]

Noting that $a = 6$, $b = 31$, and $c = 40$, we're ready to apply the Quadratic Formula:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{(31)^2 - 4(6)(40)}}{2(6)}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{961 - 960}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{1}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm 1}{12}$$

The two solutions are given by

$$\frac{-31 + 1}{12} = \frac{-30}{12} = -\frac{5}{2}$$

and $$\frac{-31 - 1}{12} = \frac{-32}{12} = -\frac{8}{3}$$

$$\boxed{n = -\frac{5}{2}, -\frac{8}{3}}$$

EXAMPLE 3: **Solve for p: $2p^2 = 200$**

Solution: Bringing the 200 over to the left side of the equation gives us our quadratic equation in standard form:

$$2p^2 - 200 = 0$$

which, if it helps you, we can view as

$$2p^2 + 0p - 200 = 0$$

Notice that $a = 2$, $b = 0$, and $c = -200$. Substituting these values into the Quadratic Formula yields our two solutions for p:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad p = \frac{-0 \pm \sqrt{(0)^2 - 4(2)(-200)}}{2(2)}$$

$$\Rightarrow \quad p = \frac{0 \pm \sqrt{0 - 4(2)(-200)}}{4}$$

$$\Rightarrow \quad p = \frac{\pm\sqrt{1600}}{4}$$

$$\Rightarrow \quad p = \frac{\pm 40}{4} = \pm 10$$

Thus, the solutions are $\boxed{p = \pm 10}$

Note: At or near the beginning of this problem we could have divided each side of the equation by 2. The Quadratic Formula, although containing different values for *a* and *c*, would nevertheless have yielded the same answers, ±10. Try it.

EXAMPLE 4: **Solve for u: $5u^2 = 9u$**

Solution: First convert to standard form: $5u^2 - 9u = 0$, from which we deduce that $a = 5$, $b = -9$, and $c = 0$. Plugging these three values into the Quadratic Formula gives:

$$u = \frac{-(\mathbf{-9}) \pm \sqrt{(\mathbf{-9})^2 - 4(\mathbf{5})(\mathbf{0})}}{2(\mathbf{5})} = \frac{9 \pm \sqrt{81 - 0}}{10} = \frac{9 \pm 9}{10}$$

No radicals remain, so that issue is settled; using the plus and minus signs separately yields the two solutions:

$$\frac{9+9}{10} = \frac{18}{10} = \frac{9}{5} \quad \text{and} \quad \frac{9-9}{10} = \frac{0}{10} = 0$$

Final answer: $\boxed{u = \frac{9}{5},\ 0}$

EXAMPLE 5: **Solve for u: $28u - 4u^2 = 49$**

Solution: The standard quadratic form requires that the quadratic term (the squared variable) be in front, followed by the linear term, followed by the constant, all set equal to zero; that is, $ax^2 + bx + c = 0$. We achieve this goal with the following steps:

$$28u - 4u^2 = 49 \quad \text{(the given equation)}$$

$$\Rightarrow \quad -4u^2 + 28u = 49 \quad \text{(commutative property)}$$

$$\Rightarrow \quad -4u^2 + 28u - 49 = 0 \quad \text{(subtract 49 from each side)}$$

At this point it's in a good form for the Quadratic Formula, but it's traditional to disallow the leading coefficient (the –4) to be negative. One way to change –4 into 4 is to multiply it by –1; but, of course, we will have to do this to each side of the equation.

$$\Rightarrow \quad \mathbf{-1}[-4u^2 + 28u - 49] = \mathbf{-1}[0] \quad \text{(multiply each side by –1)}$$

$$\Rightarrow \quad 4u^2 - 28u + 49 = 0 \quad \text{(distribute)}$$

When written this way,

$$4u^2 - 28u + 49 = 0,$$

we deduce that $a = 4$, $b = -28$, and $c = 49$. On to the Quadratic Formula:

$$u = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(4)(49)}}{2(4)}$$

$$= \frac{28 \pm \sqrt{784 - 784}}{8}$$

$$= \frac{28 \pm \sqrt{0}}{8} = \frac{28 \pm 0}{8} = \frac{28}{8} = \frac{7}{2}$$

And we're done: $u = \frac{7}{2}$

Notice that although all the previous quadratic equations had two solutions, this equation has just <u>one</u> solution. (Or does it have two solutions that are the same?)

<u>EXAMPLE 6:</u> **Solve for z: $z^2 + 3z + 3 = 0$**

<u>Solution:</u> This quadratic equation seems innocent enough, so let's take the values $a = 1$, $b = 3$, and $c = 3$ and place them in their proper places in the Quadratic Formula:

$$z = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 12}}{2} = \frac{-3 \pm \sqrt{-3}}{2}$$

Houston . . . we have a problem. We know that we cannot consider the square root of a negative number in this class. If we try to take the square root of –3, the calculator will almost for sure indicate an error. We can't finish this calculation (at least not in Algebra 1), so we must agree that there is simply **NO SOLUTION** to the equation $z^2 + 3z + 3 = 0$.

Homework

6. Solve each quadratic equation:

a. $x^2 - 3x - 10 = 0$

b. $6n^2 + 19n - 7 = 0$

c. $d^2 + 7d = 0$

d. $2u^2 + u + 1 = 0$

e. $z^2 - 49 = 0$

f. $t^2 + 16 = 0$

g. $n^2 - 10n = 0$

h. $w^2 - 144 = 0$

7. Solve each quadratic equation:

a. $n^2 + 10n = -25$
b. $4x^2 = 4x - 1$
c. $3m^2 = 8m$
d. $x^2 = -5 + 3x$
e. $-x^2 - 5x + 6 = 0$
f. $2x^2 - 5x - 3 = 0$

8. Solve each quadratic equation:

a. $2w^2 - 70w + 500 = 0$
b. $w^2 + 2w + 5 = 0$
c. $9w^2 - 24w + 16 = 0$
d. $7n^2 = 10n$
e. $2x^2 - 15x + 7 = 0$
f. $15x^2 - 8 = 2x$
g. $4a^2 + 35a = -24$
h. $21t^2 = 20t + 9$

9. Solve each quadratic equation:

a. $w^2 - 12w + 35 = 0$
b. $2x^2 - 13x + 15 = 0$
c. $n^2 - 20n + 150 = 0$
d. $6a^2 - 31a + 40 = 0$
e. $-2x^2 - 10x + 28 = 0$
f. $25y^2 = 4$
g. $2z^2 + 2 = -4z$
h. $-4a^2 = 4a - 3$
i. $6u^2 = 47u + 8$
j. $0 = 4t^2 + 8t + 3$
k. $-n^2 + n + 56 = 0$
l. $-14x^2 = -40 + 46x$

Review Problems

10. Solve for y: $15y^2 + 2y - 8 = 0$

11. Solve for a: $-49a^2 - 9 = -42a$

12. Solve for x: $4x^2 = 1$

13. Solve for z: $10z^2 = 9z$

14. Solve for n: $3n^2 = 7n - 5$

15. Solve for w: $-2w^2 + 5w + 3 = 0$

Solutions

1. The plus/minus sign: ±

2. Since $\sqrt{100} = 10$, and since there's a plus/minus sign in front this, there will be two solutions, one using the plus sign and one using the minus sign.

3. Since $\sqrt{0} = 0$, and since ±0 is just the single number 0, the plus/minus sign and the radical sign basically disappear, leaving a single solution.

4. Since $\sqrt{-9}$ does not exist in Algebra 1, the Quadratic Formula has no meaning, and so there are no solutions.

5. First, if $a = 0$, the first term becomes $ax^2 = 0(x^2) = 0$, and the quadratic equation $ax^2 + bx + c = 0$ simplifies to $bx + c = 0$, which is NOT quadratic anymore. Indeed, we learned weeks ago how to solve this equation:

$$bx + c = 0 \Rightarrow bx = -c \Rightarrow x = -\frac{c}{b}.$$

Second, if we ignore the above fact and use $a = 0$ in the Quadratic Formula anyway, the denominator becomes $2a = 2(0) = 0$. That is, we're dividing by zero, which you know darned well is undefined!
No matter how you look at it, the Quadratic Formula with $a = 0$ simply makes no sense.

6. a. 5, –2 b. $\frac{1}{3}, -\frac{7}{2}$ c. 0, –7 d. No solution
e. ± 7 f. No solution g. 10, 0 h. ± 12

7. a. –5 b. $\frac{1}{2}$ c. $0, \frac{8}{3}$ d. No solution e. 1, –6
f. $3, -\frac{1}{2}$

8. a. 25, 10 b. No solution c. $\frac{4}{3}$ d. $0, \frac{10}{7}$
e. $7, \frac{1}{2}$ f. $-\frac{2}{3}, \frac{4}{5}$ g. $-8, -\frac{3}{4}$ h. $\frac{9}{7}, -\frac{1}{3}$

9. a. 5, 7 b. $5, \frac{3}{2}$ c. No solution d. $\frac{8}{3}, \frac{5}{2}$
e. 2, –7 f. $\pm\frac{2}{5}$ g. –1 h. $\frac{1}{2}, -\frac{3}{2}$
i. $8, -\frac{1}{6}$ j. $-\frac{1}{2}, -\frac{3}{2}$ k. 8, –7 l. $-4, \frac{5}{7}$

10. $\frac{2}{3}, -\frac{4}{5}$

11. $\frac{3}{7}$

12. $\pm\frac{1}{2}$

13. $0, \frac{9}{10}$

14. No solution

15. $3, -\frac{1}{2}$

❑ *TO ∞ AND BEYOND!*

Solve for *x*: $\pi x^2 = \sqrt{2}\,x - \sqrt{\pi}$

Solve for *x*: $ax^2 + bx^2 + cx + dx + \pi^2 - \pi = 0$

"Education's purpose is to replace an empty mind with an open one."

Malcolm Forbes