VARIABLES AND IDENTITIES

❑ *INTRODUCTION*

First we need to know that there are many ways to indicate **multiplication**; for example, the product of 5 and 7 can be written in a variety of ways:

$$5 \times 7 \qquad 5 \cdot 7 \qquad 5(7) \qquad (5)7 \qquad (5)(7)$$

Each product is 35

Repeated multiplication has its own terminology and notation. To ***square*** a number means to multiply that number by itself. For example, the "square of 5" is $5 \cdot 5 = 25$, and we can write $5^2 = 25$. This is read either "5 squared" or "5 to the second power" or sometimes just "5 to the second." In general, the square of n is written n^2. In the expression n^2, we call n the **base** and 2 the **exponent**.

> Numbers that are multiplied together are called ***factors***.

As for what operations we do first, second, and so forth, we might recall from Pre-Algebra that one of the rules contained in the Order of Operations tells us that multiplication is done before addition and subtraction. Another part of the Order of Operations tells us that parentheses are always done first.

Here are three examples:

$7 + 2 \times 3 = 7 + 6 = 13$	Multiply before adding
$(3 + 10)(9 - 7) = (13)(2) = 26$	Parentheses before anything
$(8 + 2)^2 = 10^2 = 100$	Parentheses before anything

❑ *IDENTITIES*

We continue with the following important definition:

An ***identity*** in algebra is a statement
that two expressions are equal.

If there are any letters in the expressions, then the two expressions must be equal no matter what numbers the letters might represent.

EXAMPLE 1: **Prove that $a + b = b + a$ is an identity.**

Solution: This statement is true because if a and b are replaced with any numbers at all, the two expressions will be equal. For instance, if we let $a = 10$ and $b = 5$, then the statement becomes

$$10 + 5 = 5 + 10$$

which is certainly true because each side of the identity is equal to 15. Since there are infinitely many numbers that a and b could be, we could never really prove that it's an identity with a slew of examples, so we're happy with a few examples that demonstrate its truth. In short, $a + b = b + a$ is an identity.

There's a fancy word we use to describe the fact that addition can be done in either order. What has a judge done when she **commutes** a death sentence and changes it to life in prison? She has *reversed* the original sentence.

When your dad **commutes** to work, he goes in the morning and then *reverses* direction to get home in the evening, ending up where he started.

Thus, to commute means to switch, or reverse. Since the statement $a + b = b + a$ is a switching of the order in which we add the a and the b, we say that addition is a ***commutative*** (accent on the '*mut*') operation.

<u>Note #1:</u> There's another commutative property; it's for multiplication. By using a pair of (different) numbers, convince yourself that $xy = yx$ is a true statement for <u>any</u> values of x and y. In short, multiplication is also a commutative operation.

<u>Note #2:</u> There would be no need for any mention of the commutative properties of addition and multiplication were it not for the fact that there are operations which are <u>not</u> commutative. For example, consider the division problem $\frac{6}{3}$, which equals 2. If we "commute" the top and bottom, we get the division problem $\frac{3}{6}$, which equals $\frac{1}{2}$. Since switching, or reversing the two numbers in the division problem resulted in different answers, we conclude that division is <u>not</u> a commutative operation. Any ideas about whether subtraction is, or is not, a commutative operation?

<u>EXAMPLE 2:</u> **Prove that $(x + 5)(x - 5) = x^2 - 25$ is an identity.**

<u>Solution:</u> Let's plug the value $x = 8$ (I just chose this number off the top of my head) into both sides of this statement:

$(x + 5)(x - 5)$	$x^2 - 25$
$(\mathbf{8} + 5)(\mathbf{8} - 5)$	$\mathbf{8}^2 - 25$
$(13)(3)$	$64 - 25$
39	**39** ✓

Both sides came out to the same number. Although we used only one value of x to test the statement, it truly is an identity; <u>any</u> number you use for x will result in equal numbers. Later in the course, we'll see a way to prove conclusively that $(x + 5)(x - 5) = x^2 - 25$ without using any number values of x at all!

On the other hand, to prove that something is <u>not an identity</u>, we need only give one example where the statement <u>fails</u>. Consider the next example, showing how we demonstrate that something is not an identity.

<u>EXAMPLE 3:</u> **Prove that $(x + 3)(x + 2) = x^2 + 6$ is <u>not</u> an identity.**

<u>Solution:</u> Let $x = 10$ (it's as good a number as any). Then, working each side of the statement separately, we obtain:

$(x + 3)(x + 2)$	$x^2 + 6$
$(\mathbf{10} + 3)(\mathbf{10} + 2)$	$\mathbf{10}^2 + 6$
$(13)(12)$	$100 + 6$
156	106 ✘

An example which makes a statement <u>fail</u> is called a ***counterexample***.

We conclude that $(x + 3)(x + 2) = x^2 + 6$ is <u>not</u> an identity.

<u>EXAMPLE 4:</u> **Prove that $(x + 3)^2 = x^2 + 6x + 9$ is an identity.**

<u>Solution:</u> 7 is a nice number. Let's use it to test the statement:

$(x + 3)^2$	$x^2 + 6x + 9$	
$(\mathbf{7} + 3)^2$	$\mathbf{7}^2 + 6(\mathbf{7}) + 9$	
10^2	$49 + 42 + 9$	(multiply before adding)
100	100 ✓	

<u>A Final Reminder:</u> Proving that an identity is true would theoretically require that every possible number value for each letter be tested. Since this is impossible to carry out, we are happy in this chapter to pick some numbers out of thin air and test them. If they work, we will assume that we have a genuine identity.

On the other hand, proving that an identity is false is relatively easy. We find one ***counterexample***, and that conclusively proves the identity false, since that one exception would certainly prove that the identity can't be true for all numbers. By the way, try to avoid using the numbers 0, 1, and 2 for your experiments; these numbers can easily lead to bogus conclusions, as you will see in the following homework.

Homework

1. By using numbers, show that the statement $a(b + c) = ab + c$ is <u>false</u> (that is, show that it's <u>not</u> an identity).

   Note the Order of Operations: When you calculate the left side, do the parentheses first by adding *b* and *c*, and then multiply by *a*. On the right side, multiply and then add.

2. By using numbers, give some evidence that the statement

   $a(b + c) = ab + ac$ is <u>true</u> (that is, show that it's an identity).

3. You were advised above to avoid using the numbers 0, 1, or 2 to test whether a statement is an identity. Here's why:

   a. Show that if $x = 0$, the statement $4x = 3x$ is true.

      Now show that the statement $4x = 3x$ is not an identity.

   b. Show that if $n = 1$, the statement $n^2 = n^3$ is true.

      Now show that the statement $n^2 = n^3$ is not an identity.

   c. Show that if $a = 2$, the statement $a^2 = 2a$ is true.

      Now show that the statement $a^2 = 2a$ is not an identity.

4. Prove that the following is <u>not</u> an identity:

$$(a + b)^2 = a^2 + b^2$$

5. Give some evidence that the following <u>is</u> an identity:

$$(a + b)^2 = a^2 + 2ab + b^2$$

6. Is $(a + b)(a - b) = a^2 - b^2$ an identity? Prove your contention.

7. Consider the statement: $(x + y)^3 = x^3 + y^3$. Decide if it's an identity, and give your reasoning.

Determine whether or not each statement is an identity:

8. $(x + 3)(x - 3) = x^2 - 6$ (choose x to be bigger than 3)

9. $(x + 3)(x - 3) = 2x - 9$ (choose x to be bigger than 3)

10. $(x + 3)(x - 3) = x^2 - 9$ (choose x to be bigger than 3)

11. $(a + b)(a - b) = 2a - 2b$ (choose a to be bigger than b)

12. $(a + b)(a - b) = a^2 + b^2$ (choose a to be bigger than b)

13. $(a + b)(a - b) = (a - b)^2$ (choose a to be bigger than b)

14. $(a + b)(a - b) = a^2 - b^2$ (choose a to be bigger than b)

15. $(n + 3)(n + 2) = 2n + 6$

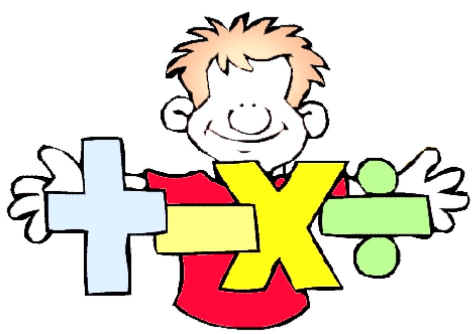
16. $(n + 3)(n + 2) = n^2 + 6$

17. $(n + 3)(n + 2) = n^2 + 5n + 6$

18. $(a + 5)^2 = 2a + 10$

19. $(a + 5)^2 = a^2 + 25$

20. $(a + 5)^2 = a^2 + 10a + 25$



❑ *CONSTANTS AND VARIABLES*

Quantities like 7, –80, π, and $\frac{2}{3}$ are called ***constants*** because their values never change. But quantities like x, y, N, and σ (Greek sigma) are called ***variables*** because we don't know their values, and so they could be just about any number at all. But the point is, although we write variables in algebra using letters of some alphabet (usually English, sometimes Greek), they're not really letters — *they're just symbols to represent numbers that we don't know*.

Sometimes we don't even care to find the value of a variable; for example, when we say that "the sum of a and b is written $a + b$," we're trying to make a statement that is valid for any numerical values of a and b.

Other times, for instance in the equation "$x + 9 = 11$," x is certainly a variable, but the goal is to figure out what x is, and in this case x turns out to be the constant 2.

Review Problems

21. Determine whether each statement is an identity, and prove your conclusion. Remember our agreement that you should not use 0, 1 or 2 for the values of your variables:

    a. $(w + 10)^2 = w^2 + 100$

    b. $(x + 1)(x + 2) = x^2 + 3x + 2$

    c. $x(y + z) = xy + z$

    d. $x(y + z) = xy + xz$

    e. $(a + b)(a - b) = a^2 - b^2$ (choose *a* bigger than *b*)

Solutions

1. For instance, let $a = 2$, $b = 3$, and $c = 4$. Then

$$a(b + c) = 2(3 + 4) = 2(7) = 14;$$

but, $$ab + c = 2(3) + 4 = 6 + 4 = 10.$$ ☹

2. Using the same numbers as above:

$$a(b + c) = 2(3 + 4) = 2(7) = 14;$$

and, $$ab + ac = 2(3) + 2(4) = 6 + 8 = 14$$ ☺

Note: Just because these numbers made the formula work doesn't necessarily mean that the formula works all the time (although it actually does work all the time). All we've done at this point is give some justification for the formula.

3. a. Since 4(0) is 0, and 3(0) is 0, it's true that $4x = 3x$ when $x = 0$. But if we choose x to be 5, for instance, $4(5) = 20$, while $3(5) = 15$. Hence, $4x = 3x$ is not an identity, because it's not true for all values of x. I'd like to see what you come up with for the rest of the problem.

4. Choose $a = 3$ and $b = 4$. (Choose any numbers you want, but avoid using 0, 1, and 2.)

$(a + b)^2$	$a^2 + b^2$
$(3 + 4)^2$	$3^2 + 4^2$
7^2	$9 + 16$
49	25 ☹

5. Choose a pair of numbers for a and b. Then test those numbers in the statement and show that you get the same result for each side.

6. Test this theory as in the previous problem.

7. It's not an identity. For example, if we let $x = 2$ and $y = 3$, we get

$$(x + y)^3 = (2 + 3)^3 = 5^3 = 125;$$

but, $$x^3 + y^3 = 2^3 + 3^3 = 8 + 27 = 35.$$

8. Let $x = 5$, for instance. Then

$$(x + 3)(x - 3) = (\mathbf{5} + 3)(\mathbf{5} - 3) = (8)(2) = 16$$

$$x^2 - 6 = \mathbf{5}^2 - 6 = 25 - 6 = 19$$ ☹

The statement is therefore not an identity.

9. Not an identity

10. Let $x = 10$, for example. Then

$$(x + 3)(x - 3) = (\mathbf{10} + 3)(\mathbf{10} - 3) = (13)(7) = 91$$

$$x^2 - 9 = \mathbf{10}^2 - 9 = 100 - 9 = 91$$ ☺

This situation (letting $x = 10$) does not prove it conclusively; after all, maybe it works for $x = 10$ but not for some other value of x. Indeed, we should try the expression for other values of x — the more x's we use, the more sure we can be of our conclusion. But for this chapter, one good example is good enough.

11. No **12**. No **13**. No **14**. Yes **15**. No **16**. No

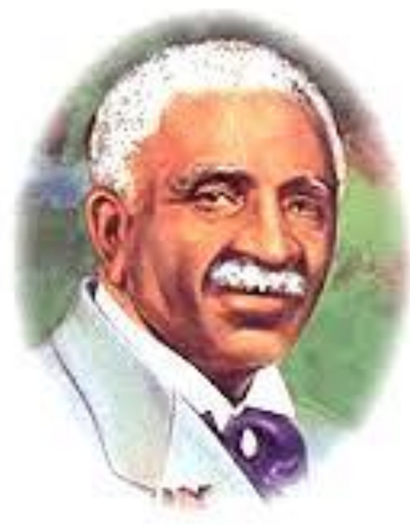
17. Yes **18**. No **19**. No **20**. Yes

21. a. No b. Yes c. No d. Yes e. Yes

Question: "How does a math teacher scold her child?"

Answer: "If I've told you *n* times, I've told you *n* + 1 times!"

“Education is the key to unlock the golden door of freedom.”



– George Washington Carver