CHALLENGE PROBLEMS

1. In base 10, the number π is written 3.1415926 . . .

   In base π,

   a. how would π be written?
   b. how would π^3 be written?
   c. how would $\pi + 1/\pi$ be written?

2. How many 3-character passwords can be created if the first character is an upper-case letter, the second is a digit from 0 to 9, and the third is a digit from 0 to 9, but is <u>not</u> equal to the second character?

3. Give an irrational approximation of 3.1415926.

4. We define a 3-digit number to be any whole number from 100 to 999. How many 3-digit numbers contain 3 distinct (different) digits?

5. How many ways can 3 people be chosen from a group of 5 married couples such that there is no married couple among the 3 chosen?

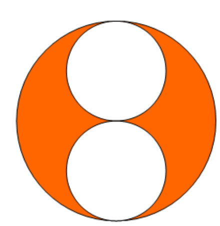
6. You have been promoted to pet detective while Ace Ventura guides some tourists on a safari. From the following clues, you must figure out who is the owner of each pet, and where the pet got lost.

   A rabbit and a dog are two of the lost pets.
   The pet lost in the garden is owned by Mary.
   Robert does not own a dog.
   John's pet was lost in the woods.
   The cat was not lost in the woods or in the park.

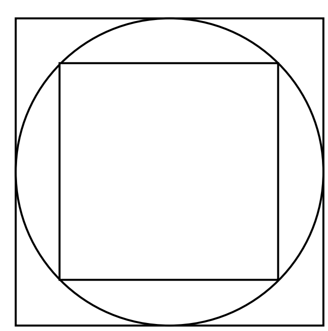
7. [Very hard] Find two rectangles such that the <u>perimeter</u> of the 1st rectangle is twice the perimeter of the 2nd rectangle, while the <u>area</u> of the 2nd rectangle is twice the area of the 1st rectangle.

8. *I* can be this, and *you* can be this. And, yes, *we* can be this. *He* can't be this and *she* can't be this. And no, *they* can't be this. *Dogs* can't be this, but *cats* can. And a *kitten* can't be this, but a *puppy* can. *Givers* can't be this, but *beggars* can. And *humility* can't be this, but *greed* can. Not even *peace* can be this, but *fear* can. What is "this"?

9. Prove that the total area of the two smaller circles is one-half the area of the larger circle.



10. Prove that the area of the big square is twice the area of the small square.



11. Solve for x:

$$-3(2x + 5) + (-3x - 1) - (4 - 10x) \;=\; 4(2x - 1) - 7(3x - 2) + 2(6x + 4) - 9x$$

12. *Squaring the Circle*

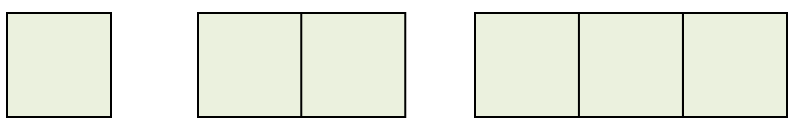
A circle has radius r. What square has the same area as the circle?

13. *Circling the Square*

Each side of a square is s. What circle has the same area as the square?

14. Line segments:

1 square | 2 squares | 3 squares



4 segments | 7 segments | 10 segments

a. How many segments are needed to make 100 squares?

b. How many segments are needed to make n squares?

15. Janie drove from home to college at an average speed of 40 mph, and returned home (same distance) at an average speed of 60 mph. What was Janie's average speed for the entire trip?

16.

A – 3
E – 4
F – 3
H – 3
K – 3
L – 2
M – ???

17. Astrid, Bettina, Chaunce, and David are friends. List them in order from youngest to oldest using the following facts:

- Betinna is younger than Astrid.
- David is older than Chaunce.
- David is not the oldest.
- Bettina is the second youngest.

18. The product of my number and twice my number is 72. What is half my number?

19. Mary got either a 90 or a 100 on each of her five math tests. If the average score was 98, how many 90s did she get?

20. Of the set of 100 numbers from 1 to 100, how many are both 5 more than some number in the set <u>and</u> 5 less than some number in the set?

21. If $2+4+6+\cdots$ $=$ $2{,}250$, then $1+3+5+\cdots$ _________

22. If 84 players split themselves into teams, how many more teams can they form by splitting into teams of 4 instead of teams of 6?

23. If each side of a square is a whole number, its perimeter could be

a. 33 b. 44 c. 55 d. 66

24. Find n: $\sqrt{100} = \sqrt{36} + \sqrt{n}$

25. At most, how many students can sit in a row of 25 chairs if the seated students must be separated by at least one empty chair?

26. Find the difference between $\frac{5}{6}$ and its reciprocal.

27. If I divide my age by 5, the remainder is 3. Your age is twice mine. If I divide your age by 5, what is the remainder?

28. A rectangle has a perimeter of 30 cm and an area of 56 cm^2. What is the difference between the length and the width of the rectangle?

29. The first 12 contestants won an average of $80. The next 20 contestants won an average of $70. What was the average winnings of all the contestants?

30. The first n_1 contestants won an average of d_1 dollars. The next n_2 contestants won an average of d_2 dollars. What was the average winnings of all the contestants?

31. Today is my birthday. My age today, in months, is 72 times my age 5 years ago, in years. What is my age today, in years?

32. The product of 2,005 whole numbers is even. At most, how many of the whole numbers can be odd?

33. What number equals one-fourth of its own reciprocal?

34. (301 + 302 + 303 + ... + 325) – (1 + 2 + 3 + ... + 25) =

35. The diameter of the $24 pizza is double the diameter of the $8 pizza. Which pizza is the better deal?

36. Each of the rooms from room #150 through room #500 of the hotel was occupied by a family of four. How many people were staying in those rooms?

37. The analog clock's reflection in the mirror seemed to indicate a time of 5:20. What time is it?

38. Imagine a room with 4 people. How many handshakes are possible? Let's call the people A, B, C, and D. Here are the ways we can pair the people to shake hands:

AB, AC, AD, BC, BD, CD

Notice that BB does not count as a handshake, since we're assuming that it takes two people to shake hands. Also notice that CA, for instance, does not count as an additional handshake, since it's the same pairing as AC. So we have a total of 6 handshakes.

a. How many handshakes are possible with 10 people in the room?

b. How many handshakes are possible with n people in the room?

39. An archaeologist claims to have discovered a coin dated 45 B.C. Do you believe her? [Assume coins were made back then, and assume they would have survived these past two millennia.]

40. Joe can paint the house in 10 hours, while Mary can paint the house in 15 hours. Working together, how many hours will it take them to paint the house?

a. 6 hrs   b. 12.5 hrs   c. 25 hrs

41. Manny spent $4.37 (no sales tax) on some pencils (he bought more than one, and they were all the same price). All he remembered was that the price of each pencil was more than 20¢. How many pencils did he buy?

42. If today is Sunday, what day of the week will it be 1,000 days from now?

43. *Goldbach's Conjecture* states that every even number starting at 4 can be written as the sum of two primes. Verify Goldbach's Conjecture for 80.

44. The sum of the first 10 natural numbers is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \mathbf{55}$$

a. Find the sum of the first 1000 natural numbers. That is, find the sum:

$$1+2+3+4+\cdots \qquad 998+999+1000$$

b. Find the sum of the first n natural numbers. That is, find the sum:

$$1+2+3+4+\cdots$$

45. The (non-negative) *powers* of 10 are the numbers 10^0, 10^1, 10^2, 10^3, 10^4, etc. That is, the powers of 10 are 1, 10, 100, 1000, 10000, etc.

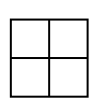
Find the smallest power of 10 that is divisible by 160.

46. A 10×10 chessboard has 36 squares on its border. Sketch the chessboard and confirm this fact.

a. Consider a 20×20 chessboard. How many squares are on the border?

b. Consider an $n \times n$ chessboard. How many squares are on the border?

47. There are 5 squares in the following 2×2 chessboard (the whole square and the 4 small ones):



a. Verify that there are 14 squares in a 3×3 chessboard.

b. How many squares are there in a standard 8×8 chessboard?

c. How many squares are there in a 12×12 chessboard?

d. How many squares are there in an $n \times n$ chessboard?

48. Answer “even” or “odd” for each of the following questions:

a. The sum of two even numbers is _______.

b. The sum of two odd numbers is _______.

c. The sum of three odd numbers is _______.

d. The sum of 168 odd numbers is _______.

e. The sum of 281 odd numbers is _______.

49. A boat was traveling at a rate of 88 feet per second. How fast was the boat traveling in *miles per hour*?

50. A cargo plane can transport a maximum of 6 tanks. How many trips will the plane have to make to transport 73 tanks?

51. The postage is $0.42 for the first ounce, and $0.34 for each additional ounce. Find the postage for a letter that weighs 11 ounces.

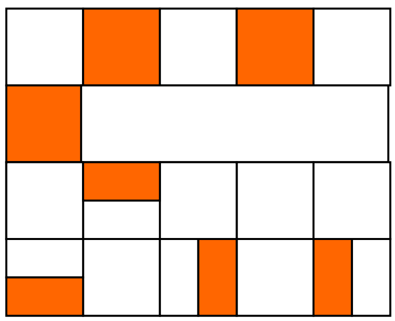
52. a. Find the difference between the <u>sum</u> of the first five positive even numbers (2, 4, 6, 8, 10) and the <u>sum</u> of the first five odd numbers (1, 3, 5, 7, 9).

b. Find the difference between the sum of the first million positive even numbers and the sum of the first million odd numbers.

c. Find the difference between the sum of the first k positive even numbers and the sum of the first k odd numbers.

53. What time is halfway between 10:49 am and 1:57 pm (later that day)?

54. What percent of the figure is shaded?



55. Consider the infinite sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$.
As you go farther and farther into the sequence, the terms are getting infinitely close to what number?

56. The infinite decimal 0.66666 . . . can be written in fraction form as $\frac{2}{3}$. (Divide 3 into 2 to confirm this fact.)

a. What is the fraction form of the infinite decimal 0.444 . . . ?

b. What is the fraction form of the infinite decimal 0.535353 . . . ?

c. What is the fraction form of the infinite decimal 0.8171717 . . . ?

57. *Twin primes* are prime numbers that have a difference of 2. For example, 11 and 13 are twin primes. Find the smallest twin primes greater than 100.

58. Notice that 2 and 5 are primes whose difference is 3. Prove that this is the <u>only</u> pair of primes whose difference is 3.

59. A *Pythagorean Triple* is a triple of numbers, written (a, b, c), such that

$$a^2 + b^2 = c^2$$

a. Prove that (3, 4, 5) is a Pythagorean Triple.

b. Consider the triple of number (6, 8, 10), obtained from part a. by multiplying each of the numbers by 2. Prove that (6, 8, 10) is also a Pythagorean Triple.

c. Suppose (ka, kb, kc) is a Pythagorean Triple. Prove that (a, b, c) is also a Pythagorean Triple.

d. Find a Pythagorean Triple which is <u>not</u> a multiple of (3, 4, 5).

60. The *arithmetic mean* (accent on the 'met') of two numbers a and b is found by calculating:

$$\frac{a+b}{2}$$

The *geometric mean* of a and b is found by calculating:

$$\sqrt{ab}$$

a. Find two numbers where the two means have the same value.

b. Find two numbers where the geometric mean is less than the arithmetic mean.

61. If the current time is 7:03 a.m. on a 12-hour clock, what time will it be in 553 minutes?

62. A set of five distinct (different) positive integers has a mean of 1,000 and a median of 100. What is the largest possible integer that could be included in the set?

63. On Claudia's birthday in 2004, her age was four times her brother's age on that day. On her birthday in 2005, her age was three times her brother's age on that day. In what year will Claudia's age, on her birthday, be twice her brother's age on that day?

64. Thomas took three 100-point tests, and the mean score for the three tests was 87 points. If the highest score is taken away, the mean of the remaining two scores would be 83.5 points. What is the highest score?

65. In a group of cows and ducks, the number of legs is 14 more than twice the number of heads. How many cows are there in the group?

66. The mean of three numbers is m. If two of the numbers are x and y, express the third number in terms of m, x, and y.

67. What is the greatest product obtainable from two numbers whose sum is 50?

68. a. In how many different ways can four students stand in a straight line?

    b. In how many different ways can four students stand in a straight line if two of the students refuse to stand next to each other?

69. [Very hard] In a bowl of jelly beans, all but 16 are red, all but 16 are green, all but 16 are blue, and all but 18 are yellow. How many jelly beans of each color are in the bowl?

70. The U.S. government has approved a $170 billion rescue plan for the U.S. economy. If that amount of money consisted of one-dollar

bills, how high (in miles) would the pile of money be? Use the facts that a dollar bill is 0.010922 cm thick, and that 1 in = 2.54 cm.

71. If 1 ziffer = 8 dufits and 5 ziffers = 7 binkies, then how many binkies are there in 20 dufits?

72. One bacterium is placed in a jar at 9 am. The number of bacteria in the jar doubles every minute, and at noon the jar is full. At what time was the jar half full? At what time was the jar half empty?

73. Find a number which satisfies <u>both</u> of the following conditions:

1. When the number is divided by 3, the remainder is 1.
2. When the number is divided by 7, the remainder is 3.

74. Five students have heights of 60, 61, 62, 63 and 64 inches. Four students are randomly chosen. What is the probability that the mean (average) height is at least 62 inches?

75. [Open-ended] Create a formula for a timed quiz that takes into account both the amount of time taken to complete the quiz and the number of problems right.

76. [Open-ended] The formula for Batting Average $\left[\text{Ave} = \frac{\text{Hits}}{\text{At Bats}}\right]$ does not take into account whether a "hit" was a single, a double, a triple, or a home run. Clearly, 3 hits consisting of 2 doubles and a home run should count a lot more than 3 hits consisting of 3 singles. Create a reasonable formula which <u>does</u> take into account the type of hit that occurred.

77. A child is running up a flight of stairs containing 10 steps. He can jump 1 step at a time or 2 steps at a time. How many different ways are there to climb the stairs?

78. Janie drove from home to college at an average speed of 40 mph, and returned home (same distance) at an average speed of 60 mph. What was Janie's average speed for the entire trip?

79. The EPA has given the XQ7 Roadster a mileage rating of 30 mpg (mi/gal). To prepare a report for European customers, you need to convert that to km/L (kilometers per liter). You may use the following facts: 1 mi = 1.61 km and 1 L = 1.06 qt.

80. The *factorial* of a positive whole number is the product of the whole numbers from the number down to 1. For example, 5 factorial (written 5!) is the product

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

[Note that the 1 in the product is not needed, but it's nice to have.]

a. Without calculating 10!, prove that 10! ends in exactly two zeros.

b. Prove that the factorial of a number can <u>never</u> end in exactly five zeros.

P.S. Even 0 has a factorial; by definition, 0! = 1.

81. Let N represent the product of three positive integers; that is, N is the product of three numbers chosen from the set {1, 2, 3, 4, . . .}. Assume:

a. N is six times the sum of the three numbers.

b. One of the three numbers is the sum of the other two.

Find one value of *N*.

$0.\bar{9}$

Infinite hotel rooms -- a new customer arrives

Warm-up Puzzles

Old Coin

Socks

10 each -- min # to guarantee a matching pair

6, 7, 8 -- min # to guarantee at least one of each color

Hotel Rooms

Farmer, Fox, Hen, Bag of Corn

Clock Arithmetic

The Knight's Tour

Cows and Ducks - e.g. 10 heads, 26 legs

Triangle Inequality

Asteroid and Comet

The price of a can of root beer is more than \$0.20. How many cans of root beer could you buy for exactly \$4.37? (There's no tax involved.)

15 circles: o o o o o o o o o o o o o o o

Take turns crossing out one circle. Winner: first person to cross out a circle so that 3 consecutive circles are crossed out.

One pile of tiles -- select 1 or 2 or . . . at a time. Last to pick = winner

Nim: 3 or more piles -- any number of tiles from ONE pile. Winner = last to pick last tile.

Fifty: two players start from 0 and alternatively add a number from 1 to 10 to the sum. The player who reaches 50 wins.

Tictactoe board -- cross out 1, 2 or 3 squares, but from same row or column Winner??

Border Squares

Total number of Squares on $n \times n$ chessboard

Four 4's

Ken-Ken

Krypto

Crossword Puzzle

Handshakes

Diagonals

180° in a triangle

Magic Squares

Sum of Consecutive Integers

Divisibility Rules (2, 3, and 5)

Factors

Odds and Evens

Perfect Numbers

Primes
    Tile rectangles OR   Primes vs non-primes

Prime factoring

The 7 Bridges of Königsberg

Goldbach's Conjecture

Binary

Cryptography

Counting
- the counting principle
- passwords
- lining up items
- factorials

Flipping Coins e.g. HHTH

Sequences

Pythagorean Theorem

Find Two Numbers

I'm Thinking of a Number

Equations

Roll 2 Dice
Sample space
And vs Or
double-counting

Division by 0
Pizza, better deal
Lining up, permutations
Equations
Sequences
Socks, 10 each -- min # to guarantee a matching pair
6, 7, 8 -- min # to guarantee at least one of each color
Border Squares
$0.\overline{9}$

Hotel Rooms

Day of the Week

"Any number times 1 is that number" $\Rightarrow$ $N \cdot 1 = N$ for any number N

Find 2 numbers : Write using variables

Square roots and equations

Fractional equations

pizza toppings, combinations
sum 1 to 100
Binary
$(a + b)^2 = \ldots$
Go 40 mph return 60 mph

Infinite hotel rooms

One new customer arrives

An infinite number of new customers arrive

Old Coin

Farmer, Fox, Hen, Bag of Corn

Bacteria in the Jar

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The Buzzer Game

The Google Game

The Whiteboard Contest