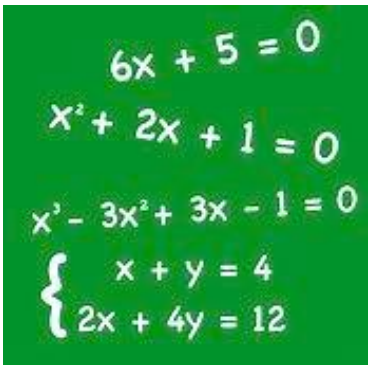
CUBIC AND QUARTIC EQUATIONS

❑ *INTRODUCTION*

The chapter is a collection of additional equations: Cubic (third degree) and Quartic (fourth degree). They can all be solved using the Factoring method. (You can never be too good at factoring!)



❑ *CUBIC EQUATIONS USING THE GCF*

EXAMPLE 1: **Solve the cubic equation: $30x^3 + 99x^2 = 21x$**

Solution: Since factoring is the way we solved quadratic equations earlier in the course, let's give factoring a try here. The factoring method requires one side of the equation to be 0:

$$30x^3 + 99x^2 - 21x = 0$$

Another theme in this chapter is that factoring always begins with an attempt to factor out the GCF, which in this case is $3x$:

$$3x(10x^2 + 33x - 7) = 0$$

Further factoring of the trinomial in the parentheses gives the final factorization of the left side our equation:

$$3x(5x - 1)(2x + 7) = 0$$

Noting that $3x$ is one of the factors, we have three factors whose product is 0; we therefore set each of the three factor to 0:

$$3x = 0 \quad \text{or} \quad 5x - 1 = 0 \quad \text{or} \quad 2x + 7 = 0$$

Solving each of these three linear equations gives us three solutions to our cubic equation:

$$x = 0, \frac{1}{5}, -\frac{7}{2}$$

Homework

1. Solve each cubic equation:

   a. $x^3 + 3x^2 + 2x = 0$

   b. $4n^3 - 18n^2 + 8n = 0$

   c. $x^3 = 16x$

   d. $3y^3 = -30y^2 - 75y$

   e. $a^3 + 9a = 0$

   f. $30x^3 + 25x^2 - 30x = 0$

2. Solve for x: $x^2(x + 1)(2x - 3)(x + 7)^3(x^2 - 4)(x^2 - 5x + 6) = 0$

❑ *CUBIC EQUATIONS USING GROUPING*

EXAMPLE 2: **Solve the cubic equation: $x^3 + 5x^2 = 9x + 45$**

Solution: The first step is to bring the terms on the right side of the equation to the left:

$$x^3 + 5x^2 - 9x - 45 = 0$$

Factor the GCF in the first pair of terms and the last pair of terms:

$$x^2(x + 5) - 9(x + 5) = 0$$

Pull out the GCF, $x + 5$:

$$(x + 5)(x^2 - 9) = 0$$

Continue the factoring of the difference of squares:

$$(x + 5)(x + 3)(x - 3) = 0$$

Set each factor to 0:

$$x + 5 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$

Solving each linear equation gives:

$$x = -5 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 3$$

And now we have our three solutions:

$$\boxed{x = -5, -3, 3}$$

❑ *QUARTIC EQUATIONS*

EXAMPLE 3: **Solve the quartic equation: $x^4 - 26x^2 + 25 = 0$**

Solution: The factoring we learned in Chapter 19 is exactly what we need here:

$$x^4 - 26x^2 + 25 = 0$$

$$\Rightarrow \quad (x^2 - 1)(x^2 - 25) = 0$$

$$\Rightarrow \quad (x + 1)(x - 1)(x + 5)(x - 5) = 0$$

$$\Rightarrow \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 5 = 0 \quad \text{or} \quad x - 5 = 0$$

$$\Rightarrow \quad x = -1 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -5 \quad \text{or} \quad x = 5$$

$$\boxed{x = \pm 1, \pm 5}$$

EXAMPLE 4: **Solve for n: $2n^4 - 15n^2 = 27$**

Solution: The factoring technique requires one side of the equation to be 0, so our first step is to make that happen, by subtracting 27 from each side of the equation:

$$2n^4 - 15n^2 - 27 = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n^2 - 9) = 0$$

$$\Rightarrow \quad (2n^2 + 3)(n + 3)(n - 3) = 0$$

$$\Rightarrow \quad 2n^2 + 3 = 0 \text{ or } n + 3 = 0 \text{ or } n - 3 = 0$$

Let's first try to solve the first equation:

$$2n^2 + 3 = 0 \Rightarrow 2n^2 = -3 \Rightarrow n^2 = -\frac{3}{2} \Rightarrow n = \pm\sqrt{-\frac{3}{2}},$$

which are numbers that are NOT in the set of real numbers, $\mathbb{R}$. In this class, we conclude that the equation $2n^2 + 3 = 0$ has No Solution. The other two equations should be easy for you by now. The final solution is

$$\boxed{x = \pm 3}$$

EXAMPLE 5: **Solve for a: $a^4 = -13a^2 - 36$**

Solution:

$$a^4 = -13a^2 - 36$$

$$\Rightarrow \quad a^4 + 13a^2 + 36 = 0$$

$$\Rightarrow \quad (a^2 + 9)(a^2 + 4) = 0$$

$$\Rightarrow \quad a^2 + 9 = 0 \text{ or } a^2 + 4 = 0$$

I hope it's clear that neither of these last two equations has a solution in the real numbers, $\mathbb{R}$. We're done:

No Solution

Homework

3. Solve each equation:

a. $x^3 + x^2 - 16x = 16$

b. $n^4 = 13n^2 - 36$

c. $2a^4 - 49a^2 = 25$

d. $x^4 + 17x^2 + 16 = 0$

Practice Problems

Solve and check each equation:

4. $30x^3 = 2x^2 + 4x$

5. $n^3 + 5 = 5n^2 + n$

6. $x^4 + 900 = 109x^2$

7. $a^4 + 3a^2 - 4 = 0$

8. $t^4 + 49 + 50t^2 = 0$

9. $w^3 + 2w^2 - 12w - 9 = 0$ [Hard]
Hint: To factor, divide by $w - 3$.

Solutions

1. a. $x = 0, -1, -2$ b. $n = 0, \frac{1}{2}, 4$ c. $x = 0, 4, -4$

d. $y = 0, -5$ e. $a = 0$ f. $x = 0, \frac{2}{3}, -\frac{3}{2}$

2. There are quite a few solutions.

3. a. $x = \pm 4, -1$ b. $n = \pm 2, \pm 3$ c. $a = \pm 5$ d. No solution

4. $x = 0, -\frac{1}{3}, \frac{2}{5}$ **5**. $n = 5, \pm 1$ **6**. $x = \pm 3, \pm 10$

7. $a = \pm 1$ **8**. No solution

9. After you divide the cubic by $w - 3$, the other factor should be a quadratic, but it is NOT factorable. So the only solution is $w = 3$. Note: The solutions provided by the quadratic can be determined four or five chapters from now.

Edith Hamilton:

"To be able to be caught up into the world of thought – that is educated."