THE FIVE LAWS OF EXPONENTS

❑ *EXPONENTS WITH VARIABLES*

It's now time for a change in tactics, in order to give us a deeper understanding of exponents. For each of the following five examples, we will "stretch-and-squish," and then we'll generalize what we observe to the official ***Five Laws of Exponents***.

I. We start by finding the product of x^3 and x^4:

$$x^3x^4 = (xxx)(xxxx) = xxxxxxx = x^7$$

Notice that the bases (the x's) are the same, and it's a multiplication problem. As long as the bases are the same, and it's a multiplication problem, it appears that we merely need to write down the base, and then <u>add</u> the exponents together to get the exponent of the answer. That is, $x^ax^b = x^{a+b}$.

$$\boxed{x^3x^4 = x^7}$$

II. For our second example, let's raise a power to a power:

$$(x^4)^2 = (xxxx)^2 = (xxxx)(xxxx) = xxxxxxxx = x^8$$

We appear to have a shortcut at hand. Simply <u>multiply</u> the two exponents together and we're done. So, to raise a power to a power, we can write a general rule: $(x^a)^b = x^{ab}$.

$$\boxed{(x^4)^2 = x^8}$$

III. Now we're to try raising a product to a power; for instance,

$$(ab)^5 = (ab)(ab)(ab)(ab)(ab) = (aaaaa)(bbbbb) = a^5b^5$$

In general, when raising a product to a power, raise each factor to the power: $(xy)^n = x^n y^n$.

$$(ab)^5 = a^5b^5$$

Note that the quantity in the parentheses is a single term -- there's no adding or subtracting in the parentheses. In fact, if there are two or more terms in the parentheses, this law of exponents does not apply.

IV. Next we divide powers of the same base. We'll need two examples for this law of exponents.

A. $$\frac{x^6}{x^2} = \frac{xxxxxx}{xx} = \frac{\cancel{x}\cancel{x}xxxx}{\cancel{x}\cancel{x}} = x^4$$

B. $$\frac{y^4}{y^6} = \frac{yyyy}{yyyyyy} = \frac{\cancel{y}\cancel{y}\cancel{y}\cancel{y}}{\cancel{y}\cancel{y}\cancel{y}\cancel{y}yy} = \frac{1}{y^2}$$

$$\frac{x^6}{x^2} = x^4 \qquad \frac{y^4}{y^6} = \frac{1}{y^2}$$

In general, when dividing powers of the same base, subtract the exponents, leaving the remaining factors on the top if the top exponent is bigger, and on the bottom if the bottom exponent is bigger.

V. Our last example in this section is the process of raising a quotient to a power. As usual, we stretch and squish; then we generalize to a law of exponents.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{aaaa}{bbbb} = \frac{a^4}{b^4}$$

$$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$$

In general, we can raise a quotient to a power by raising both the top and bottom to the power: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

❑ *SUMMARY OF THE FIVE LAWS OF EXPONENTS*

Exponent Law	Example
$x^a x^b = x^{a+b}$	$x^2 x^6 = x^8$
$(x^a)^b = x^{ab}$	$(a^4)^3 = a^{12}$
$(xy)^n = x^n y^n$	$(wz)^7 = w^7 z^7$
For $a > b$, $\frac{x^a}{x^b} = x^{a-b}$ For $b > a$, $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$	$\frac{x^{10}}{x^2} = x^8$ $\frac{a^3}{a^7} = \frac{1}{a^4}$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$

❑ *SINGLE-STEP EXAMPLES*

EXAMPLE 1:

A. $A^7 A^5 = A^{7+5} = A^{12}$

The bases are the same, and it's a multiplication problem. So we can simply write the base and add the exponents.

B. $x^2x^3x^4 = x^{2+3+4} = x^9$

All the bases are the same, and it's a multiplication problem, and so we simply add the exponents.

C. $(x+y)^4(x+y)^9 = (x+y)^{13}$

It doesn't matter what the base is, as long as we're multiplying powers of the same base.

EXAMPLE 2:

A. $(c^{10})^2 = c^{20}$

Raising a base to a power, and then raising that result to a further power requires simply that we multiply the exponents.

B. $\left((x^2)^3\right)^4 = x^{24}$

Power to a power to a power? Just multiply all three exponents.

EXAMPLE 3:

A. $(ax)^5 = a^5x^5$

It's a power of a product (a single term). So just raise each factor to the 5th power.

B. $(abc)^7 = a^7b^7c^7$

Even a term with three factors can be raised to the 7th power by raising each factor to the 7th power.

EXAMPLE 4:

A. $\dfrac{x^7}{x^5} = x^2$

Since 7 > 5, we divide powers of the same base by subtracting the exponents.

B. $\dfrac{w^{15}}{w^{25}} = \dfrac{1}{w^{10}}$

Since the bigger exponent is on the bottom, we subtract 15 from 25 and leave that power of w on the bottom.

EXAMPLE 5:

A. $\left[\dfrac{x}{z}\right]^7 = \dfrac{x^7}{z^7}$

To raise a quotient to a power, just raise both the top and bottom to the 7th power.

B. $\left(\dfrac{a+b}{u-w}\right)^{23} = \dfrac{(a+b)^{23}}{(u-w)^{23}}$

Just raise top and bottom to the 23rd power.

Homework

1. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^4 b. $x^5x^6x^2$ c. y^3y^3 d. $z^{12}z$

e. $(x^3)^4$ f. $(z^8)^2$ g. $(n^{10})^{10}$ h. $(a^1)^7$

i. $(ab)^3$ j. $(xyz)^5$ k. $(RT)^1$ l. $(math)^5$

m. $\dfrac{a^8}{a^2}$ n. $\dfrac{b^3}{b^9}$ o. $\dfrac{w^5}{w^5}$ p. $\dfrac{Q^{100}}{Q^{50}}$

q. $\left(\dfrac{k}{w}\right)^4$ r. $\left(\dfrac{a}{b}\right)^{99}$ s. $\left(\dfrac{1}{m}\right)^{20}$ t. $a(bc)^2$

2. Use the Five Laws of Exponents to simplify each expression:

a. a^3a^5 b. $u^5u^7u^2$ c. $y^{30}y^{30}$ d. $z^{14}z$

e. $(x^4)^5$ f. $(z^9)^2$ g. $(n^{100})^{10}$ h. $(a^1)^9$

i. $(xy)^4$ j. $(abc)^{17}$ k. $(pn)^1$ l. $(love)^4$

m. $\dfrac{a^{10}}{a^2}$ n. $\dfrac{b^3}{b^{12}}$ o. $\dfrac{w^9}{w^9}$ p. $\dfrac{Q^{100}}{Q^{20}}$

q. $\left(\dfrac{x}{w}\right)^3$ r. $\left(\dfrac{a}{b}\right)^{999}$ s. $\left(\dfrac{1}{z}\right)^{22}$ t. $w(xy)^3$

❑ *WHEN NOT TO USE THE FIVE LAWS OF EXPONENTS*

$\boldsymbol{a^5b^6}$ cannot be simplified. Although the first law of exponents demands that the expressions be multiplied -- and they are -- it also requires that the bases be the same -- and they aren't.

$\boldsymbol{x^3 + x^4}$ cannot be simplified. Even though the bases are the same, the first law of exponents requires that the two powers of x be multiplied.

$\boldsymbol{w^3 + w^3}$ can be simplified, but not by the first law of exponents, since the powers of w are not being multiplied. But the two terms are *like terms*, which means we simply add them together to get $\boldsymbol{2w^3}$.

$\boldsymbol{(a+b)^{23}}$ does <u>not</u> equal $a^{23}+b^{23}$. You may think that the third law of exponents, $(xy)^n = x^n y^n$, might apply, but it does not, and that's because xy is a single term, whereas $a+b$ consists of two terms. You'll have to wait until Intermediate Algebra to learn a clever way to calculate the 23rd power of $a+b$. Also, you may have already learned in this class that $(a+b)^2$ is actually equal to $a^2+2ab+b^2$, and so again, $(a+b)^n \neq a^n + b^n$.

Homework

3. Simplify each expression:

a. y^4y^4	b. a^3b^4	c. $x^4x^3x^2$	d. $p^3t^2p^2$
e. a^3+a^5	f. a^3a^5	g. n^4+n^4	h. x^3-x^3
i. $(x+y)^{55}$	j. Q^2+Q^2	k. u^5w^6	l. h^6-h^2
m. $(a-b)^2$	n. $(ab)^2$	o. $(x^3)^3$	p. x^4+x^5
q. $x^{14}+x^{14}$	r. $y^{12}-y^{12}$	s. a^8+a^9	t. $a^{10}+a^{10}$
u. $(xy)^2$	v. $(x+y)^2$	w. a^3b^4	x. a^3+b^4
y. $a(a^2)(b^2)b$	z. n^6+n^6		

❑ *MULTI-STEP EXAMPLES*

<u>EXAMPLE 6:</u>

A. $(-3x^2y^3)(-5xy^7) = (-3)(-5)\,(x^2x)(y^3y^7) = 15x^3y^{10}$

B. $-2x^2y(xy-4x^3y^4) = -2x^3y^2+8x^5y^5$

C. $\left(2a^2b^3\right)^4 = 2^4(a^2)^4(b^3)^4 = 16a^8b^{12}$

D. $7\left(xy^{10}\right)^5 = 7x^5(y^{10})^5 = 7x^5y^{50}$

E. $\left(\dfrac{a^2}{b^3}\right)^7 = \dfrac{(a^2)^7}{(b^3)^7} = \dfrac{a^{14}}{b^{21}}$

F. $\left(\dfrac{x^3y^9}{xy^{12}}\right)^5 = \left(\dfrac{x^2}{y^3}\right)^5 = \dfrac{(x^2)^5}{(y^3)^5} = \dfrac{x^{10}}{y^{15}}$

Homework

4. Simplify each expression:

a. $(-5a^3b^4)(-2a^2b)$

b. $(7xy)(-7x^2y^5)$

c. $(-2uw)(2uw)$

d. $x^3(2x^2 - x - 1)$

e. $3y^2(3y^2 - y + 3)$

f. $\left(a^2b^3\right)^4$

g. $\left(-5m^3n^{10}\right)^3$

h. $\left[-3p^3q^3\right]^4$

i. $4\left(xy^7\right)^{10}$

j. $-10(-2c^3y^4)^3$

k. $\left(\dfrac{a^3}{c^2}\right)^{10}$

l. $\left[\dfrac{2x^3}{3xy^4}\right]^3$

m. $\left(\dfrac{a^2b^3}{a^4b}\right)^5$

n. $2(3x^2y^3)^4$

❑ *ZERO AS AN EXPONENT*

We've already learned that anything to the zero power is 1 (as long as it's not zero to the zero). We reached this conclusion after we learned that $2^0 = 1$, and figured it might be true for any base. Now we try to verify this; that is, what is x^0? Consider the expression

$$x^3x^0 \qquad \text{where we assume } x \neq 0.$$

To figure out the meaning of x^0, we can use the first law of exponents to calculate

$$x^3x^0 = x^{3+0} = x^3$$

That is,

$$x^3x^0 = x^3$$

Now "isolate" the x^0, since that's what we're trying to find the value of. We do this by dividing each side of the equation by x^3:

$$\frac{x^3x^0}{x^3} = \frac{x^3}{x^3},$$

It's legal to divide by x^3, since we've stipulated that $x \neq 0$.

which implies that

$$x^0 = 1,$$

and we're done:

Any number (except 0) raised to the zero power is 1.

<u>EXAMPLE 7:</u>

A. $(x - 3y + z)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

B. $(abc)^0 = \mathbf{1}$ (any quantity ($\neq 0$) to the zero power is 1)

C. $a + b^0 = \boldsymbol{a + 1}$ (the exponent is on the b only)

D. $uw^0 = u(1) = \boldsymbol{u}$ (the exponent is on the w only)

E. $(-187)^0 = \mathbf{1}$ (the exponent is on the –187)

F. $-14^0 = \mathbf{-1}$ (the exponent in on the 14, not on the minus sign)

Homework

5. Evaluate each expression:

a. $2^0 + 3^0$ b. $4^0 \cdot 5^0$ c. $6^0 + 2^1 + 2^2 + 2^3 + 2^4$

d. $(1 + 12)^0$ e. $2^5 - 2^3 + 2^1 - 2^0$ f. $(8 - 5)^0 + (10 - 8)^1$

g. $2^0 \times 2^1 \times 2^2 \times 2^3 \times 2^4$

h. $\left(\frac{12}{29}\right)^0 + \left(\frac{100}{77}\right)^0 - (20 - \pi - 3)^0 + \left(3^2 - 7\right)^1$

6. Simplify each expression:

a. x^0 b. xy^0 c. $x + y^0$ d. $(x + y)^0$

e. $(ab)^0$ f. $\left(\frac{a^2}{b^3}\right)^0$ g. $\frac{(x^2)^0}{y^3}$ h. m^0m

i. $x^0 + x^0$ j. Q^0Q^0 k. $a^0 - a^0$ l. $\left(\frac{2x^2y^0}{-3ab^{10}}\right)^0$

Review Problems

7. Simplify: $\left(-3x^3y^4x^7\right)^3$

8. Simplify: $-3\left(x^5x^4x^8\right)^3$

9. Simplify: $\dfrac{a^2b^3c^9}{ab^4c^3}$

10. Simplify: $-2y^3(3y^4 - 2y^3 + 1)$

11. Simplify: $x^{12} + x^{14}$

12. Simplify: $u^{22} + u^{22}$

13. Simplify: $abcd^0e^0$

14. Simplify: $\left[\dfrac{10^0a^0b^{14}}{a^3b^7}\right]^5$

Solutions

1. a. a^7 b. x^{13} c. y^6 d. z^{13}
e. x^{12} f. z^{16} g. n^{100} h. a^7
i. a^3b^3 j. $x^5y^5z^5$ k. RT l. $m^5a^5t^5h^5$
m. a^6 n. $\dfrac{1}{b^6}$ o. 1 p. Q^{50}
q. $\dfrac{k^4}{w^4}$ r. $\dfrac{a^{99}}{b^{99}}$ s. $\dfrac{1}{m^{20}}$ t. ab^2c^2

2. a. a^8 b. u^{14} c. y^{60} d. z^{15}
e. x^{20} f. z^{18} g. n^{1000} h. a^9
i. x^4y^4 j. $a^{17}b^{17}c^{17}$ k. pn l. $l^4o^4v^4e^4$
m. a^8 n. $\dfrac{1}{b^9}$ o. 1 p. Q^{80}

q. $\frac{x^3}{w^3}$ r. $\frac{a^{999}}{b^{999}}$ s. $\frac{1}{z^{22}}$ t. wx^3y^3

3. a. y^8 b. As is c. x^9 d. p^5t^2
e. As is f. a^8 g. $2n^4$ h. 0
i. As is (for now) j. $2Q^2$ k. As is l. As is
m. $a^2 - 2ab + b^2$ n. a^2b^2 o. x^9 p. As is
q. $2x^{14}$ r. 0 s. As is t. $2a^{10}$
u. x^2y^2 v. $x^2 + 2xy + y^2$ w. As is
x. As is y. a^3b^3 z. $2n^6$

4. a. $10a^5b^5$ b. $-49x^3y^6$ c. $-4u^2w^2$ d. $2x^5 - x^4 - x^3$
e. $9y^4 - 3y^3 + 9y^2$ f. a^8b^{12} g. $-125m^9n^{30}$
h. $81p^{12}q^{12}$ i. $4x^{10}y^{70}$ j. $80c^9y^{12}$ k. $\frac{a^{30}}{c^{20}}$
l. $\frac{8x^6}{27y^{12}}$ m. $\frac{b^{10}}{a^{10}}$ n. $162x^8y^{12}$

5. a. 2 b. 1 c. 31 d. 1
e. 25 f. 3 g. 1024 h. 3

6. a. 1 b. x c. $x + 1$ d. 1 e. 1 f. 1
g. $\frac{1}{y^3}$ h. m i. 2 j. 1 k. 0 l. 1

7. $-27x^{30}y^{12}$ **8.** $-3x^{51}$ **9.** $\frac{ac^6}{b}$

10. $-6y^7 + 4y^6 - 2y^3$ **11.** As is **12.** $2u^{22}$

13. abc **14.** $\frac{b^{35}}{a^{15}}$

I have no particular talent.
I am merely inquisitive.

Albert Einstein