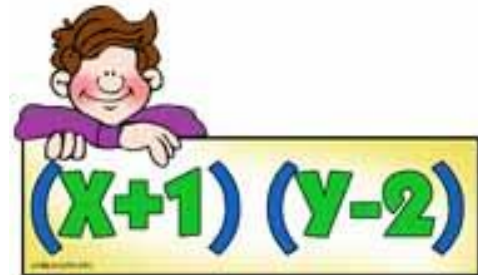

FACTORING QUADRATICS: THE REAL DEAL

□ INTRODUCTION

Hopefully we have grasped the concept of factoring. We now present a series of examples which try to turn the random method into something a little more systematic; but no matter what method is used, it's essentially a matter of *trial-and-error*. (Some teachers call it *guess-and-check*.)



□ EXAMPLES

EXAMPLE 1: **Factor:** $6x^2 - 7x - 5$

Solution: We first think about the ways that $6x^2$ can be broken up. We see $6x \cdot x$ and $3x \cdot 2x$.

Using $6x$ and x , we start with

$$(6x \quad \quad)(x \quad \quad)$$

Now put numbers in the slots whose product is -5 :

$$(6x + 5)(x - 1) = 6x^2 - 6x + 5x - 5 = 6x^2 - x - 5 \quad \text{⊗}$$

$$(6x - 5)(x + 1) = 6x^2 + 6x - 5x - 5 = 6x^2 + x - 5 \quad \text{⊗}$$

$$(6x + 1)(x - 5) = 6x^2 - 30x + x - 5 = 6x^2 - 29x - 5 \quad \text{⊗}$$

$$(6x - 1)(x + 5) = 6x^2 + 30x - x - 5 = 6x^2 + 29x - 5 \quad \text{⊗}$$

That's it for the -5 ; we've tried everything.

Now we'll start again, using $3x$ and $2x$ in the front slots:

$$(3x \quad \quad)(2x \quad \quad)$$

Now try the same combinations as above for the -5 :

$$(3x + 5)(2x - 1) = 6x^2 - 3x + 10x - 5 = 6x^2 + 7x - 5 \quad \ominus$$

$$(3x - 5)(2x + 1) = 6x^2 + 3x - 10x - 5 = \underline{6x^2 - 7x - 5} \quad \text{☺}$$

Although there are other ways to arrange the 5 and the 1, there's no need to go any further; we've found what we were looking for.

Therefore, the final factorization of $6x^2 - 7x - 5$ is

$$\boxed{(3x - 5)(2x + 1)}$$

EXAMPLE 2: **Factor:** $n^2 - 5n + 6$

Solution: As before, we focus on the first and last terms. There's only one way to break up the n^2 , namely $n \cdot n$. So we start with

$$(n \quad \quad)(n \quad \quad)$$

But there are a few ways to break up the 6. Noting that 6 is positive -- and therefore the signs of its factors must be the same -- let's begin experimenting:

$$(n + 6)(n + 1) = n^2 + n + 6n + 6 = n^2 + 7n + 6 \quad \ominus$$

$$(n + 1)(n + 6) = n^2 + 6n + n + 6 = n^2 + 7n + 6 \quad \ominus$$

$$(n - 6)(n - 1) = n^2 - n - 6n + 6 = n^2 - 7n + 6 \quad \ominus$$

$$(n - 1)(n - 6) = n^2 - 6n - n + 6 = n^2 - 7n + 6 \quad \ominus$$

$$(n + 3)(n + 2) = n^2 + 2n + 3n + 6 = n^2 + 5n + 6 \quad \ominus$$

$$(n + 2)(n + 3) = n^2 + 3n + 2n + 6 = n^2 + 5n + 6 \quad \ominus$$

$$(n - 3)(n - 2) = n^2 - 2n - 3n + 6 = \underline{n^2 - 5n + 6} \quad \text{☺}$$

and thus our final factorization is

$$(n - 3)(n - 2)$$

Did we really need all those attempts to find the answer? NO.

Isn't it possible to see that the factors of 6 must be negative, considering the $-5n$ term? OF COURSE.

You need to use only as many options as needed until you reach the required goal. Some students can just "see" what combination works, while others need a lot of experimenting. The more you practice, the quicker you'll find the right factors.

EXAMPLE 3: **Factor:** $y^2 + 7y + 14$

Solution: There's really only one way to begin; we split the y^2 term into y and y :

$$(y \quad \quad)(y \quad \quad)$$

Now let's work on the 14:

$$(y + 14)(y + 1) = y^2 + y + 14y + 14 = y^2 + 15y + 14 \quad \text{⊗}$$

$$(y + 1)(y + 14) = y^2 + 14y + y + 14 = y^2 + 15y + 14 \quad \text{⊗}$$

$$(y - 14)(y - 1) = y^2 - y - 14y + 14 = y^2 - 15y + 14 \quad \text{⊗}$$

$$(y - 1)(y - 14) = y^2 - 14y - y + 14 = y^2 - 15y + 14 \quad \text{⊗}$$

$$(y + 7)(y + 2) = y^2 + 2y + 7y + 14 = y^2 + 9y + 14 \quad \text{⊗}$$

$$(y + 2)(y + 7) = y^2 + 7y + 2y + 14 = y^2 + 9y + 14 \quad \text{⊗}$$

$$(y - 7)(y - 2) = y^2 - 2y - 7y + 14 = y^2 - 9y + 14 \quad \text{⊗}$$

$$(y - 2)(y - 7) = y^2 - 7y - 2y + 14 = y^2 - 9y + 14 \quad \text{⊗}$$

Nothing but sad faces, and we've tried every possible arrangement. This can mean only one thing: There's no way to factor the expression $y^2 + 7y + 14$. The expression is

$y^2 + 7y + 14$ is called **prime** because, like the prime number 13, it can't be factored.

Not factorable

EXAMPLE 4: **Factor:** $w^2 - 25$

Solution: This quadratic expression has only two terms, but which one is missing? When a quadratic starts with w^2 , we expect the next term to contain a w . So it's the middle term that is missing. But the middle term is not the one we focus on anyway, so let's begin the usual process.

The binomial $w^2 - 25$ is called a **difference of squares**, because it's w -squared minus 5-squared.

$$(w + 25)(w - 1) = w^2 - w + 25w - 25 = w^2 + 24w - 25 \quad \text{☹}$$

$$(w - 25)(w + 1) = w^2 + w - 25w - 25 = w^2 - 24w - 25 \quad \text{☹}$$

$$(w + 1)(w - 25) = w^2 - 25w + w - 25 = w^2 - 24w - 25 \quad \text{☹}$$

$$(w - 1)(w + 25) = w^2 + 25w - w - 25 = w^2 + 24w - 25 \quad \text{☹}$$

$$(w + 5)(w - 5) = w^2 - 5w + 5w - 25 = \underline{w^2 - 25} \quad \text{☺}$$

Therefore, the expression $w^2 - 25$ factors into

$$(w + 5)(w - 5)$$

EXAMPLE 5: Factor: $9u^2 + 12u + 4$

Solution: The quadratic term, $9u^2$, can be broken down two ways, $9u \times u$ and $3u \times 3u$.

Let's start the multiplications using $9u$ and u :

$$(9u + 4)(u + 1) = 9u^2 + 9u + 4u + 4 = 9u^2 + 13u + 4 \quad \ominus$$

$$(9u + 1)(u + 4) = 9u^2 + 36u + u + 4 = 9u^2 + 37u + 4 \quad \ominus$$

$$(9u + 2)(u + 2) = 9u^2 + 18u + 2u + 4 = 9u^2 + 20u + 4 \quad \ominus$$

That's about it for the $9u$ and u combination. Now for $3u$ and $3u$:

$$(3u + 4)(3u + 1) = 9u^2 + 3u + 12u + 4 = 9u^2 + 15u + 4 \quad \ominus$$

$$(3u + 1)(3u + 4) = 9u^2 + 12u + 3u + 4 = 9u^2 + 15u + 4 \quad \ominus$$

$$(3u + 2)(3u + 2) = 9u^2 + 6u + 6u + 4 = \underline{9u^2 + 12u + 4} \quad \text{☺}$$

Eureka! The factorization of $9u^2 + 12u + 4$ is $(3u + 2)(3u + 2)$, which we can write more succinctly as

$(3u + 2)^2$

EXAMPLE 6: Factor: $a^2 + 49$

Solution: Let's start experimenting right away:

$$(a + 7)(a + 7) = a^2 + 7a + 7a + 49 = a^2 + 14a + 49 \quad \ominus$$

$$(a + 49)(a + 1) = a^2 + a + 49a + 49 = a^2 + 50a + 49 \quad \ominus$$

$$(a + 1)(a + 49) = a^2 + 49a + a + 49 = a^2 + 50a + 49 \quad \ominus$$

$$(a - 7)(a - 7) = a^2 - 7a - 7a + 49 = a^2 - 14a + 49 \quad \ominus$$

$$(a - 49)(a - 1) = a^2 - a - 49a + 49 = a^2 - 50a + 49 \quad \ominus$$

$$(a - 1)(a - 49) = a^2 - 49a - a + 49 = a^2 - 50a + 49 \quad \ominus$$

Have we tried everything? It appears we have, and no smiley face! (As we saw before, not every expression can be factored.) So we say that $a^2 + 49$ is

Not factorable

First Notice: Even though $a^2 + 49$ is not factorable, $a^2 - 49$ is factorable, since $a^2 - 49 = (a + 7)(a - 7)$. The difference between a plus sign and a minus sign makes all the difference in the world -- so be careful!

Second Notice: Some students jump to the conclusion that when two terms are connected by a plus sign, the expression is not factorable. But consider $4x^2 + 16$. It may not factor with two sets of parentheses like we're learning in this chapter, but it does have a greatest common factor (GCF) of 4, which can be factored out to produce $4(x^2 + 4)$. Thus $4x^2 + 16$ is factorable.

Homework

1. Factor each expression:

- | | | |
|----------------------|--------------------|----------------------|
| a. $2x^2 + 3x + 1$ | b. $3n^2 - 7n + 2$ | c. $5a^2 + 3a - 2$ |
| d. $3m^2 - 11m - 20$ | e. $4x^2 - 3x - 1$ | f. $6u^2 + 7u - 10$ |
| g. $4z^2 - 4z - 3$ | h. $6y^2 - 5y - 6$ | i. $7n^2 - 45n + 18$ |
| j. $a^2 + 16a + 63$ | k. $2x^2 + x - 3$ | l. $7z^2 - 12z + 5$ |

2. Factor each expression:

- | | | |
|----------------------|---------------------|---------------------|
| a. $x^2 + 5x + 6$ | b. $x^2 - 5x + 6$ | c. $x^2 - 5x - 6$ |
| d. $x^2 + 5x - 6$ | e. $n^2 + 10n + 9$ | f. $z^2 - 4z - 5$ |
| g. $t^2 - 20t + 96$ | h. $u^2 - 6u - 16$ | i. $Q^2 + 34Q - 72$ |
| j. $x^2 + 25x + 156$ | k. $a^2 - 17a + 70$ | l. $t^2 + t - 110$ |

3. Factor each expression:

- | | | |
|----------------------|---------------------|-----------------------|
| a. $x^2 + 8x + 16$ | b. $y^2 - 10y + 25$ | c. $a^2 + 18a + 81$ |
| d. $b^2 - 20b + 100$ | e. $4z^2 + 4z + 1$ | f. $9n^2 - 24n + 16$ |
| g. $25x^2 - 30x + 9$ | h. $x^2 + 6x + 36$ | i. $2t^2 + 33t + 100$ |
| j. $16a^2 + 8a + 1$ | k. $9u^2 - 12u + 4$ | l. $w^2 + 50w + 625$ |

4. Factor each expression:

- | | | | |
|----------------|---------------|----------------|---------------|
| a. $p^2 - 1$ | b. $c^2 - 4$ | c. $R^2 - 16$ | d. $z^2 - 36$ |
| e. $x^2 - 25$ | f. $y^2 - 81$ | g. $n^2 - 10$ | h. $w^2 + 16$ |
| i. $a^2 - 144$ | j. $e^2 - 72$ | k. $m^2 + 100$ | l. $W^2 - 1$ |
| m. $x^2 - 9$ | n. $a^2 - 49$ | o. $g^2 - 64$ | p. $y^2 + 4$ |

5. Factor each expression:

- | | | | |
|-----------------|-------------------|------------------|------------------|
| a. $4x^2 - 9$ | b. $9y^2 - 49$ | c. $u^2 - 2$ | d. $v^2 + 1$ |
| e. $16z^2 - 49$ | f. $49w^2 - 16$ | g. $49a^2 - 144$ | h. $121b^2 - 64$ |
| i. $9x^2 + 2$ | j. $1 - x^2$ | k. $16 - n^2$ | l. $25 - 4g^2$ |
| m. $9 + t^2$ | n. $144N^2 - 169$ | o. $225a^2 - 1$ | |

6. Factor each expression:

- | | | |
|-----------------------|-----------------------|------------------------|
| a. $3x^2 + 10x - 8$ | b. $t^2 - 121$ | c. $y^2 + 10y + 25$ |
| d. $16a^2 - 121$ | e. $b^2 - 20$ | f. $n^2 + 121$ |
| g. $x^2 + 3x + 1$ | h. $12q^2 - 23q + 5$ | i. $6a^2 - 13a + 6$ |
| j. $x^2 + 14x + 13$ | k. $4y^2 - 49$ | l. $9Q^2 + 12Q + 4$ |
| m. $25z^2 - 10z + 1$ | n. $16x^2 + 34x - 15$ | o. $16x^2 + 118x - 15$ |
| p. $16x^2 - 77x - 15$ | q. $16x^2 - 72x + 45$ | r. $16a^2 - 8a + 1$ |
| s. $x^2 + 7x + 5$ | t. $8c^2 + 2c - 21$ | u. $8c^2 - 13c - 21$ |
| v. $3a^2 - 5a - 12$ | w. $6x^2 + 17x - 14$ | x. $2y^2 + 7y - 9$ |

Review Problems

7. Factor each expression:

a. $x^2 + 17x + 72$

b. $y^2 - 9y + 8$

c. $N^2 + 100$

d. $N^2 - 100$

e. $x^2 - 18x + 81$

f. $a^2 + 10a + 25$

g. $t^2 + 4t - 45$

h. $a^2 - 21a - 22$

i. $a^2 - 9a - 22$

j. $2x^2 - 9x - 5$

k. $6x^2 + x - 40$

l. $6x^2 - 13x - 15$

m. $6x^2 + 11x - 17$

n. $R^2 - 144$

o. $25n^2 - 30n + 9$

p. $T^2 + 144$

q. $49w^2 + 70w + 25$

r. $9a^2 - 9a + 2$

s. $9a^2 - 19a + 2$

t. $x^2 + 16x - 36$

u. $x^2 + 37x + 36$

v. $30c^2 - 11c + 1$

w. $16a^2 - 6a - 1$

x. $36q^2 + 60q + 25$

y. $x^2 + 8x + 18$

z. $32n^2 - 66n + 27$

Solutions

1. a. $(2x + 1)(x + 1)$

b. $(3n - 1)(n - 2)$

c. $(5a - 2)(a + 1)$

d. $(3m + 4)(m - 5)$

e. $(4x + 1)(x - 1)$

f. $(6u - 5)(u + 2)$

g. $(2z + 1)(2z - 3)$

h. $(2y - 3)(3y + 2)$

i. $(7n - 3)(n - 6)$

j. $(a + 9)(a + 7)$

k. $(2x + 3)(x - 1)$

l. $(7z - 5)(z - 1)$

2. a. $(x + 3)(x + 2)$ b. $(x - 2)(x - 3)$ c. $(x - 6)(x + 1)$
 d. $(x + 6)(x - 1)$ e. $(n + 9)(n + 1)$ f. $(z - 5)(z + 1)$
 g. $(t - 12)(t - 8)$ h. $(u - 8)(u + 2)$ i. $(Q + 36)(Q - 2)$
 j. $(x + 13)(x + 12)$ k. $(a - 10)(a - 7)$ l. $(t + 11)(t - 10)$
3. a. $(x + 4)^2$ b. $(y - 5)^2$ c. $(a + 9)^2$
 d. $(b - 10)^2$ e. $(2z + 1)^2$ f. $(3n - 4)^2$
 g. $(5x - 3)^2$ h. Not factorable i. $(2t + 25)(t + 4)$
 j. $(4a + 1)^2$ k. $(3u - 2)^2$ l. $(w + 25)^2$
4. a. $(p + 1)(p - 1)$ b. $(c + 2)(c - 2)$ c. $(R + 4)(R - 4)$
 d. $(z + 6)(z - 6)$ e. $(x + 5)(x - 5)$ f. $(y + 9)(y - 9)$
 g. Not factorable h. Not factorable i. $(a + 12)(a - 12)$
 j. Not factorable k. Not factorable l. $(W + 1)(W - 1)$
 m. $(x + 3)(x - 3)$ n. $(a + 7)(a - 7)$ o. $(g + 8)(g - 8)$
 p. Not factorable
5. a. $(2x + 3)(2x - 3)$ b. $(3y + 7)(3y - 7)$ c. Not factorable
 d. Not factorable e. $(4z + 7)(4z - 7)$ f. $(7w + 4)(7w - 4)$
 g. $(7a + 12)(7a - 12)$ h. $(11b + 8)(11b - 8)$ i. Not factorable
 j. $(1 + x)(1 - x)$ k. $(4 + n)(4 - n)$ l. $(5 + 2g)(5 - 2g)$
 m. Not factorable n. $(12N + 13)(12N - 13)$ o. $(15a + 1)(15a - 1)$
6. a. $(3x - 2)(x + 4)$ b. $(t + 11)(t - 11)$ c. $(y + 5)^2$
 d. $(4a + 11)(4a - 11)$ e. Not factorable f. Not factorable
 g. Not factorable h. $(3q - 5)(4q - 1)$ i. $(2a - 3)(3a - 2)$
 j. $(x + 1)(x + 13)$ k. $(2y + 7)(2y - 7)$ l. $(3Q + 2)^2$
 m. $(5z - 1)^2$ n. $(8x - 3)(2x + 5)$ o. $(8x - 1)(2x + 15)$
 p. $(16x + 3)(x - 5)$ q. $(4x - 3)(4x - 15)$ r. $(4a - 1)^2$

- s. Not factorable t. $(4c + 7)(2c - 3)$ u. $(8c - 21)(c + 1)$
 v. $(3a + 4)(a - 3)$ w. $(2x + 7)(3x - 2)$ x. $(2y + 9)(y - 1)$
7. a. $(x + 9)(x + 8)$ b. $(y - 1)(y - 8)$ c. Not factorable
 d. $(N + 10)(N - 10)$ e. $(x - 9)^2$ f. $(a + 5)^2$
 g. $(t + 9)(t - 5)$ h. $(a - 22)(a + 1)$ i. $(a - 11)(a + 2)$
 j. $(2x + 1)(x - 5)$ k. $(3x + 8)(2x - 5)$ l. $(6x + 5)(x - 3)$
 m. $(6x + 17)(x - 1)$ n. $(R + 12)(R - 12)$ o. $(5n - 3)^2$
 p. Not factorable q. $(7w + 5)^2$ r. $(3a - 1)(3a - 2)$
 s. $(9a - 1)(a - 2)$ t. $(x + 18)(x - 2)$ u. $(x + 36)(x + 1)$
 v. $(5c - 1)(6c - 1)$ w. $(8a + 1)(2a - 1)$ x. $(6q + 5)^2$
 y. Not factorable z. $(16n - 9)(2n - 3)$

“The school is the last expenditure on which America should be willing to economize.”

Franklin D. Roosevelt

